Hans-Jürgen Schmeisser; Dorothee D. Haroske

Extrapolation of function spaces and related topics

In: Jiří Rákosník (ed.): Nonlinear Analysis, Function Spaces and Applications, Proceedings of the International School held in Třešť, September 11-17, 2010, Vol. 9. Institute of Mathematics AS CR, Praha, 2011. pp. 271--301.

Persistent URL: http://dml.cz/dmlcz/702644

EXTRAPOLATION OF FUNCTION SPACES AND RELATED TOPICS

Dorothee D. Haroske and Hans-Jürgen Schmeisser

Abstract. We survey recent contributions dealing with function spaces of Lorentz-Zygmund type and Lipschitz type which can be obtained by extrapolation techniques, and study equivalent characterizations. Furthermore, we collect results connected with embeddings and decompositions in such spaces, as well as the case of missing derivatives. We only present some model proofs and describe ideas how to deal with such questions; otherwise we refer to the original papers for details and related topics.

1. Introduction

Extrapolation theory in the proper sense is usually considered to start with the paper by Yano [103], although Titchmarsh [90], [91] and Zygmund [105], [106] had considered some similar results before. Strong motivation came from harmonic analysis, in particular, mapping properties of operators; for further historic details and explanations we refer to [74] and [10]. Meanwhile this topic is well-established and studied in detail, we refer to the papers and books by Jawerth, Milman [56], [57], Milman [71], [72], Karadzhov, Milman [58], and, recently, by Cobos, Fernández-Cabrera, Manzano, Martínez [11] for the abstract background. Our interest comes from the investigation of embeddings in limiting situations, such as the famous Sobolev embedding [87] or the Brézis-Wainger embedding [6] which attracted a lot of interest in the past. But we have also other limiting situations in mind, where the identification of some source or target space as extrapolation space is often an essential tool to study further questions like compactness or decompositions in such situations. Quite

2010 *Mathematics Subject Classification.* 46E35.

Key words and phrases. extrapolation, Lorentz-Zygmund spaces, Sobolev spaces, reduced Sobolev spaces, Besov spaces, Lipschitz spaces, limiting embeddings, case of missing derivatives.

recently applications (to spectral theory, approximation problems) require precise knowledge about spaces that arise by extrapolation procedures – and similar to interpolation theory, one wants to know under what assumptions the resulting space 'inherits' features from the extrapolated spaces. Though one can think of many different parameters and properties to be interpolated, we shall mainly concentrate on function spaces where extrapolation concerns their integrability (parameters) or smoothness (parameters).

We are certainly not experts on the abstract extrapolation theory, but have worked in this area (in concrete situations) for a couple of years. There is also joint interest with our friend and colleague Miroslav Krbec on which we shall comment in the following: this short survey paper was presented by the second author at the *Spring School on Nonlinear Analysis, Function Spaces and Applications 9* (NAFSA-9) in Třešť in September 2010, where in a special session the 60th birthday of our colleague was marked. Over the last years there developed a wide and intensive collaboration on related topics between groups in Prague, Brighton and Cardiff, Aveiro and Coimbra, and Jena (and further colleagues), as can easily be observed from the list of references as well. Another reason for us to return to this subject recently is connected with wavelet decomposition techniques in extrapolation spaces, we shall give some more details in the end.

Apart from the abstract approach there is a long-standing and fruitful collaboration of Czech mathematicians with a large number of different colleagues studying spaces of logarithmic and more general smoothness and integrability with respect to embeddings, norms, mapping properties, compactness assertions, interpolation; we give some references below.

The paper is organised as follows. In Section 2 we concentrate on Lorentz-Zygmund spaces, their definition, basic properties, extrapolation results and connection to limiting embeddings. Similarly we proceed in Section 3 with Lipschitz spaces, where now – in contrast to Section 2 – smoothness is extrapolated. Finally, in Section 4 we discuss some further settings in view of spaces on $\mathbb{R}^n$, Sobolev spaces and Besov spaces.

2. Lorentz-Zygmund spaces

First we fix some notation. By $\mathbb{N}$ we denote the set of natural numbers, by $\mathbb{N}_0$ the set $\mathbb{N} \cup \{0\}$, and by $\mathbb{Z}^n$ the set of all lattice points in $\mathbb{R}^n$ having integer components. The positive part of a real function f is given by $f_+(x) = \max(f(x), 0)$, the integer part of $a \in \mathbb{R}$ by $\lfloor a \rfloor = \max\{k \in \mathbb{Z} : k \le a\}$. If $0 < u \le \infty$, the number u' is given by $\frac{1}{u'} = (1 - \frac{1}{u})_+$. For two positive real sequences $\{\alpha_k\}_{k\in\mathbb{N}}$ and $\{\beta_k\}_{k\in\mathbb{N}}$ we mean by $\alpha_k \approx \beta_k$ that there exist constants $c_1, c_2 > 0$ such that $c_1\alpha_k \le \beta_k \le c_2\alpha_k$ for all $k \in \mathbb{N}$; similarly for positive functions. Given two (quasi-) Banach spaces X and Y, we write $X \hookrightarrow Y$ if $X \subset Y$ and the natural embedding of X in Y is continuous.

All unimportant positive constants will be denoted by c, occasionally with subscripts.

2.1. Lorentz-Zygmund spaces. Let $L_p(\Omega)$, $0 < p \le \infty$, be the (quasi-) Banach space with respect to Lebesgue measure, normed by

$$\| f \mid L_p(\Omega)\| = \Big(\int_\Omega |f(x)|^p \,\mathrm{d}x\Big)^{1/p}, \tag{2.1}$$

(with the usual modification for $p = \infty$), where $\Omega \subset \mathbb{R}^n$ stands for a domain in $\mathbb{R}^n$. A natural refinement of this scale of Lebesgue spaces are the spaces $L_p(\log L)_a(\Omega)$ being the set of all measurable functions $f\colon \Omega \to \mathbb{C}$ such that

$$\int_\Omega |f(x)|^p \log^{ap}(2 + |f(x)|)\,\mathrm{d}x < \infty. \tag{2.2}$$

This definition (2.2) for spaces $L_p(\log L)_a(\Omega)$ may be found in the book of Bennett and Sharpley in [4, Ch. 4, Def. 6.11] where $1 < p < \infty$, $a \in \mathbb{R}$, and $\Omega \subset \mathbb{R}^n$ with $|\Omega| < \infty$. They are called *Zygmund spaces* there. We use an alternative definition (admitting also parameters $0 < p \le 1$ and $p = \infty$), presented in Definition 2.1 below.

In [4, Ch. 4, Lemma 6.12] it is shown that $f \in L_p(\log L)_a(\Omega)$, $1 < p < \infty$, $a \in \mathbb{R}$, if, and only if,

$$\left(\int_0^{|\Omega|} \big[(1 + |\log t|)^a f^*(t)\big]^p \mathrm{d}t\right)^{1/p} < \infty, \tag{2.3}$$

where f^* denotes the non-increasing rearrangement of f, as usual,

$$f^*(t) = \inf\big\{s > 0 : |\{x \in \Omega : |f(x)| > s\}| \le t\big\}, \quad t > 0. \tag{2.4}$$

There is a plenty of literature on this topic; we refer to [4, Ch. 2], [16, Ch. 2, §2], for instance. In view of (2.3) we come to an alternative definition of $L_p(\log L)_a(\Omega)$, which simultaneously extends it to parameters $0 < p \le \infty$.

Definition 2.1. Let $\Omega \subset \mathbb{R}^n$, and $0 < p, q \le \infty$ and $a \in \mathbb{R}$. The *Lorentz-Zygmund space* $L_{p,q}(\log L)_a = L_{p,q}(\log L)_a(\Omega)$ consists of all measurable functions $f\colon \Omega \to \mathbb{C}$ for which

$$\| f \mid L_{p,q}(\log L)_a(\Omega)\| = \begin{cases} \left(\displaystyle\int_0^{|\Omega|} \left[t^{\frac{1}{p}} (1+|\log t|)^a f^*(t)\right]^q \frac{\mathrm{d}t}{t} \right)^{1/q}, & 0<q<\infty, \\ \displaystyle\sup_{0<t<|\Omega|} t^{\frac{1}{p}} (1+|\log t|)^a f^*(t), & q=\infty \end{cases} \tag{2.5}$$

is finite.

Remark 2.2. The above definition may be found in [4, Ch. 4, Def. 6.13] and in [3, (1.4), (1.14)]. Note that $L_{p,p}(\Omega) = L_p(\Omega)$ are the usual Lebesgue spaces, $0 < p \leq \infty$, and $L_{p,q}(\log L)_0(\Omega) = L_{p,q}(\Omega)$ are the *Lorentz spaces*. The spaces $L_{p,q}(\log L)_a(\Omega)$ are trivial when $p = \infty$, $0 < q < \infty$, and $a + 1/q \geq 0$, or $p = q = \infty$, but $a > 0$; thus in case of $p = \infty$ we only study spaces $L_{\infty,q}(\log L)_a$ in the sequel, where $a + 1/q < 0$ for $0 < q < \infty$, or $a \leq 0$ for $q = \infty$, respectively.

It is sometimes more convenient to work with the discretisation of (2.5),

$$\| f \mid L_{p,q}(\log L)_a(\Omega)\| \approx \left(\sum_{k=-\infty}^{\infty} \left[\mathrm{e}^{-k/p} (1+|k|)^a f^*(\mathrm{e}^{-k})\right]^q \right)^{1/q} \tag{2.6}$$

where one benefits from the monotonicity of f^*.

Note that (2.5) does not give a norm in any case, not even for $p, q \geq 1$. However, replacing the non-increasing rearrangement f^* in (2.5) by its maximal function

$$f^{**}(t) = \frac{1}{t} \int_0^t f^*(s)\,\mathrm{d}s, \quad t > 0, \tag{2.7}$$

one obtains for $1 < p < \infty$, $1 \leq q \leq \infty$, or $p = q = \infty$, a norm in that way, the corresponding expressions (2.5) with f^* and f^{**}, respectively, being equivalent; cf. [4, Ch. 4, Lemma 4.5, Thm. 4.6].

