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NUMERICAL SOLUTION OF A NEW HYDRODYNAMIC MODEL OF FLOCKING

Václav Kučera, Andrea Živčáková

Charles University in Prague, Faculty of Mathematics and Physics
Sokolovská 83, 186 75 Praha, Czech Republic
kucera@karlin.mff.cuni.cz, zivcakova@karlin.mff.cuni.cz

Abstract

This work is concerned with the numerical solution of a hydrodynamic model of the macroscopic behavior of flocks of birds due to Fornasier et al., 2011. The model consists of the compressible Euler equations with an added nonlocal, nonlinear right-hand side. As noticed by the authors of the model, explicit time schemes are practically useless even on very coarse grids in 1D due to the nonlocal nature of the equations. To this end, we apply a semi-implicit discontinuous Galerkin method to solve the equations. We present a simple numerical test of the resulting scheme.

1. Continuous problem

In [4], a new hydrodynamic limit of a modification of the famous Cucker-Smale model was derived. The equations describe, using macroscopic quantities, the dynamics of flocks of birds or other self-organizing entities. The equations are highly nonlinear and nonlocal and are therefore extremely expensive to treat numerically. In [4] a first simple simulation was performed using the finite volume method. Here, we discretize the model more efficiently using the discontinuous Galerkin method.

Let $\Omega = (0,1) \subset \mathbb{R}$ and for $0 < M < +\infty$, we set $Q_M := \Omega \times (0, M)$. We treat the following problem written in conservative variables. Find $\boldsymbol{w} : Q_M \to \mathbb{R}^3$ such that

$$\frac{\partial \boldsymbol{w}}{\partial t} + \frac{\partial \boldsymbol{f}(\boldsymbol{w})}{\partial x} = \boldsymbol{g}(\boldsymbol{w}) \ \text{ in } Q_M, \tag{1}$$

where $\boldsymbol{w} = (\rho, \rho u, E)^\top \in \mathbb{R}^3$ is the *state vector* and

$$\begin{aligned} \boldsymbol{f}(\boldsymbol{w}) &= \big(f_1(\boldsymbol{w}), f_2(\boldsymbol{w}), f_3(\boldsymbol{w})\big)^\top = \big(\rho u, \rho u^2 + p, (E+p)u\big)^\top, \\ \boldsymbol{g}(\boldsymbol{w}) &= \lambda\big(0, \mathcal{A}(\boldsymbol{w}), \mathcal{B}(\boldsymbol{w})\big)^\top. \end{aligned} \tag{2}$$

Here ρ denotes the density, u velocity, E energy and p pressure. The right-hand side functions $\mathcal{A}$ and $\mathcal{B}$ are given by

$$\begin{aligned} \mathcal{A}(\boldsymbol{w})(x,t) &= \int_{\mathbb{R}} b(|x-y|)\Big(u(y,t) - u(x,t)\Big)\rho(x,t)\rho(y,t)\,\mathrm{d}y, \\ \mathcal{B}(\boldsymbol{w})(x,t) &= \int_{\mathbb{R}} b(|x-y|)\rho(x,t)\Big(\rho(y,t)u(x,t)u(y,t) - 2E(y,t)\Big)\,\mathrm{d}y, \end{aligned} \tag{3}$$

where

$$b(|x-y|) = \frac{K}{(\lambda + |x-y|^2)^{\beta+1}}, \tag{4}$$

and $K, \lambda > 0$ and $\beta \geq 0$ are given constants. The relations between E, p are the classical laws of a perfect gas,

$$E = \rho\left(\frac{3}{2}T + \frac{u^2}{2}\right), \quad p = \rho T, \tag{5}$$

where T is the thermodynamic temperature.

In (3), we write the right-hand side terms $\mathcal{A}, \mathcal{B}$ as functions of $\boldsymbol{w}$, although the integrals in (3) are written terms of the nonconservative variables ρ, u, T. Expressing $\mathcal{A}, \mathcal{B}$ in $\boldsymbol{w}$ in a suitable way is a key ingredient in our scheme and will be described in detail in Section 2.3. System (1) is equipped with the initial condition $\boldsymbol{w}(x,0) = \boldsymbol{w}^0(x)$ and periodic boundary conditions.