Remark 2.3. The spaces $L_{p,q}(\log L)_a(\Omega)$ are monotonically ordered in q (for fixed p and a) as well as in a (when p, q are fixed). In particular, for $a_1, a_2 \in \mathbb{R}$, $a_2 < a_1$,

$$L_p(\log L)_{a_1}(\Omega) \hookrightarrow L_p(\log L)_{a_2}(\Omega). \tag{2.8}$$

Moreover, when $|\Omega| < \infty$, then there is also some monotonicity in p, i.e. we obtain for any $0 < \varepsilon < p$ and all $a > 0$,

$$L_{p+\varepsilon}(\Omega) \hookrightarrow L_p(\log L)_a(\Omega) \hookrightarrow L_p(\Omega) \hookrightarrow L_p(\log L)_{-a}(\Omega) \hookrightarrow L_{p-\varepsilon}(\Omega), \tag{2.9}$$

see [35, Rem. 2.1/2] and [36, Prop. 2.6.1/1 (i)]. Otherwise, when $|\Omega| = \infty$, there is no monotonicity in p. But for fixed p, there is an interplay between q and a; cf. [3, Thms. 9.3, 9.5]: let $0 < p, q, r \le \infty$, $a, b \in \mathbb{R}$, with $a + \frac{1}{q} < 0$, $b + \frac{1}{r} < 0$ if $r, q < p = \infty$, or $a \le 0$ when $p = q = \infty$, $b \le 0$ if $p = r = \infty$, respectively. Then

$$L_{p,q}(\log L)_a(\Omega) \hookrightarrow L_{p,r}(\log L)_b(\Omega) \quad \text{if} \quad \begin{cases} \text{either} & q \le r,\ a \ge b \\ \text{or} & q > r,\ a + \frac{1}{q} > b + \frac{1}{r}. \end{cases} \tag{2.10}$$

Moreover, for $0 < q \le r \le \infty = p$, this can be extended to

$$L_{\infty,q}(\log L)_a(\Omega) \hookrightarrow L_{\infty,r}(\log L)_b(\Omega) \quad \text{if} \quad a + \frac{1}{q} = b + \frac{1}{r}. \tag{2.11}$$

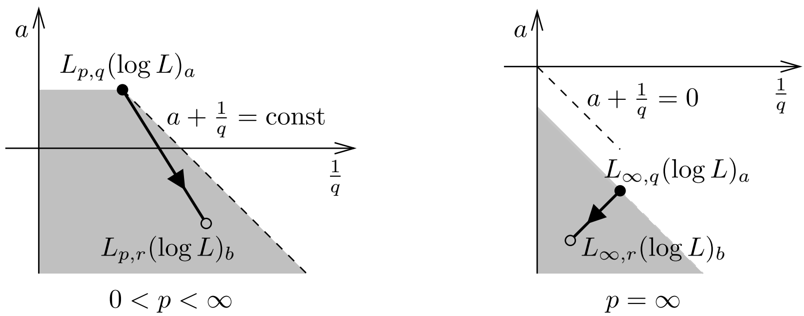


Figure 1

In other words, spaces $L_{\infty,q}(\log L)_a(\Omega)$ are ordered along the "diagonals" $a + 1/q = \text{const.}$, see also Figure 1 where we indicated in the shaded areas admitted parameters $(\frac{1}{r}, b)$ for target spaces $L_{p,r}(\log L)_b(\Omega)$ such that for a fixed source space we have $L_{p,q}(\log L)_a(\Omega) \hookrightarrow L_{p,r}(\log L)_b(\Omega)$. These conditions can, in general, not be relaxed, see [3, Rem. 9.4].

Remark 2.4. In case of $|\Omega| < \infty$, say, $|\Omega| = 1$, and $p = q = \infty$, $a \ge 0$, one has

$$L_{\infty,\infty}(\log L)_{-a}(\Omega) = L_{\exp,a}(\Omega), \tag{2.12}$$

where the latter are the *exponential* (or *Zygmund*) *spaces* consisting of all measurable functions f on Ω for which there is a constant $\lambda = \lambda(f) > 0$ such that

$$\int_\Omega \exp(\lambda |f(x)|)^{1/a} \, \mathrm{d}x < \infty, \tag{2.13}$$

(if $a = 0$, this is interpreted as f is bounded, i.e. $L_{\exp,0} = L_\infty$); see [4, Ch. 4, Def. 6.11, Lemma 6.12].

For later use, let us mention two special cases separately (concerning spaces 'close' to L_1 and L_∞, respectively). Assume $a \in \mathbb{R}$, $1 < q < \infty$, $b > \frac{1}{q}$. Then

$$\| f \mid L_1(\log L)_a(\Omega)\| = \int_0^{|\Omega|} (1 + |\log t|)^a f^*(t)\,\mathrm{d}t, \tag{2.14}$$

$$\| f \mid L_{\infty,q}(\log L)_{-b}(\Omega)\| = \left(\int_0^{|\Omega|} \left[\frac{f^*(t)}{(1+|\log t|)^b}\right]^q \frac{\mathrm{d}t}{t}\right)^{1/q}. \tag{2.15}$$

2.2. Extrapolation. We consider the situation of domains $\Omega \subset \mathbb{R}^n$ with finite measure $|\Omega| < \infty$ in further detail. Let for $0 < p < \infty$, $j \in \mathbb{N}$, the numbers p_j and p_{-j} be given by

$$\frac{1}{p_j} := \frac{1}{p} + 2^{-j} \quad \text{and} \quad \frac{1}{p_{-j}} := \frac{1}{p} - 2^{-j}, \tag{2.16}$$

where we may always assume that $j \geq j_0 = j_0(p)$ such that $p_{-j} \in (0,\infty)$ as well.

Theorem 2.5. *Let $\Omega \subset \mathbb{R}^n$ be a domain with $|\Omega| < \infty$, $0 < p < \infty$, $0 < q \leq \infty$.*

(i) *If $a < 0$, then $f \in L_{p,\,q}(\log L)_a(\Omega)$ if, and only if,*

$$\Big(\sum_{j=j_0}^{\infty} 2^{jaq} \| f \mid L_{p_j,q}(\Omega)\|^q\Big)^{1/q} \tag{2.17}$$

(with the usual modification if $q = \infty$) is finite, where (2.17) *is an equivalent quasi-norm in $L_{p,\,q}(\log L)_a(\Omega)$.*

(ii) *If $a > 0$, then $f \in L_{p,q}(\log L)_a(\Omega)$ if, and only if, it can be represented as*

$$f = \sum_{j=j_0}^{\infty} f_j \quad \textit{with } f_j \in L_{p_{-j},q}(\Omega),\ j \geq j_0, \tag{2.18}$$

and such that

$$\Big(\sum_{j=j_0}^{\infty} 2^{jaq} \| f_j \mid L_{p_{-j},q}(\Omega)\|^q\Big)^{1/q} \tag{2.19}$$

is finite (with the usual modification if $q = \infty$). The infimum of the expressions (2.19) *taken over all admissible representations* (2.18) *is an equivalent quasi-norm on $L_{p,q}(\log L)_a(\Omega)$.*

Remark 2.6. This result in the above version was proved by Cobos, Fernández-Cabrera, Manzano, Martínez in [11] using interpolation methods, see also the approach by Karadzhov, Milman in [58] and the recent survey [2]. Note that in case of $p = q$ one can replace $L_{p_j,q}$ in (i) by L_{p_j}, and $L_{p_{-j},q}$ in (ii) by $L_{p_{-j}}$, respectively. If $p = q = \infty$, $a < 0$, then there is the obvious counterpart of (2.17).

Theorem 2.5 extends quite a number of forerunners to the full range of parameters: the basic assertion for $1 < p = q < \infty$ is due to Edmunds, Triebel [34–37]. This was extended in [49] to $p \neq q$ and by Edmunds, Gurka, Opic in [22] to $p = q = \infty$. Further progress was made by Fiorenza, Krbec in [40], [41] concerning $p = q = 1$, $a > 0$, and by Edmunds, Krbec in [30] and Cruz-Uribe, Krbec in [14] including values $p = \infty$, $q \le \infty$. We return to this last case in Theorem 2.10 below.

2.3. Yano's theorem. As briefly mentioned in our introduction, Yano's paper [103] can be seen as some starting point of extrapolation theory, though there are earlier results of similar type by Titchmarsh [90], [91] and Zygmund [105], [106]. We refer to [74] and [10] for some further (historical) description of those results. Roughly speaking, the theorem of Yano can be described as follows: If T is a bounded linear operator on $L_p(\Omega)$ for $p > 1$ with $\|T\|_{\mathcal{L}(L_p(\Omega))} = \mathcal{O}((p-1)^a)$ as $p \downarrow 1$ for some $a > 0$, then these estimates can be extrapolated to obtain assertions for $T \in \mathcal{L}\big(L_1(\log L)_a(\Omega), L_1(\Omega)\big)$. More general formulations and other cases can be found in [108, XII.4.11], [92, Theorem IV.5.3] (for sub-linear T), [40] (for sub-additive T), and [88, p. 23], [36, p. 74] (for T being the Hardy-Littlewood maximal operator). In those latter results a decomposition approach is used instead of weak type inequalities and the Marcinkiewicz interpolation theorem. We formulate a version below that suits our idea to demonstrate (in its proof) the strength and interplay of decomposition, localisation and extrapolation techniques. The case $a = 1$ is due to Yano in [103].

Theorem 2.7. *Let $a > 0$, $\Omega \subset \mathbb{R}^n$ be a domain with $|\Omega| < \infty$, $1 < p < \infty$, and T sub-linear such that for some $p_0 > 1$,*

$$\| Tf \mid L_p(\Omega)\| \le c\Big(\frac{1}{p-1}\Big)^a \| f \mid L_p(\Omega)\|, \quad 1 < p < p_0. \tag{2.20}$$

Then

$$T\colon L_1(\log L)_a(\Omega) \to L_1(\Omega) \quad \textit{is bounded.}$$

Proof. We follow the elementary proof given by EDMUNDS, KRBEC in [32]. Let us for simplicity assume that $|\Omega| = 1$ and denote by $I_k = \left[\mathrm{e}^{-k}, \mathrm{e}^{-k+1}\right]$, $k \in \mathbb{N}$, hence $|I_k| = (\mathrm{e}-1)\mathrm{e}^{-k}$. Assume that $f = \sum_{k=1}^{\infty} f_k$ such that

$$\begin{cases} f_k^*(t) \in \left[f^*(\mathrm{e}^{-k+1}), f^*(\mathrm{e}^{-k})\right] & \text{if } 0 < t \le |I_k|, \\ f_k^*(t) = 0 & \text{if } t \ge |I_k|. \end{cases} \tag{2.21}$$

Then the sub-linearity of T together with Hölder's inequality and $|\Omega| = 1$ imply

$$\begin{aligned} \| Tf \mid L_1(\Omega)\| &\le \sum_{k=1}^{\infty} \| Tf_k \mid L_1(\Omega)\| \\ &\le \sum_{k=1}^{\infty} \| Tf_k \mid L_{1+\frac{1}{k}}(\Omega)\| \\ &\le c \sum_{k=1}^{\infty} k^a \| f_k \mid L_{1+\frac{1}{k}}(\Omega)\| \\ &= c \sum_{k=1}^{\infty} k^a \left(\int_0^{|I_k|} [f_k^*(t)]^{\frac{k+1}{k}} \,\mathrm{d}t \right)^{\frac{k}{k+1}} \end{aligned}$$

in view of (2.20). Now the decomposition of f and monotonicity lead to

$$\begin{aligned} \| Tf \mid L_1(\Omega)\| &\le c_1 \sum_{k=1}^{\infty} k^a f^*(\mathrm{e}^{-k}) (\mathrm{e}^{-(k+1)})^{\frac{k}{k+1}} \\ &= c_1 \sum_{k=1}^{\infty} k^a \mathrm{e}^{-k} f^*(\mathrm{e}^{-k}) \\ &\le c_2 \int_0^1 (1 + |\log t|)^a f^*(t) \,\mathrm{d}t \\ &= c_2 \| f \mid L_1(\log L)_a(\Omega)\|, \end{aligned}$$

where we applied (2.6) and (2.14) in the end. □

Remark 2.8. In the course of the above argument it turns out that

$$\sum_{k=1}^{\infty} k^a \| f_k \mid L_{1+\frac{1}{k}}(\Omega)\| \le c \int_0^1 (1 + |\log t|)^a f^*(t) \,\mathrm{d}t.$$

We can even show the equivalence of both terms.

Corollary 2.9. *Let $a > 0$ and $\Omega \subset \mathbb{R}^n$ be a domain with $|\Omega| = 1$. Then*

$$\sum_{k=1}^{\infty} k^a \| f_k \mid L_{1+\frac{1}{k}}(\Omega)\| \approx \int_0^1 (1 + |\log t|)^a f^*(t)\,\mathrm{d}t,$$

where $f = \sum_{k\in\mathbb{N}} f_k$ is the decomposition described in (2.21).

Proof. We use the optimal decomposition of $f = \sum f_k$ as above, that is, $f^*(\mathrm{e}^{-k}) \le f^*_{k+1}(t)$ for $0 < t \le |I_{k+1}|$. Thus integration yields

$$\begin{aligned} f^*(\mathrm{e}^{-k}) &\le |I_{k+1}|^{-\frac{k+1}{k+2}} \left(\int_0^{|I_{k+1}|} f^*_{k+1}(t)^{\frac{k+2}{k+1}}\,\mathrm{d}t\right)^{\frac{k+1}{k+2}} \\ &\le c\mathrm{e}^k \| f^*_{k+1} \mid L_{1+\frac{1}{k+1}}(0,1)\| \\ &= c\mathrm{e}^k \| f_{k+1} \mid L_{1+\frac{1}{k+1}}(\Omega)\| \end{aligned}$$

for all $0 < t \le |I_{k+1}|$ such that discretisation yields

$$\begin{aligned} \int_0^1 (1 + |\log t|)^a f^*(t)\,\mathrm{d}t &\le c_1 \sum_{k=1}^{\infty} (1+k)^a f^*(\mathrm{e}^{-k})\mathrm{e}^{-k} \\ &\le c_2 \sum_{k=1}^{\infty} (1+k)^a \| f_{k+1} \mid L_{1+\frac{1}{k+1}}(\Omega)\| \end{aligned}$$

as requested. □

2.4. Decompositions in exponential spaces. Assume $|\Omega| = 1$, $1 < q < \infty$. In view of Remark 2.4 and (2.11) we have

$$L_{\infty,q}(\log L)_{-1}(\Omega) \hookrightarrow L_\infty(\log L)_{-1/q'}(\Omega) = L_{\exp,1/q'}(\Omega), \tag{2.22}$$

i.e., by (2.5) and (2.15) with $b = 1$,

$$\sup_{0<t<1} \frac{f^*(t)}{(1 + |\log t|)^{1/q'}} \le c\left(\int_0^{|\Omega|} \left[\frac{f^*(t)}{1 + |\log t|}\right]^q \frac{\mathrm{d}t}{t}\right)^{1/q}.$$

Plainly, the case $q = \infty$ can be included. Now we come to the counterpart of Theorem 2.5 in case of $1 < q \le \infty = p$. Recall our notation $I_k = \left[\mathrm{e}^{-k}, \mathrm{e}^{-k+1}\right]$, $k \in \mathbb{N}$.

Theorem 2.10. *Let $\Omega \subset \mathbb{R}^n$ be a domain with $|\Omega| = 1$, $1 < q \le \infty$, and $a > \frac{1}{q}$. Then the following assertions are equivalent:*

(i) $f \in L_{\infty,q}(\log L)_{-a}(\Omega)$,

(ii) $\displaystyle\sum_{k=1}^{\infty}\left[\frac{\| f \mid L_k(\Omega)\|}{k^a}\right]^q < \infty$,

(iii) $\displaystyle\sum_{k=1}^{\infty}\left[\frac{\| f \mid L_{k,q}(\Omega)\|}{k^a}\right]^q < \infty$,

(iv) $\displaystyle\sum_{k=1}^{\infty}\left[\frac{\| f^* \mid L_k(I_k)\|}{k^a}\right]^q < \infty$,

where we always have the usual modification if $q = \infty$.

Remark 2.11. For later use, let us explicitly state the equivalence of (i) and (ii) in case of $q = \infty$ in the following form, recall (2.12),

$$\| f \mid L_{\exp,a}(\Omega)\| \approx \sup_{k\in\mathbb{N}} k^{-a}\| f \mid L_k(\Omega)\|, \quad a > 0. \tag{2.23}$$

As already mentioned, Theorem 2.10 is due to Edmunds, Krbec [30], and Cruz-Uribe, Krbec [14]. As a consequence, it admits a simple proof of the famous Brézis-Wainger embedding

$$W_p^m(\Omega) \hookrightarrow L_{\infty,p}(\log L)_{-1}(\Omega) \quad \text{if } m = \frac{n}{p} \in \mathbb{N}, \tag{2.24}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain with sufficiently smooth boundary, and $W_p^m(\Omega)$, $m \in \mathbb{N}$, $1 \le p < \infty$, are the classical Sobolev spaces. We return to this point below. Generalisations dealing with *Generalized Lorentz-Zygmund spaces* (GLZ) and *Lorentz-Karamata spaces* were obtained by Edmunds, Evans, Gogatishvili, Gurka, Krbec, Neves, Opic, Pick in an impressive number of papers: [18–21], [23–25], [42–46], [74–76], [78], see also [17, Sects. 3.4, 3.6] for a recent survey. There are further, closely linked topics studied in this field, such as mapping properties of convolution operators, Moser's Lemma (see Edmunds, Krbec [31]), and interpolation arguments [39]. However, this list is by no means complete.

2.5. Limiting embeddings in the critical case. As is well-known, the history of such questions starts in the 1930s with Sobolev's famous embedding theorem [87]

$$W_p^m(\Omega) \hookrightarrow L_r(\Omega), \tag{2.25}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain with sufficiently smooth boundary, $m \in \mathbb{N}$, $1 \le p < \infty$ with $m < \frac{n}{p}$, and $1 \le r \le \infty$ such that $\frac{m}{n} - \frac{1}{p} \ge -\frac{1}{r}$. In particular, in the limiting case $m = \frac{n}{p} \in \mathbb{N}$, the inclusion (2.25) does not hold for $r = \infty$. Thus, to obtain further refinements of the limiting case of (2.25) it became necessary to deal with wider classes of function spaces. In the late 1960s PEETRE [81], TRUDINGER [102], MOSER [73], and POHOZHAEV [83] independently found refinements of (2.25) expressed in terms of Orlicz spaces of exponential type, see also STRICHARTZ [89], YUDOVICH [104], HEMPEL, MORRIS and TRUDINGER [55], BENNETT, RUDNICK [3]; this was followed by many contributions in the last decades investigating problems related to (2.25) in detail. In 1979 HANSSON [48] and BRÉZIS, WAINGER [6] showed independently (2.24), see also HEDBERG [54], and sharper results by MAZ'YA [69] and [70] dealing with capacitary estimates. Recently we noticed a revival of interest in limiting embeddings of Sobolev spaces indicated by a considerable number of publications devoted to this subject; in addition to the papers referred to in Remark 2.11 let us only mention the important contributions by EDMUNDS and KRBEC [29], EDMUNDS, KERMAN and PICK [28], CWIKEL, PUSTYLNIK [15], and – also from the standpoint of applications to spectral theory – the publications [35] and [94] by EDMUNDS and TRIEBEL. This list is by no means complete, but reflects the increased interest in related questions in the last years. There are a lot of different approaches to the modification of (2.25) in order to get – in the adapted framework – appropriately optimal assertions. We especially recommend the very interesting detailed survey paper by PICK [82].

Dealing with Sobolev spaces of fractional order $H^s_p(\Omega)$, $s \in \mathbb{R}$, $1 < p < \infty$, the BRÉZIS-WAINGER result (2.24) can be extended to

$$\begin{aligned} H^{n/p}_p(\Omega) &\hookrightarrow L_{\infty,\,p}(\log L)_{-1}(\Omega) \\ &\hookrightarrow L_\infty(\log L)_{-1/p'}(\Omega) = L_{\exp,1/p'}(\Omega), \end{aligned} \tag{2.26}$$

recall also (2.22). Here $H^s_p(\mathbb{R}^n)$, $s \in \mathbb{R}$, $1 < p < \infty$, are the well-known (fractional) Sobolev spaces of all measurable functions $f\colon \mathbb{R}^n \to \mathbb{C}$, normed by

$$\| f \mid H^s_p(\mathbb{R}^n)\| = \| \mathcal{F}^{-1}(1+|\xi|^2)^{s/2}\mathcal{F}f \mid L_p(\mathbb{R}^n)\|,$$

using the Fourier transform $\mathcal{F}$ and their inverse $\mathcal{F}^{-1}$. They naturally extend the scale of (classical) Sobolev spaces $W^m_p(\mathbb{R}^n)$ since

$$W^m_p(\mathbb{R}^n) = H^m_p(\mathbb{R}^n), \quad m \in \mathbb{N}_0,\ 1 < p < \infty. \tag{2.27}$$

For bounded domains $\Omega \subset \mathbb{R}^n$ the spaces $H_p^s(\Omega)$ are defined by restriction; we shall mainly work with the following closed subspaces of $H_p^s(\mathbb{R}^n)$,

$$\widetilde{H}_p^s(\Omega) = \{f \in H_p^s(\mathbb{R}^n) : \operatorname{supp} f \subset \overline{\Omega}\}. \tag{2.28}$$

Recall that for $a > 0$,

$$\| f \mid L_{\exp,a}(\Omega)\| \approx \sup_{k\in\mathbb{N}} k^{-a} \| f \mid L_k(\Omega)\|, \quad a > 0,$$

by (2.23), cf. [36, Thm. 2.6.2/1]. Moreover, when $1 < p < \infty$, $k \in \mathbb{N}$, $k \geq p$, then

$$\left\| \mathrm{id}\colon \widetilde{H}_p^{n/p}(\Omega) \to L_k(\Omega) \right\| \approx k^{\frac{1}{p'}},$$

cf. [36, Thm. 2.7.2] and [94]. Using the extrapolation characterisation [29] one can now conclude that

$$\widetilde{H}_p^{n/p}(\Omega) \hookrightarrow L_{\exp,a}(\Omega) \quad \text{is compact if, and only if,} \quad a > \frac{1}{p'}, \tag{2.29}$$

see also [36, Thm. 2.7.3].