2. Discretization

We shall use the multidimensional notation for $\Omega \subset \mathbb{R}^d$, although in our computations we have $d = 1$. Let $\mathcal{T}_h$ be triangulation of Ω and $\mathcal{F}_h$ the system of all faces (nodes in 1D) of $\mathcal{T}_h$. For each $\Gamma \in \mathcal{F}_h$ we choose a unit normal $n_\Gamma = \pm 1$, which, for $\Gamma \subset \partial\Omega$, has the same orientation as the outer normal to Ω. For each *interior* face $\Gamma \in \mathcal{F}_h$ there exist two neighbours $K_\Gamma^{(L)}$, $K_\Gamma^{(R)} \in \mathcal{T}_h$ such that n_Γ is the outer normal to $K_\Gamma^{(L)}$. For v piecewise defined on $\mathcal{T}_h$ and $\Gamma \in \mathcal{F}_h$ we introduce $v|_\Gamma^{(L)}$ is the trace of $v|_{K_\Gamma^{(L)}}$ on Γ, $v|_\Gamma^{(R)}$ is the trace of $v|_{K_\Gamma^{(R)}}$ on Γ, $\langle v \rangle_\Gamma = \frac{1}{2}\big(v|_\Gamma^{(L)} + v|_\Gamma^{(R)}\big)$ and $[v]_\Gamma = v|_\Gamma^{(L)} - v|_\Gamma^{(R)}$. On $\partial\Omega$, we define $v|_\Gamma^{(L)}, v|_\Gamma^{(R)}$ using periodic boundary conditions. If $[\cdot]_\Gamma, \langle\cdot\rangle_\Gamma, v|_\Gamma^{(L)}, v|_\Gamma^{(R)}$ appear in an integral over $\Gamma \in \mathcal{F}_h$, we omit the subscript Γ.

Let $p \in \mathbb{N}$ and let $P^p(K)$ be the space of polynomials on $K \in \mathcal{T}_h$ of degree $\leq p$. The approximate solution will be sought in the space of discontinuous piecewise polynomial functions

$$\boldsymbol{S}_h := [S_h]^3, \text{ where } S_h = \{v;\, v|_K \in P^p(K), \forall K \in \mathcal{T}_h\}.$$

2.1. Discontinuous Galerkin space semidiscretization

The discrete problem is derived in the following way. We multiply (1) by a test function $\boldsymbol{\varphi}_h \in \boldsymbol{S}_h$, integrate over $K \in \mathcal{T}_h$ and apply Green's theorem in the convective terms. Summing over $K \in \mathcal{T}_h$ and rearranging the boundary terms, we obtain

$$\int_\Omega \frac{\partial \boldsymbol{w}}{\partial t} \cdot \boldsymbol{\varphi}\,\mathrm{d}x + \sum_{\Gamma\in\mathcal{F}_h}\int_\Gamma \boldsymbol{f}(\boldsymbol{w})n\cdot[\boldsymbol{\varphi}]\,\mathrm{d}S - \sum_{K\in\mathcal{T}_h}\int_K \boldsymbol{f}(\boldsymbol{w})\cdot\frac{\partial\boldsymbol{\varphi}}{\partial x}\,\mathrm{d}x = \int_\Omega \boldsymbol{g}(\boldsymbol{w})\cdot\boldsymbol{\varphi}\,\mathrm{d}x. \tag{6}$$

Since $\boldsymbol{w}$ will be approximated by a function from $\boldsymbol{S}_h$, which are discontinuous on edges, we approximate the physical flux $\boldsymbol{f}(\boldsymbol{w})n$ through an edge $\Gamma \in \mathcal{F}_h$ by a so-called *numerical flux* $\mathbf{H}(\boldsymbol{w}^{(L)}, \boldsymbol{w}^{(R)}, n)$ as in the finite volume method. In our computations

we use the *Vijayasundaram* numerical flux, cf. [5, 2]. Now we can define the following forms defined for $\boldsymbol{w}, \boldsymbol{\varphi} \in \boldsymbol{S}_h$:
Convective form:

$$b_h(\boldsymbol{w}, \boldsymbol{\varphi}) = \sum_{\Gamma \in \mathcal{F}_h} \int_\Gamma \mathbf{H}(\boldsymbol{w}^{(L)}, \boldsymbol{w}^{(R)}, n) \cdot [\boldsymbol{\varphi}] \, \mathrm{d}S - \sum_{K \in \mathcal{T}_h} \int_K \boldsymbol{f}(\boldsymbol{w}) \cdot \frac{\partial \boldsymbol{\varphi}}{\partial x} \, \mathrm{d}x,$$

right-hand side source term form:

$$l_h(\boldsymbol{w}, \boldsymbol{\varphi}) = -\int_\Omega \boldsymbol{g}(\boldsymbol{w}) \cdot \boldsymbol{\varphi} \, \mathrm{d}x.$$