Remark 2.12. There are extensions of assertion (2.29) in various directions: as far as function spaces are concerned, there are results for Besov spaces in [36, Thm. 2.7.3] and in [52, Cor. 8.21], the latter also dealing with spaces of Triebel-Lizorkin type. This is closely connected with the concept of growth envelopes studied in [52], see also [95].

In view of applications also the degree of compactness of embeddings like (2.29) is of special interest, characterised by the asymptotic behaviour of its corresponding entropy or approximation numbers. We do not go into further detail but refer to the results in [7], [9], [33], [36], [49], [50], [67], [68], [94] which mainly rely on extrapolation arguments.

2.6. The case of missing derivatives. We briefly discuss some phenomenon which is connected with critical embeddings for so-called reduced Sobolev spaces and spaces with dominating mixed smoothness; this is based on our joint papers with Krbec [60], [65], see also [59], [63], [64].

Let for a multi-index $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}_0^n$ its length be given by $|\alpha| = \alpha_1 + \cdots + \alpha_n$, as usual, and define for $m \in \mathbb{N}$ the set

$$M(n;m) = \{\alpha \in \mathbb{N}_0^n : |\alpha| = m,\ \alpha_i \in \{0,1\},\ i = 1, \ldots, n\}.$$

For some numbers $d_1, \ldots, d_m \in \mathbb{N}$ with $d_1 + \cdots + d_m = n$, we consider the subset

$$S(n; d_1, \ldots, d_m) = \big\{\alpha = (\alpha^1, \ldots, \alpha^m) \in \mathbb{N}_0^n : \alpha^j = (\alpha_1^j, \ldots, \alpha_{d_j}^j) \in \mathbb{N}_0^{d_j}, \\ |\alpha^j| = 1, j = 1, \ldots, m\big\}.$$

If $d_1 = \cdots = d_m = d$ with $n = md$, then we shall write $S(n; m) = S(n; d, \ldots, d)$. Obviously we always assume that $m \leq n$.

Definition 2.13. Let $m \in \mathbb{N}$, $\mathcal{M} \subseteq M(n; m)$, $1 < p < \infty$, and $\Omega \subseteq \mathbb{R}^n$. Then the *reduced Sobolev spaces* are given by

$$W_p^{\mathcal{M}}(\Omega) = \{f \in L_p(\Omega) : \mathrm{D}^\beta f \in L_p(\Omega) \text{ for all } \alpha \in \mathcal{M} \\ \text{and all } \beta \in \mathbb{N}_0^n \text{ with } \beta \leq \alpha\}.$$

Plainly,

$$W_p^m(\Omega) \hookrightarrow W_p^{M(n;m)}(\Omega) \hookrightarrow W_p^{\mathcal{M}}(\Omega), \quad 1 < p < \infty. \tag{2.30}$$

A remarkable result, proved by Adams in [1] says that in analogy to (2.25),

$$W_p^{M(n;m)}(\Omega) \hookrightarrow L_r(\Omega) \tag{2.31}$$

if $m < \frac{n}{p}$, $1 \leq r < \infty$, and $\frac{1}{r} = \frac{1}{p} - \frac{m}{n}$. In other words, the Sobolev embedding remains true for reduced Sobolev spaces.

A deeper look shows that reduced Sobolev spaces are closely related to Sobolev spaces with dominating mixed derivatives. More precisely, if $n = dm$, then (2.31) can be refined by

$$W_p^{M(n;m)}(\mathbb{R}^n) \hookrightarrow W_p^{S(n;m)}(\mathbb{R}^n) = S_p^1 W(\mathbb{R}^d \times \cdots \times \mathbb{R}^d) \hookrightarrow L_r(\mathbb{R}^n),$$

where $\frac{1}{r} \geq \frac{1}{p} - \frac{m}{n} = \frac{1}{p} - \frac{1}{d}$. Details about the spaces $S_p^1 W(\mathbb{R}^d \times \cdots \times \mathbb{R}^d)$ and related spaces of functions with dominating mixed smoothness can be found in the survey article [84] and the references given there.

In the general case, for example, when $\mathcal{M} = S(n; d_1, \ldots, d_m)$, the situation is more sophisticated, cf. [60], and leads to Lebesgue spaces with mixed norms as target spaces. Here we restrict ourselves to the above model case $\mathcal{M} = S(n; m)$ for convenience, where $n = dm$. Thus in the critical case $m = \frac{n}{p}$ this leads to $p = d > 1$. The following extrapolation result can be found in [60, Thm. 3.2, Cor. 4.1].

Proposition 2.14. *Let $1 < p < \infty$, $m \in \mathbb{N}$. Then there exists some number $c > 0$ such that for all $f \in W_p^{S(n;m)}(\mathbb{R}^n)$,*

$$\sup_{k>p} k^{-m/p'} \| f \mid L_k(\mathbb{R}^n)\| \le c\| f \mid W_p^{S(n;m)}(\mathbb{R}^n)\|. \tag{2.32}$$

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with $|\Omega| = 1$. In analogy to (2.28) we set

$$\widetilde{W}_p^{\mathcal{M}}(\Omega) = \{f \in W_p^{\mathcal{M}}(\mathbb{R}^n) : \operatorname{supp} f \subset \overline{\Omega}\}.$$

Then (2.32) implies the counterpart of (2.26),

$$\widetilde{W}_p^{S(n;m)}(\Omega) \hookrightarrow L_\infty(\log L)_{-m/p'}(\Omega) = L_{\exp,m/p'}(\Omega). \tag{2.33}$$

Moreover, in [60] it is shown that this is sharp in the scale of exponential spaces as target spaces. Hence, the situation is different in the critical case if one replaces the standard Sobolev spaces by reduced Sobolev spaces.

Remark 2.15. In the frame of Zygmund spaces the embeddings in (2.33) can be refined in the spirit of (2.26) using multivariate rearrangements combined with atomic representations and extrapolation arguments. This has been done in [65]. In particular, we proved in [65, Thm. 5.3] for $m = 2$ that

$$\widetilde{W}_p^{S(n;m)}(\Omega) \hookrightarrow L_{\infty,p}(\log L)_{-m}(\Omega).$$

Let us further mention that in case of general subsets $\mathcal{M} \subset M(n;m)$, but with the additional condition that the coordinate-wise sum of all multi-indices is uniformly bounded,

$$\sum_{\alpha\in\mathcal{M}} \alpha_i = k_i \le K, \quad i = 1, \dots, n,$$

then there is some counterpart of (2.33) in [60, Cor. 4.3] if one modifies the target space of exponential type appropriately.

3. Lipschitz spaces

3.1. Definitions. Let $C(\mathbb{R}^n)$ be the space of all complex-valued bounded uniformly continuous functions on $\mathbb{R}^n$, equipped with the sup-norm as usual. Recall the concept of the modulus of continuity,

$$\omega(f,t) = \sup_{|h|\le t} \| \Delta_h f \mid C(\mathbb{R}^n)\|, \quad t > 0, \tag{3.1}$$

where Δ_h, $h \in \mathbb{R}^n$, is the usual difference operator $(\Delta_h f)(x) = f(x+h) - f(x)$, $x \in \mathbb{R}^n$.

Remark 3.1. For simplicity we restrict ourselves to first differences (and derivatives) here considered in the sup-norm. In general one has to study the *r-th modulus of smoothness* of a function $f \in L_p(\mathbb{R}^n)$, $0 < p \le \infty$, $r \in \mathbb{N}$, defined by

$$\omega_r(f,t)_p = \sup_{|h| \le t} \| \Delta_h^r f \mid L_p(\mathbb{R}^n)\|, \quad t > 0, \tag{3.2}$$

where the iterated differences Δ_h^m, $m \in \mathbb{N}_0$, are given by

$$(\Delta_h^1 f)(x) = f(x+h) - f(x), \quad (\Delta_h^{m+1} f)(x) = \Delta_h^1(\Delta_h^m f)(x), \tag{3.3}$$

for $x, h \in \mathbb{R}^n$; cf. [4, Ch. 5, Def. 4.2] or [16, Ch. 2, §7]. An essential feature of these moduli is given by Marchaud's inequality,

$$\omega_r(f,t)_p \le \frac{r}{\log 2} t^r \int_t^\infty \frac{\omega_{r+1}(f,u)_p}{u^r} \frac{\mathrm{d}u}{u}, \tag{3.4}$$

where $f \in L_p(\mathbb{R}^n)$, $1 \le p \le \infty$, $t > 0$, and $r \in \mathbb{N}$; see [4, Ch. 5, (4.11)] or [16, Ch. 2, Thm. 8.1] (for the one-dimensional case).

Next we need the notions of Hölder-Zygmund and Besov spaces and spaces of Lipschitz type. Again we restrict ourselves to smoothness parameters $s \in (0,1]$ and $p = \infty$.

Definition 3.2. Let $0 < s < 1$.

(i) The *Hölder-Zygmund space* $\mathcal{C}^s(\mathbb{R}^n)$ consists of all $f \in C(\mathbb{R}^n)$, such that

$$\| f \mid \mathcal{C}^s(\mathbb{R}^n)\| = \| f \mid C(\mathbb{R}^n)\| + \sup_{h \ne 0} \frac{\| \Delta_h f \mid C(\mathbb{R}^n)\|}{|h|^s} \tag{3.5}$$

is finite.

(ii) The *Besov space* $B^s_{\infty,q}(\mathbb{R}^n)$ consists of all functions $f \in L_\infty(\mathbb{R}^n)$ such that

$$\| f \mid B^s_{\infty,q}(\mathbb{R}^n)\| = \| f \mid L_\infty(\mathbb{R}^n)\| + \left(\int_0^1 \left[t^{-s} \omega(f,t) \right]^q \frac{\mathrm{d}t}{t} \right)^{1/q} \tag{3.6}$$

(with the usual modification if $q = \infty$) is finite.

Remark 3.3. Note that Besov spaces $B^s_{p,q}(\mathbb{R}^n)$ of positive smoothness $s > 0$ can always be given by differences as subspaces of $L_p(\mathbb{R}^n)$, but they do not coincide with their Fourier-analytically defined counterparts (in the framework of tempered distributions $\mathcal{S}'(\mathbb{R}^n)$) in all cases. However, for $s \ge 1$ and

$p < \infty$ the characterisation (3.6) has to be modified appropriately; that is, for $0 < p \le \infty$, $s > n(\frac{1}{p} - 1)_+$, $0 < q \le \infty$, and $r \in \mathbb{N}$ with $r > s$,

$$\| f \mid B^s_{p,q}(\mathbb{R}^n)\| \approx \| f \mid L_p(\mathbb{R}^n)\| + \Big(\int_0^1 \big[t^{-s}\omega_r(f,t)_p\big]^q \frac{\mathrm{d}t}{t}\Big)^{1/q} \tag{3.7}$$

(with the usual modification if $q = \infty$), where $\omega_r(f,t)_p$ is given by (3.2). We refer to [4, Ch. 5, Def. 4.3], [16, Ch. 2, §10] (where the Besov spaces are defined like in (3.7)) for the Banach case, and [93, Thm. 2.5.12], [98, Sect. 9], [86] for the equivalence and distinction of both approaches. Obviously, with $q = \infty$, one recovers in (ii) the Hölder-Zygmund spaces $\mathcal{C}^s(\mathbb{R}^n) = B^s_{\infty,\infty}(\mathbb{R}^n)$, $0 < s < 1$ (in the sense of equivalent norms), i.e.

$$\| f \mid \mathcal{C}^s(\mathbb{R}^n)\| \approx \| f \mid C(\mathbb{R}^n)\| + \sup_{0<t<1} \frac{\omega(f,t)}{t^s}. \tag{3.8}$$

We also consider spaces of Lipschitz type which are "close" to Lip^a, $0 < a \le 1$. For convenience, we deal with $p = \infty$ exclusively.

Definition 3.4. Let $0 < a \le 1$.

(i) The *Lipschitz space* $\mathrm{Lip}^a(\mathbb{R}^n)$ is defined as the set of all $f \in C(\mathbb{R}^n)$ such that

$$\| f \mid \mathrm{Lip}^a(\mathbb{R}^n)\| = \| f \mid C(\mathbb{R}^n)\| + \sup_{0<t<1} \frac{\omega(f,t)}{t^a} \tag{3.9}$$

is finite.

(ii) Let $0 < q \le \infty$, and

$$\begin{cases} \alpha \in \mathbb{R} & \text{if } 0 < a < 1,\ 0 < q \le \infty \\ \alpha > \frac{1}{q} & \text{if } a = 1,\ 0 < q < \infty \\ \alpha \ge 0 & \text{if } a = 1,\ q = \infty. \end{cases} \tag{3.10}$$

The space $\mathrm{Lip}^{(a,-\alpha)}_{\infty,q}(\mathbb{R}^n)$ is defined as the set of all $f \in C(\mathbb{R}^n)$ such that

$$\| f \mid \mathrm{Lip}^{(a,-\alpha)}_{\infty,q}(\mathbb{R}^n)\| = \| f \mid C(\mathbb{R}^n)\| + \Big(\int_0^{\frac{1}{2}} \Big[\frac{\omega(f,t)}{t^a |\log t|^\alpha}\Big]^q \frac{\mathrm{d}t}{t}\Big)^{1/q} \tag{3.11}$$

(with the usual modification if $q = \infty$) is finite.

Remark 3.5. Note that the restriction for a in (i) and for α in (ii) are quite natural as otherwise the spaces are trivial; when $a = 1$ one recovers the classical Lipschitz space $\mathrm{Lip}^1(\mathbb{R}^n)$ in (i) which are different from $\mathcal{C}^1(\mathbb{R}^n)$ (unlike in case of $0 < a < 1$, $\mathrm{Lip}^a(\mathbb{R}^n) = \mathcal{C}^a(\mathbb{R}^n)$). The above spaces $\mathrm{Lip}^{(a,-\alpha)}_{\infty,q}(\mathbb{R}^n)$ first appeared (in this notation) in [51] in connection with limiting embeddings, extending the case $a = 1$ studied in [26], [27]; in the latter case we also investigated spaces $\mathrm{Lip}^{(1,-\alpha)}_{p,q}(\mathbb{R}^n)$ in [51] in detail,

$$\left\| f \mid \mathrm{Lip}^{(1,-\alpha)}_{p,q}(\mathbb{R}^n)\right\| = \| f \mid L_p(\mathbb{R}^n)\| + \left(\int_0^{\frac{1}{2}} \left[\frac{\omega(f,t)_p}{t\,|\log t|^\alpha}\right]^q \frac{\mathrm{d}t}{t}\right)^{1/q} \tag{3.12}$$

(with the usual modification if $q = \infty$), where $1 \le p \le \infty$, $0 < q \le \infty$, and α according to (3.10).

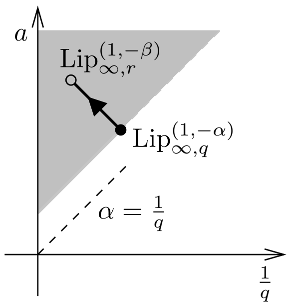
Remark 3.6. The spaces $\mathrm{Lip}^{(1,-\alpha)}(\mathbb{R}^n) = \mathrm{Lip}^{(1,-\alpha)}_{\infty,\infty}(\mathbb{R}^n)$, $\alpha \ge 0$, equipped with the norm

$$\| f \mid \mathrm{Lip}^{(1,-\alpha)}(\mathbb{R}^n)\| = \| f \mid C(\mathbb{R}^n)\| + \sup_{0<t<\frac{1}{2}} \frac{\omega(f,t)}{t|\log t|^\alpha},$$

can be obtained as special cases of the more general spaces $C^{0,\sigma(t)}(\overline{\Omega})$, $\Omega \subseteq \mathbb{R}^n$, introduced by Kufner, John and Fučík (see [66, Def. 7.2.12]), whereas $\mathrm{Lip}^{(1,0)}_{p,\infty} = \mathrm{Lip}(1, L_p)$ are considered by DeVore and Lorentz in [16, Ch. 2, §9]. In [26] we further studied *Zygmund spaces* of type $\mathcal{C}^{(1,-\alpha)}(\mathbb{R}^n)$, $\alpha \ge 0$, as refinements of $\mathcal{C}^1(\mathbb{R}^n)$, given by Definition 3.2, and counterparts of the spaces $\mathrm{Lip}^{(1,-\alpha)}(\mathbb{R}^n)$.

Remark 3.7. We recall a sharp embedding result for logarithmic Lipschitz spaces $\mathrm{Lip}^{(a,-\alpha)}_{p,q}$ from [51] and restrict ourselves to $p = \infty$, $a = 1$ for convenience. Let $0 < q, r \le \infty$, $\alpha > \frac{1}{q}$, $\beta > \frac{1}{r}$. Then

$$\begin{gathered}\mathrm{Lip}^{(1,-\alpha)}_{\infty,q}(\mathbb{R}^n) \hookrightarrow \mathrm{Lip}^{(1,-\beta)}_{\infty,r}(\mathbb{R}^n) \quad \text{if, and only if,}\\ \text{either}\ \ r \ge q,\ \beta - \frac{1}{r} \ge \alpha - \frac{1}{q}, \ \ \text{or}\ \ r < q,\ \beta - \frac{1}{r} > \alpha - \frac{1}{q}.\end{gathered} \tag{3.13}$$



$0 < p < \infty$
Figure 2

The similarity to (2.10), (2.11) is obvious. Let us again point out the somehow astonishing result that one can "compensate" some gain of logarithmic smoothness $-\beta > -\alpha$ by "paying" with the additional index q, that is, as long as $(-\beta) - (-\alpha) \leq \frac{1}{q} - \frac{1}{r}$, $r \geq q$. We also refer to Figure 1 for the counterpart of limiting embeddings in terms of integrability instead of smoothness. For later use, we explicate (3.13) in case of $\alpha = 1$, $r = \infty$, and $\beta = \frac{1}{q'}$, that is

$$\mathrm{Lip}^{(1,-1)}_{\infty,q}(\mathbb{R}^n) \hookrightarrow \mathrm{Lip}^{(1,-1/q')}_{\infty,\infty}(\mathbb{R}^n) = \mathrm{Lip}^{(1,-1/q')}(\mathbb{R}^n). \tag{3.14}$$

3.2. Extrapolation. The following extrapolation type result for spaces $\mathrm{Lip}^{(a,-\alpha)}_{p,q}(\mathbb{R}^n)$ was obtained in [27], [51]; for convenience we recall the case $p = \infty$, $a = 1$ only.

Proposition 3.8. (i) *Let $q = \infty$, $\alpha > 0$. Then $f \in \mathrm{Lip}^{(1,-\alpha)}(\mathbb{R}^n) = \mathrm{Lip}^{(1,-\alpha)}_{\infty,\infty}(\mathbb{R}^n)$ if, and only if, f belongs to $C(\mathbb{R}^n)$ and there is some $c > 0$ such that for all λ, $0 < \lambda < 1$,*

$$\sup_{0<t<1/2} \frac{\omega(f,t)}{t^{1-\lambda}} \leq c\lambda^{-\alpha}.$$

Moreover, we obtain as an equivalent norm in $\mathrm{Lip}^{(1,-\alpha)}(\mathbb{R}^n)$,

$$\| f \mid \mathrm{Lip}^{(1,-\alpha)}(\mathbb{R}^n)\| \approx \| f \mid C(\mathbb{R}^n)\| + \sup_{0<\lambda<1} \lambda^{\alpha} \sup_{0<t<1/2} \frac{\omega(f,t)}{t^{1-\lambda}}. \tag{3.15}$$

(ii) *Let $0 < q < \infty$, $\alpha > \frac{1}{q}$. Then $f \in \mathrm{Lip}^{(1,-\alpha)}_{\infty,q}(\mathbb{R}^n)$ if, and only if, f belongs to $C(\mathbb{R}^n)$ and there is some $c > 0$ such that*

$$\int_0^1 \lambda^{\alpha q} \int_0^{\frac{1}{2}} \Big[\frac{\omega(f,t)}{t^{1-\lambda}}\Big]^q \frac{\mathrm{d}t}{t} \frac{\mathrm{d}\lambda}{\lambda} \leq c.$$

Moreover,

$$\begin{aligned} \Big\| f \;\Big|\; \mathrm{Lip}^{(1,-\alpha)}_{\infty,q}(\mathbb{R}^n)\Big\| &\approx \| f \mid C(\mathbb{R}^n)\| \\ &\quad + \Big(\int_0^1 \lambda^{\alpha q} \int_0^{\frac{1}{2}} \Big[\frac{\omega(f,t)}{t^{1-\lambda}}\Big]^q \frac{\mathrm{d}t}{t} \frac{\mathrm{d}\lambda}{\lambda}\Big)^{1/q}. \end{aligned} \tag{3.16}$$

Remark 3.9. Part (i) can be found in [62, Prop. 2.5], see also [64], which was also the motivation for the above extension in [27], [51].