Finally, we introduce the space semi-discrete problem: We seek $\boldsymbol{w}_h \in C^1([0, M]; \boldsymbol{S}_h)$:

$$\frac{d}{dt}(\boldsymbol{w}_h(t), \boldsymbol{\varphi}_h) + b_h(\boldsymbol{w}_h(t), \boldsymbol{\varphi}_h) + l_h(\boldsymbol{w}_h(t), \boldsymbol{\varphi}_h) = 0, \ \forall \boldsymbol{\varphi}_h \in \boldsymbol{S}_h, \ \forall t \in (0, M). \quad (7)$$

2.2. Time discretization

Equation (7) represents a system of nonlinear ordinary differential equations, which must be discretized in time. Due to extreme time step restrictions caused by the nonlocal right-hand side terms, cf. [4], we want to avoid using an explicit scheme. However an implicit time discretization is also very expensive due to its nonlinearity. Therefore we use the semi-implicit scheme of [3] and apply it to our problem.

Let $0 = t_0 < t_1 < t_2 < \dots$ be a partition of time interval $[0, M]$ and define $\tau_k = t_{k+1} - t_k$. We approximate $\boldsymbol{w}_h^k \approx \boldsymbol{w}_h(t_k)$, where $\boldsymbol{w}_h^k \in \boldsymbol{S}_h$. We use a first order backward difference approximation for the time derivative. Following [3], the nonlinear convective terms $b_h(\boldsymbol{w}_h^{k+1}, \boldsymbol{\varphi}_h)$ are linearized as

$$\begin{aligned} \tilde{b}_h(\boldsymbol{w}_h^k, \boldsymbol{w}_h^{k+1}, \boldsymbol{\varphi}_h) = &- \sum_{K \in \mathcal{T}_h} \int_K \mathbb{A}\,(\boldsymbol{w}_h^k) \boldsymbol{w}_h^{k+1} \cdot \frac{\partial \boldsymbol{\varphi}_h}{\partial x} \, \mathrm{d}x \\ &+ \int_{\mathcal{F}_h} \Big(\mathbb{P}^+\big(\langle \boldsymbol{w}_h^k \rangle, n\big) \boldsymbol{w}_h^{k+1,(L)} + \mathbb{P}^-\big(\langle \boldsymbol{w}_h^k \rangle, n\big) \boldsymbol{w}_h^{k+1,(R)} \Big) \cdot [\boldsymbol{\varphi}_h] \, \mathrm{d}S, \end{aligned} \quad (8)$$

where $\mathbb{A} = \frac{D\boldsymbol{f}}{D\boldsymbol{w}}$ and $\mathbb{P}^+, \mathbb{P}^-$ are matrices defining the Vijayasundaram numerical flux, cf. [3] for details.

As for the source terms, again we linearize them to obtain the approximation $l_h(\boldsymbol{w}_h^{k+1}, \boldsymbol{\varphi}_h) \approx \tilde{l}_h(\boldsymbol{w}_h^k, \boldsymbol{w}_h^{k+1}, \boldsymbol{\varphi}_h)$. The specific construction of this linearization is technical and will be presented separately in Section 2.3.

Collecting all the considerations, we obtain the following semi-implicit DG scheme:

Definition 1. *We say that the sequence $\boldsymbol{w}_h^k \in \boldsymbol{S}_h, k = 0, 1, \dots,$ is a semi-implicit DG solution of problem (1) if for all $\boldsymbol{\varphi}_h \in \boldsymbol{S}_h$*

$$\Big(\frac{\boldsymbol{w}_h^{k+1} - \boldsymbol{w}_h^k}{\tau_k}, \boldsymbol{\varphi}_h \Big) + \tilde{b}_h(\boldsymbol{w}_h^k, \boldsymbol{w}_h^{k+1}, \boldsymbol{\varphi}_h) + \tilde{l}_h(\boldsymbol{w}_h^k, \boldsymbol{w}_h^{k+1}, \boldsymbol{\varphi}_h) = 0. \quad (9)$$

Equation (9) represents a linear equation for the unknown $\boldsymbol{w}_h^{k+1}$. By choosing basis functions of $\boldsymbol{S}_h$ with supports on only one element, we obtain a sparse, block-tridiagonal matrix with lower left and upper right blocks corresponding to periodic boundary conditions. To solve these systems, we use the direct solver UMFPACK, [1]. It is our goal to construct $\tilde{l}_h$ in such a way so as to preserve the sparsity structure of the systems solved.