In view of Definition 3.2 one can reformulate the above result as follows to emphasise the extrapolation nature of the outcome.

Corollary 3.10. (i) *Let $\alpha > 0$. Then $f \in \mathrm{Lip}^{(1,-\alpha)}(\mathbb{R}^n)$ if, and only if, $f \in \mathcal{C}^{1-\lambda}(\mathbb{R}^n)$ for all $\lambda \in (0,1)$, and*

$$\sup_{0<\lambda<1} \lambda^a \| f \mid \mathcal{C}^{1-\lambda}(\mathbb{R}^n)\| \tag{3.17}$$

is finite, where (ex-inf) *yields an equivalent norm in* $\mathrm{Lip}^{(1,-\alpha)}(\mathbb{R}^n)$.

(ii) *Let $0 < q < \infty$, $\alpha > \frac{1}{q}$. Then $f \in \mathrm{Lip}^{(1,-\alpha)}_{\infty,q}(\mathbb{R}^n)$ if, and only if, $f \in B^{1-\lambda}_{\infty,q}(\mathbb{R}^n)$ for all $\lambda \in (0,1)$, and*

$$\left(\int_0^1 \lambda^{\alpha q} \| f \mid B^{1-\lambda}_{\infty,q}(\mathbb{R}^n)\|^q \, \frac{\mathrm{d}\lambda}{\lambda}\right)^{1/q} \tag{3.18}$$

is finite, where (3.18) *yields an equivalent norm in* $\mathrm{Lip}^{(1,-\alpha)}_{\infty,q}(\mathbb{R}^n)$.

Remark 3.11. The proof of Proposition 3.8(i) is based on the equality

$$\begin{aligned}\sup_{0<\lambda<1} \lambda^\alpha \sup_{0<t<1/2} \frac{\omega(f,\, t)}{t^{1-\lambda}} &= \sup_{0<t<1/2} \left[\frac{\omega(f,t)}{t} \sup_{0<\lambda<1} \lambda^\alpha t^\lambda\right] \\ &= \sup_{0<t<1/2} \left[\frac{\omega(f,t)}{t} \left(\frac{\alpha}{\mathrm{e}}\right)^\alpha |\log t|^{-\alpha}\right].\end{aligned}$$

However, one should always keep in mind that this extrapolation characterisation heavily depends on the use of fixed norms in (3.6), whereas equivalent norms may lead to different spaces. We refer to a more general approach by Cobos, Fernández-Cabrera, Triebel in [12].

Remark 3.12. Let us mention that extrapolation characterisations as in the above Corollary 3.10 can be extended to Lipschitz-type spaces with dominating mixed smoothness defined via mixed differences and mixed moduli of smoothness, respectively; we refer to [62, Sect. 4] for further details.

3.3. Limiting embeddings in the supercritical case. We return to the topic of limiting embeddings already explained in Section 2.5. Note that (in an appropriately modified context) it also makes sense to consider embeddings like (2.25) in "super-critical" situations, that is, when $m > \frac{n}{p}$.

Then by simple monotonicity arguments all distributions in the Sobolev space W_p^m are essentially bounded,

$$W_p^m(\mathbb{R}^n) \hookrightarrow C(\mathbb{R}^n) \tag{3.19}$$

in this case, $1 \leq p < \infty$. The counterparts of (2.25) and (2.24) then read for $m \in \mathbb{N}$, $1 \leq p < \infty$, as

$$W_p^m(\mathbb{R}^n) \hookrightarrow \mathrm{Lip}^a(\mathbb{R}^n), \quad 0 < a \leq m - \frac{n}{p} < 1,$$

and,

$$W_p^{1+n/p}(\mathbb{R}^n) \hookrightarrow \mathrm{Lip}^a(\mathbb{R}^n), \quad 0 < a < 1,\ m = 1 + \frac{n}{p} \in \mathbb{N},$$

but $W_p^{1+n/p}(\mathbb{R}^n) \not\hookrightarrow \mathrm{Lip}^1(\mathbb{R}^n)$. Using the above extended scale of Lipschitz spaces the direct counterpart of (2.26) can be written as

$$H_p^{1+n/p}(\mathbb{R}^n) \hookrightarrow \mathrm{Lip}^{(1,-1)}_{\infty,p}(\mathbb{R}^n) \hookrightarrow \mathrm{Lip}^{1,-1/p'}(\mathbb{R}^n), \tag{3.20}$$

where we used (3.14), recall (2.27). Note that the outer embedding coincides with the celebrated result of Brézis and Wainger [6] in which it was shown that every $u \in H_p^{1+n/p}$, $1 < p < \infty$, is "almost" Lipschitz-continuous, in the sense that

$$|u(x) - u(y)| \leq c|x-y| \big|\log|x-y|\big|^{1-\frac{1}{p}} \| u \mid H_p^{1+n/p}\|, \ 0 < |x-y| < \frac{1}{2}. \tag{3.21}$$

In [26], [27], [51] and [61], [62], [64] we studied the sharpness of these embeddings and found that for $1 < p < \infty$,

$$B^1_{\infty,p}(\mathbb{R}^n) \hookrightarrow \mathrm{Lip}^{(1,-1)}_{\infty,p}(\mathbb{R}^n) \hookrightarrow \mathrm{Lip}^{1,-1/p'}(\mathbb{R}^n), \tag{3.22}$$

which together with the well-known Franke-Jawerth-type sharp embedding $H_p^{1+n/p}(\mathbb{R}^n) \hookrightarrow B^1_{\infty,p}(\mathbb{R}^n)$, $1 < p < \infty$, and (3.13) implies (3.20). Note that the methods to prove such embeddings are different: in the first papers we rely on atomic decomposition techniques, whereas the latter are directly based on extrapolation ideas and Marchaud-type inequalities, recall (3.4).

Remark 3.13. The above result (in its classical form) was obtained by Hansson [48] and Brézis-Wainger [6]. In addition to the references recalled at the beginning of Section 2.5 and in Remark 2.12 (which often deal with both limiting situations) let us further mention papers by Bourdaud and Lanza de Cristoforis [5], and by Neves [74]. The borderline case was already studied by Zygmund [107], [108].

Remark 3.14. In this super-critical limiting situation we have counterparts of (2.29) like

$$B^{1+1/p}_{p,q}(\Omega) \hookrightarrow \mathrm{Lip}^{1,-\alpha}(\Omega) \ \text{ is compact if, and only if, } \ \alpha > \frac{1}{q'}, \tag{3.23}$$

where $0 < p < \infty$, $0 < q \leq \infty$, and similar results for spaces $H^{1+n/p}_p(\Omega)$. The sufficiency part can be found in [26], whereas the necessity is an easy consequence of related general results for continuity envelopes as studied in [52]. Again, further extensions in view of the function spaces can be found in [26], [27], [51], whereas the asymptotic behaviour of its corresponding entropy or approximation numbers is investigated in [13], [26], [27].

3.4. The case of missing derivatives revisited. In Section 2.6 we briefly introduced the subject of reduced Sobolev spaces. Relying on similar extrapolation techniques we proved in [61], [62] corresponding results for reduced Sobolev spaces and spaces with dominating mixed smoothness. We discuss a special case.

As a consequence of (3.20) we have in the case $m = \frac{n}{p} \in \mathbb{N}$,

$$W^{m+1}_p(\mathbb{R}^n) \hookrightarrow \mathrm{Lip}^{1,-1/p'}(\mathbb{R}^n), \quad 1 < p < \infty. \tag{3.24}$$

Observe that one can understand

$$W^{m+1}_p(\mathbb{R}^n) = \big\{ f \in L_p(\mathbb{R}^n) : \mathrm{D}^{\alpha+\eta} f \in L_p(\mathbb{R}^n) \text{ for all } \alpha \in \mathbb{N}^n_0, \\ |\alpha| \leq m, \text{ and } \eta \in \mathbb{N}^n_0 \text{ with } |\eta| = 1 \big\}.$$

Thus for a set $\mathcal{M} \subset M(n;m)$ we put

$$\mathcal{M} + 1 = \{\alpha + \eta : \alpha \in \mathcal{M},\ \eta \in \mathbb{N}^n_0 \text{ with } |\eta| = 1\}.$$

Obviously, the space $W^{\mathcal{M}+1}_p(\mathbb{R}^n)$ can be considered as a counterpart of $W^{m+1}_p(\mathbb{R}^n)$ in the context of reduced Sobolev spaces. We have

$$W^{m+1}_p(\mathbb{R}^n) \hookrightarrow W^{\mathcal{M}+1}_p(\mathbb{R}^n)$$

parallel to (2.30) and it makes sense to study sharp embeddings of spaces $W^{\mathcal{M}+1}_p(\mathbb{R}^n)$ into Lipschitz spaces $\mathrm{Lip}^{(1,-\alpha)}(\mathbb{R}^n)$ refining (3.20). However, in contrast to the approach in the critical case briefly presented in Section 2.6, it turns out now that the target space remains unchanged in (3.24) (compared with the classical Sobolev spaces). In [62, Thm. 5.11] we proved the following.

Theorem 3.15. *Let $m \in \mathbb{N}$, and $\mathcal{M} \subset M(n;m)$ such that $\sum_{\alpha\in\mathcal{M}} \alpha_i = \text{const.}$ independently of $i = 1, \dots, n$. Then for $p = n/m$,*

$$W_p^{\mathcal{M}+1}(\mathbb{R}^n) \hookrightarrow \operatorname{Lip}^{1,-1/p'}(\mathbb{R}^n). \tag{3.25}$$

In particular, if $\mathcal{M} = S(n;m)$, $n = md$, then

$$W_p^{S(n;m)+1}(\mathbb{R}^n) \hookrightarrow \operatorname{Lip}^{1,-1/p'}(\mathbb{R}^n). \tag{3.26}$$

Remark 3.16. The proof relies on the estimate

$$\| f \mid \mathcal{C}^{1-\lambda}(\mathbb{R}^n)\| \le c\lambda^{-1/p'} \| f \mid W_p^{\mathcal{M}+1}(\mathbb{R}^n)\|$$

and the extrapolation result Corollary 3.10(i). One may ask whether further reduction is possible, i.e., whether (HS-16) remains true if $\mathcal{M}+1$ is replaced by a subset $\mathcal{S} \subset \mathcal{M}+1$ containing less multi-indices of order $m+1$. This may lead to different limiting situations; several examples are discussed in [62, Sect. 5].

4. Further spaces and problems

Now we briefly sketch some further directions of research based on the extrapolation techniques.