2.3. Linearization of the source terms l_h

First, we rewrite the right-hand side integrals $\mathcal{A}, \mathcal{B}$ in terms of the conservative variables. For the integral $\mathcal{A}$, we obtain

$$\mathcal{A} = \int_{\mathbb{R}} b(|x-y|)\boldsymbol{w}(x,t) \cdot \big(w_2(y,t), -w_1(y,t), 0\big)\,\mathrm{d}y. \tag{10}$$

Similarly, we write $\mathcal{B}$ as

$$\mathcal{B} = \int_{\mathbb{R}} b(|x-y|)\boldsymbol{w}(x,t) \cdot \big(-2w_3(y,t), w_2(y,t), 0\big)\,\mathrm{d}y. \tag{11}$$

Therefore, we can rewrite the vector $\boldsymbol{g}(\boldsymbol{w})$ as

$$\boldsymbol{g}(\boldsymbol{w})(x,t) = \lambda \int_{\mathbb{R}} b(|x-y|)\mathbb{U}_2\big(\boldsymbol{w}(y,t)\big)\boldsymbol{w}(x,t)\,\mathrm{d}y, \tag{12}$$

where $\mathbb{U}_2(\boldsymbol{w}) \in \mathbb{R}^{3\times 3}$ is the matrix

$$\mathbb{U}_2(\boldsymbol{w}) = \begin{pmatrix} 0 & 0 & 0 \\ w_2 & -w_1 & 0 \\ -2w_3 & w_2 & 0 \end{pmatrix}.$$

Approximating $\boldsymbol{w}(x,t) \approx \boldsymbol{w}_h^{k+1}(x)$ and $\boldsymbol{w}(y,t) \approx \boldsymbol{w}_h^{k}(y)$, we get the linearized form

$$\tilde{l}_h(\boldsymbol{w}_h^k, \boldsymbol{w}_h^{k+1}, \boldsymbol{\varphi}_h) = \int_{\mathbb{R}} \left(\int_{\mathbb{R}} b(|x-y|)\mathbb{U}_2\big(\boldsymbol{w}_h^k(y)\big)\mathrm{d}y \right) \boldsymbol{w}_h^{k+1}(x) \cdot \boldsymbol{\varphi}_h(x)\,\mathrm{d}x. \tag{13}$$

Adding $\tilde{l}_h$ to the scheme (9) does not change the sparsity structure of the system matrix, since it contributes only to the block-diagonal. This is important, since other expressions than (12) are possible, however they lead to a full system matrix, which is undesirable. Nonetheless, the computation of these terms is extremely time consuming due to their nonlocal nature. Even if the basis functions of $\boldsymbol{S}_h$ are local, in order to evaluate $\tilde{l}_h$, we must compute the inner integral $\int_{\mathbb{R}} b(|x-y|)\mathbb{U}_2\big(\boldsymbol{w}_h^k(y)\big)\mathrm{d}y$, which is time consuming due to the slow decay of the function $b(|x-y|)$.

3. Numerical experiment

In this numerical experiment, we start at $t = 0$ with a Gaussian distribution of density $\rho(x) = \exp(-10(x - 0.5)^2)$ along with constant temperature $T = 10$ and the velocity distribution $u(x) = -\sin(2\pi x)$. The triangulation consists of 400 piecewise quadratic elements. We observe the formation of a sharp peak in ρ, as seen in Figure 1. Due to jumps in the solution, artificial diffusion was added, cf. [3].