4.1. Logarithmic Sobolev spaces. Let $0 < p < \infty$, $j \in \mathbb{N}$, recall our notation for p_j, p_{-j} in (2.16). Edmunds and Triebel introduced in [35–37] logarithmic Sobolev spaces $H_p^s(\log H)_a(\Omega)$, $1 < p < \infty$, $s \in \mathbb{R}$, $a \in \mathbb{R}$, in the same spirit as presented in Theorem 2.5, where $\Omega \subset \mathbb{R}^n$ is a bounded C^∞-domain. For instance, if $a < 0$, then

$$f \in H_p^s(\log H)_a(\Omega) \quad \text{if, and only if,} \quad \Big(\sum_{j=j_0}^{\infty} 2^{jap} \| f \mid H_{p_j}^s(\Omega)\|^p\Big)^{1/p}$$

is finite; for $a > 0$ a modification similar to Theorem 2.5(ii) is needed. If $s \in \mathbb{N}$, then as in the classical case,

$$H_p^s(\log H)_a(\Omega) = H^s(X) \text{ with } X = L_p(\log L)_a(\Omega),$$

that is, the Sobolev spaces $H_p^s(\log H)_a(\Omega)$ can be seen as the Lorentz-Zygmund space $L_p(\log L)_a(\Omega)$ ‘lifted’ with smoothness $s \in \mathbb{N}_0$ and equipped with the equivalent norm

$$\sum_{|\alpha|\le s} \| \mathrm{D}^\alpha f \mid L_p(\log L)_a(\Omega)\|,$$

see [36, Thm. 2.6.3] and [35].

4.2. Spaces on $\mathbb{R}^n$. Dealing with spaces on $\Omega = \mathbb{R}^n$, or more general, unbounded domains with $|\Omega| = \infty$, then there is no direct counterpart for the construction described in Section 2.2: it heavily relies on monotonicity arguments for the spaces like (2.9) when approaching the destination space; but this fails in this setting. To circumvent this difficulty we followed a slightly modified approach in [49], [50] and introduced additional weights (which have no influence in spaces on bounded domains). In particular, when $1 < p < \infty$ and $a < 0$, then $L_p(\log L)_a(\mathbb{R}^n)$ is the set of all measurable functions $f\colon \mathbb{R}^n \to \mathbb{C}$ such that $(1+|x|^2)^{-2^{-j}} f \in L_{p_j,p}(\mathbb{R}^n)$ for $j \geq j_0$, and

$$\| f \mid L_p(\log L)_a(\mathbb{R}^n)\| = \Big(\sum_{j=j_0}^{\infty} 2^{jap} \big\| \, (1+|x|^2)^{-2^{-j}} f \mid L_{p_j,p}(\mathbb{R}^n) \big\|^p \Big)^{1/p} \quad (4.1)$$

is finite. In case of $a > 0$ a modification similar to Theorem 2.5(ii) is necessary. This led to spaces $L_{p,q}(\log L)_a(\mathbb{R}^n)$ and $H^s_p(\log H)_a(\mathbb{R}^n)$ which inherited useful properties from their counterparts on domains when restricted, say, to dyadic annuli. So using rather mild additional logarithmic weights we found counterparts of sharp or compact embeddings like (2.29) and could – via scaling arguments – also estimate their entropy and approximation numbers, recall Remark 2.12.

On the other hand, defining the target space of type $L_{p,q}(\log L)^*_a(\mathbb{R}^n)$ directly by (2.5) causes no problems in this context, but leads to different spaces: unlike in case of bounded domains one can prove that, in general, $L_{p,q}(\log L)_a(\mathbb{R}^n)$ and $L_{p,q}(\log L)^*_a(\mathbb{R}^n)$ do not coincide. The latter approach, also in view of Bessel-potential spaces $H^s_p(\log H)^*_a(\mathbb{R}^n) = H^s(X)$ with $X = L_p(\log L)^*_a(\mathbb{R}^n)$, was studied in a series of papers by Edmunds, Gurka and Opic [18–23], [47], and by Evans, Opic, Pick and Trebels in [38], [39], [77–80].

It seems that already in this relatively simple situation on $\mathbb{R}^n$ the optimal approach to logarithmic spaces strongly depends on the intention what problems should be solved: in view of compactness, decomposition methods and some applications in spectral theory the extrapolation method appears preferable, whereas in terms of optimality of embeddings, e.g. in the context of r.i. spaces, more abstract settings, or mapping properties and connections to interpolation theory, the immediate definition (2.5) is advantageous.

4.3. Spaces on $\mathbb{T}^n$. Finally we describe some new Besov spaces obtained via extrapolation techniques and restrict ourselves to the periodic case $\Omega = \mathbb{T}^n$ for convenience. Besov spaces of type $B^s_{p,q}(\mathbb{T}^n)$ were described in [85, Ch. 3] using a Fourier-analytical approach based on smooth dyadic partitions $\varphi = (\varphi_j)_{j\in\mathbb{N}_0}$ of unity. Let $(\widehat{f}(k))_{k\in\mathbb{Z}^n}$ be the Fourier coefficients of $f \in \mathcal{D}'(\mathbb{T}^n)$ and

$$f_j^\varphi(x) = \sum_{k\in\mathbb{Z}^n} \widehat{f}(k)\varphi_j(k)\mathrm{e}^{2\pi\mathrm{i}kx}, \quad j \in \mathbb{N}_0,\ x \in \mathbb{T}^n.$$

Then for $s \in \mathbb{R}$, $0 < p, q \le \infty$,

$$\| f \mid B^s_{p,q}(\mathbb{T}^n)\| = \Big(\sum_{j=0}^\infty \| 2^{js} f_j^\varphi \mid L_p(\mathbb{T}^n)\|^q\Big)^{1/q}.$$

Equivalent characterisations, based on wavelet decompositions, can be found in [100, Sect. 1.3]. We refer to [93, Ch. 9] for the connection with weighted spaces on $\mathbb{R}^n$.

Plainly, some first approach to define Besov-Zygmund spaces consists in replacing the basic space L_p in the above definition by $L_p(\log L)_a$, that is, to understand $B^s_p(L_p(\log L)_a) = B^s_p(L_p(\log L)_a(\mathbb{T}^n))$ where $0 < p < \infty$, $s \in \mathbb{R}$, $a \in \mathbb{R}$. This idea was also used by Edmunds and Netrusov in [33] in a slightly different context. Obviously these spaces coincide for $a = 0$ with $B^s_{p,q}(\mathbb{T}^n)$.

On the other hand, using notation (2.16), we aim at descriptions similar to Theorem 2.5, that is, where

$$\| f \mid B^s_p(\log B)_a(\mathbb{T}^n)\| = \Big(\sum_{j=J}^\infty 2^{jap} \| f \mid B^s_{p_j,p}(\mathbb{T}^n)\|^p\Big)^{1/p} < \infty, \tag{4.2}$$

for $a < 0$, and a modified construction for $a > 0$. At first glance these are different spaces, $B^s_p(L_p(\log L)_a)(\mathbb{T}^n)$ and $B^s_p(\log B)_a(\mathbb{T}^n)$, but the assumption is that both spaces coincide in the sense of equivalent norms. This relies on some unpublished notes by Triebel [96], [99] and is not yet clear in all cases, e.g., when $0 < p < 1$.

As spaces of such type already appeared in some limiting embedding situations, cf. [53], the study seems quite interesting, in particular in view of (wavelet) decompositions, (compact) embeddings, entropy and approximation numbers. First partial contributions in this direction can be found in [8], [97], [100], [101].

References

[1] R. A. Adams: *Reduced Sobolev inequalities.* Canad. Math. Bull. **31** (1988), no. 2, 159–167. Zbl 0662.46036, MR0942066.

[2] P. Arzt: *Interpolation and extrapolation.* Master's thesis, Friedrich-Schiller-Universität Jena, Germany, 2009.

[3] C. Bennett and K. Rudnick: *On Lorentz-Zygmund spaces.* Dissertationes Math. (Rozprawy Mat.) **175** (1980), 67 pp. Zbl 0456.46028, MR0576995.

[4] C. Bennett and R. Sharpley: *Interpolation of operators.* Pure and Applied Mathematics, **129**. Boston: Academic Press, Inc., 1988. Zbl 0647.46057, MR0928802.

[5] G. Bourdaud and M. Lanza de Cristoforis: *Functional calculus on Hölder-Zygmund spaces.* Trans. Amer. Math. Soc. **354** (2002), no. 10, 4109–4129. Zbl 1026.47051, MR1926867.

[6] H. Brézis and S. Wainger: *A note on limiting cases of Sobolev embeddings and convolution inequalities.* Comm. Partial Differential Equations **5** (1980), no. 7, 773–789. Zbl 0437.35071, MR0579997.

[7] A. M. Caetano: *Entropy numbers of embeddings between logarithmic Sobolev spaces.* Portugal. Math. **57** (2000), no. 3, 355–379. Zbl 0974.46037, MR1781451.

[8] A. M. Caetano: *Subatomic representation of Bessel potential spaces modelled on Lorentz spaces.* In: The J. A. Sampaio Martins anniversary volume, 37–47, Textos Mat. Sér. B **34**, Coimbra: Univ. Coimbra, 2004. MR2228307.

[9] A. M. Caetano and D.D. Haroske: *Sharp estimates of approximation numbers via growth envelopes.* In: D. D. Haroske, Th. Runst and H. J. Schmeißer (eds.), Function Spaces, Differential Operators and Nonlinear Analysis. The Hans Triebel Anniversary Volume, 237–244. Basel: Birkhäuser, 2003. Zbl 1056.47016, MR1984172

[10] M. J. Carro: *New extrapolation estimates.* J. Funct. Anal., **174** (2000), no. 1, 155–166. Zbl 0962.42014, MR1761366.

[11] F. Cobos, L. M. Fernández-Cabrera, A. Manzano and A. Martínez: *Logarithmic interpolation spaces between quasi-Banach spaces.* Z. Anal. Anwend. 26 (2007), no. 1, 65–86. Zbl 1146.46013, MR2306106.

[12] F. Cobos, L.M. Fernández-Cabrera and H. Triebel: *Abstract and concrete logarithmic interpolation spaces.* J. London Math. Soc. (2) **70** (2004), no. 1, 231–243. Zbl 1066.46016, MR2064760.

[13] F. Cobos, Th. Kühn and T. Schonbek: *Compact embeddings of Brézis-Wainger type.* Rev. Mat. Iberoam. 22 (2006), no. 1, 305–322. Zbl 1109.47015, MR2268121.

[14] D. Cruz-Uribe and M. Krbec: *Localization and extrapolation in Orlicz-Lorentz spaces.* In: M. Cwikel, M. Engliš, A. Kufner, L.-E. Persson and G. Sparr (eds.), Function Spaces, Interpolation Theory and Related Topics, 273–283. Proceedings of the International Conference in honour of Jaak Peetre on his 65th birthday, Lund, Sweden, August 17–22, 2000. Berlin: de Gruyter, 2002. Zbl 1128.46302, MR1943288.