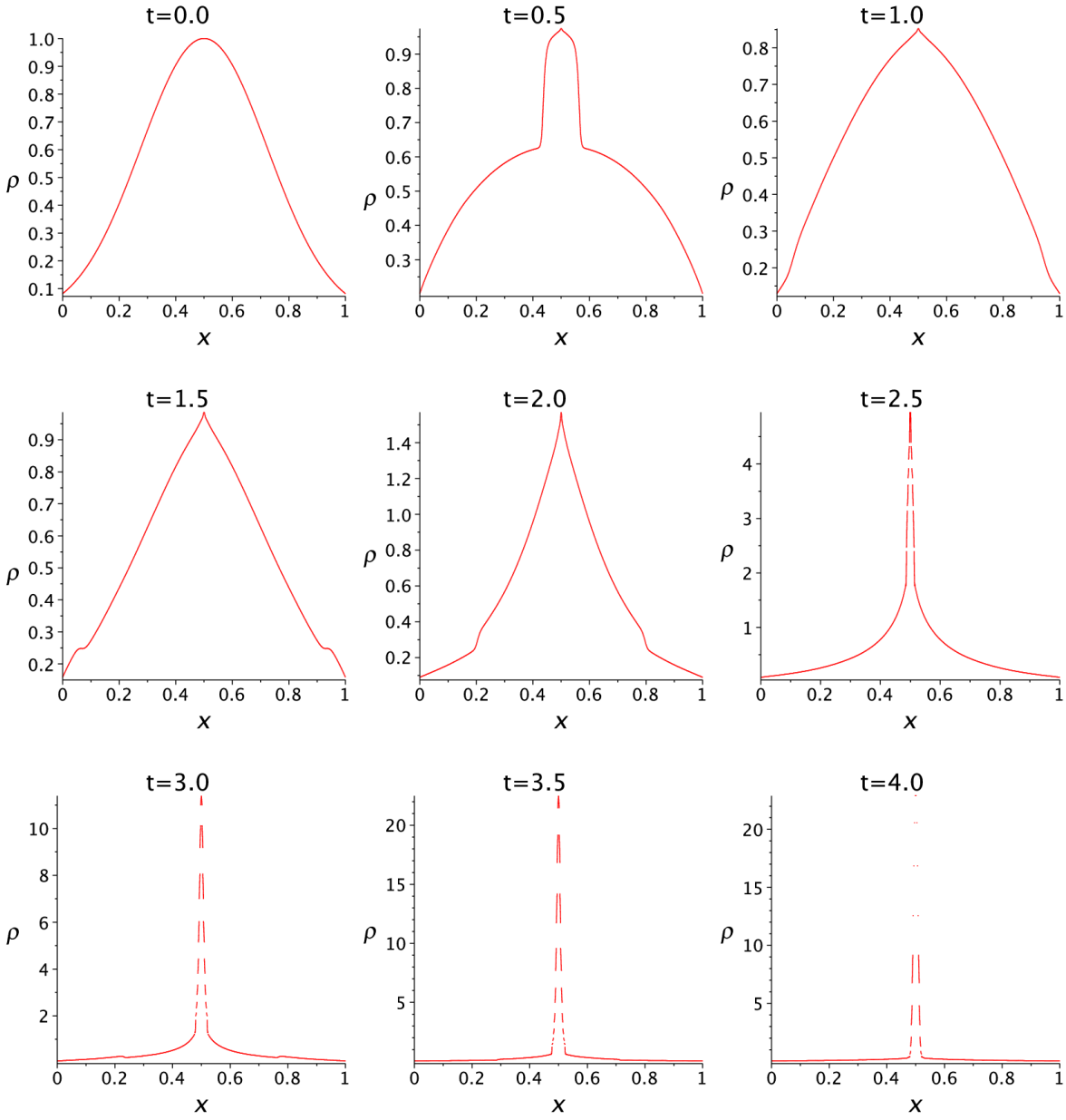

Figure 1: Numerical results for density.

Furthermore, in large regions of Ω, a state close to *vacuum* occurs, i.e. $\rho \approx 0$ and the matrices $\mathbb{A}, \mathbb{P}^+, \mathbb{P}^-$ are no longer defined. To avoid this complication, at each time step, $\boldsymbol{w}_h^k$ was *postprocessed* to avoid the vacuum state: If $\rho < \varepsilon$ or $T < \varepsilon$, then set $\rho := \varepsilon$ or $T := \varepsilon$ and recompute the energy E using relation (5). This defines a new state $\tilde{\boldsymbol{w}}_h^k$ which is used in (9) instead of $\boldsymbol{w}_h^k$ to compute $\boldsymbol{w}_h^{k+1}$. In our case, we use $\varepsilon := 10^{-5}$.

4. Conclusion

We have presented an efficient numerical method for the solution of a nonlinear and nonlocal version of the compressible Euler equations describing the dynamics of flocks of birds from [4]. To avoid severe time step restrictions, a semi-implicit discontinuous Galerkin scheme is applied. A suitable treatment of the nonlocal terms is given, which leads to sparse linear systems. Shock capturing and postprocessing of vacuum are added to obtain a stable scheme. To our knowledge, these are the first numerical results for this model, except for one test case in the original work [4].

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