[15] M. Cwikel and E. Pustylnik: *Sobolev type embeddings in the limiting case.* J. Fourier Anal. Appl. **4** (1998), no. 4–5, 433–446. Zbl 0930.46027, MR1658620.

[16] R. DeVore and G. G. Lorentz: *Constructive approximation.* Grundlehren der Mathematischen Wissenschaften **303**. Berlin: Springer-Verlag, 1993. Zbl 0797.41016, MR1261635.

[17] D. E. Edmunds and W. D. Evans: *Hardy operators, function spaces and embeddings.* Springer Monographs in Mathematics. Berlin¿ Springer, 2004. Zbl 1099.46002, MR2091115.

[18] D. E. Edmunds, P. Gurka and B. Opic: *Double exponential integrability, Bessel potentials and embedding theorems.* Studia Math. **115** (1995), no. 2, 151–181. Zbl 0829.47024, MR1347439.

[19] D. E. Edmunds, P. Gurka and B. Opic: *Double exponential integrability of convolution operators in generalized Lorentz-Zygmund spaces.* Indiana Univ. Math. J. **44** (1995), no. 1, 19–43. Zbl 0826.47021, MR1336431.

[20] D. E. Edmunds, P. Gurka and B. Opic: *Sharpness of embeddings in logarithmic Bessel potential spaces.* Proc. Roy. Soc. Edinburgh Sect. A **126** (1996), no. 5, 995–1009. Zbl 0860.46024, MR1415818.

[21] D. E. Edmunds, P. Gurka and B. Opic: *On embeddings of logarithmic Bessel potential spaces.* J. Funct. Anal. **146** (1997), no. 1, 116–150. Zbl 0934.46036, MR1446377.

[22] D. E. Edmunds, P. Gurka and B. Opic: *Norms of embeddings of logarithmic Bessel potential spaces.* Proc. Amer. Math. Soc. **126** (1998), no. 8, 2417–2425. Zbl 0895.46020, MR1451796.

[23] D. E. Edmunds, P. Gurka and B. Opic: *Optimality of embeddings of logarithmic Bessel potential spaces.* Q. J. Math. **51** (2000), no. 2, 185–209. Zbl 0957.46021, MR1765790.

[24] D. E. Edmunds, P. Gurka and B. Opic: *Compact and continuous embeddings of logarithmic Bessel potential spaces.* Studia Math. **168** (2005), no. 3, 229–250. Zbl 1083.46017, MR2146125.

[25] D. E. Edmunds, P. Gurka and B. Opic: *Non-compact and sharp embeddings of logarithmic Bessel potential spaces into Hölder-type spaces.* Z. Anal. Anwend. **25** (2006), no. 1, 73–80. Zbl 1104.46017, MR2216882.

[26] D. E. Edmunds and D. D. Haroske: *Spaces of Lipschitz type, embeddings and entropy numbers.* Dissertationes Math. (Rozprawy Mat.) **380** (1999), 43 pp. Zbl 0932.46026, MR1677961.

[27] D. E. Edmunds and D. D. Haroske: *Embeddings in spaces of Lipschitz type, entropy and approximation numbers, and applications.* J. Approx. Theory **104** (2000), no. 2, 226–271. Zbl 0997.41022, MR1761901.

[28] D. E. Edmunds, R. Kerman and L. Pick: *Optimal Sobolev imbeddings involving rearrangement-invariant quasinorms.* J. Funct. Anal. **170** (2000), no. 2, 307–355. Zbl 0955.46019, MR1740655.

[29] D. E. Edmunds and M. Krbec: *Two limiting cases of Sobolev imbeddings.* Houston J. Math. **21** (1995), no. 1, 119–128. Zbl 0835.46027, MR1331250.

[30] D. E. Edmunds and M. Krbec: *On decomposition in exponential Orlicz spaces.* Math. Nachr. **213** (2000), 77–88. Zbl 0971.46019, MR1755247.

[31] D. E. Edmunds and M. Krbec: *Decomposition and Moser's lemma.* Rev. Mat. Complut. **15** (2002), no. 1, 57–74. Zbl 1017.46020, MR1915215.

[32] D. E. Edmunds and M. Krbec: *Variations on Yano's extrapolation theorem.* Rev. Mat. Complut. **18** (2005), no. 1, 111–118. Zbl 1084.46022, MR2135534.

[33] D. E. Edmunds and Yu. Netrusov: *Entropy numbers of embeddings of Sobolev spaces in Zygmund spaces.* Studia Math. **128** (1998), no. 1, 71–102. Zbl 0919.46024, MR1489461.

[34] D. E. Edmunds and H. Triebel: *Eigenvalue distributions of some degenerate elliptic operators: an approach via entropy numbers.* Math. Ann. **299** (1994), no. 2, 311–340. Zbl 0804.35097, MR1275771.

[35] D. E. Edmunds and H. Triebel: *Logarithmic Sobolev spaces and their applications to spectral theory.* Proc. London Math. Soc. (3) **71** (1995), no. 2, 333–371. Zbl 0835.46028, MR1337470.

[36] D. E. Edmunds and H. Triebel: *Function spaces, entropy numbers, differential operators.* Cambridge Tracts in Mathematics, **120**. Cambridge: Cambridge University Press, 1996. Zbl 0865.46020, MR1410258.

[37] D. E. Edmunds and H. Triebel: *Logarithmic spaces and related trace problems.* Funct. Approximatio, Comment. Math. **26** (1998), 189–204. Zbl 0924.46029, MR1666618.

[38] W. D. Evans and B. Opic: *Real interpolation with logarithmic functors and reiteration.* Canad. J. Math. **52** (2000), no. 5, 920–960. Zbl 0981.46058, MR1782334.

[39] W. D. Evans, B. Opic and L. Pick: *Interpolation of operators on scales of generalized Lorentz-Zygmund spaces.* Math. Nachr. **182** (1996), 127–181. Zbl 0865.46016, MR1419893.

[40] A. Fiorenza and M. Krbec: *On Decompositions in $L(\log L)^\alpha$.* Preprint **129**, Math. Inst. Czech Acad. Sci., Prague, 1998.

[41] A. Fiorenza and M. Krbec: *On an optimal decomposition in Zygmund spaces.* Georgian Math. J. **9** (2002), no. 2, 271–286. Zbl 1015.46018, MR1916066.

[42] A. Gogatishvili, B. Opic and J. S. Neves: *Optimality of embeddings of Bessel-potential-type spaces into Lorentz-Karamata spaces.* Proc. Roy. Soc. Edinburgh Sect. A **134** (2004), no. 6, 1127–1147. Zbl 1076.46021, MR2107486.

[43] A. Gogatishvili, J. S. Neves and B. Opic: *Optimality of embeddings of Bessel-potential-type spaces into generalized Hölder spaces.* Publ. Mat., Barc. **49** (2005), no. 2, 297–327. Zbl 1096.46016, MR2177070.

[44] A. Gogatishvili, J. S. Neves and B. Opic: *Sharpness and non-compactness of embeddings of Bessel-potential-type spaces.* Math. Nachr. **280** (2007), no. 9–10, 1083–1093. Zbl 1137.46020, MR2334661.

[45] A. Gogatishvili, J. S. Neves and B. Opic: *Optimal embeddings and compact embeddings of Bessel-potential-type spaces.* Math. Z. **262** (2009), no. 3, 645–682. Zbl 1179.46029, MR2506313.

[46] A. Gogatishvili, J. S. Neves and B. Opic: *Optimality of embeddings of Bessel-potential-type spaces into generalized Hölder spaces involving k-modulus of smoothness.* Potential Anal. **32** (2010), no. 3, 201–228. Zbl 1197.46017, MR2595366.

[47] P. Gurka and B. Opic: *Global limiting embeddings of logarithmic Bessel potential spaces.* Math. Inequal. Appl. **1** (1998), no. 4, 565–584. Zbl 0934.46034, MR1646690.

[48] K. Hansson: *Imbedding theorems of Sobolev type in potential theory.* Math. Scand. **45** (1979), no. 1, 77–102. Zbl 0437.31009, MR0567435.

[49] D. Haroske: *Some logarithmic function spaces, entropy numbers, applications to spectral theory.* Dissertationes Math. (Rozprawy Mat.) **373** (1998), 59 pp. Zbl 0906.46027, MR1605421.

[50] D. D. Haroske: *Logarithmic Sobolev spaces on $\mathbb{R}^n$; entropy numbers, and some application.* Forum Math. **12** (2000), no. 3, 257–313. Zbl 0947.46020, MR1748482.

[51] D. D. Haroske: *On more general Lipschitz spaces.* Z. Anal. Anwend. **19** (2000), no. 3, 781–799. Zbl 0998.46013, MR1784131.

[52] D. D. Haroske: *Envelopes and sharp embeddings of function spaces.* Chapman & Hall/CRC Research Notes in Mathematics **437**. Boca Raton, FL: Chapman & Hall/CRC, 2007. Zbl 1111.46002, MR2262450.

[53] D. D. Haroske and L. Skrzypczak: *Entropy numbers of embeddings of function spaces with Muckenhoupt weights, III. Some limiting cases.* J. Funct. Spaces Appl., to appear.

[54] L. I. Hedberg: *On certain convolution inequalities.* Proc. Amer. Math. Soc. **36** (1972), 505–510. Zbl 0283.26003, MR0312232.

[55] J. A. Hempel, G. R. Morris and N. S. Trudinger: *On the sharpness of a limiting case of the Sobolev imbedding theorem.* Bull. Austral. Math. Soc. **3** (1970), 369–373. Zbl 0205.12801, MR0280998.

[56] B. Jawerth and M. Milman: *Extrapolation theory with applications.* Mem. Amer. Math. Soc. **89** (1991), no. 440, 82 pp. Zbl 0733.46040, MR1046185.

[57] B. Jawerth and M. Milman: *New results and applications of extrapolation theory.* In: Interpolation spaces and related topics (Haifa, 1990), 81–105, Israel Math. Conf. Proc. **5**. Ramat Gan: Bar-Ilan Univ., 1992. Zbl 0922.46063, MR1206493.

[58] G. E. Karadzhov and M. Milman: *Extrapolation theory: New results and applications.* J. Approx. Theory **133** (2005), no. 1, 38–99. Zbl 1081.46018, MR2122269.

[59] M. Krbec and H.-J. Schmeisser: *Extrapolation of reduced Sobolev imbeddings.* In: J. Rákosník (ed.), Function Spaces, Differential Operators and Nonlinear Analysis, 71–88. Proceedings of the Conference held in Paseky nad Jizerou, September 3–9, 1995, Prague: Prometheus Publishing House, 1996. Zbl 0879.46014, MR1480930.

[60] M. Krbec and H.-J. Schmeisser: *Limiting imbeddings. The case of missing derivatives.* Ricerche Mat. **45**(1996), no. 2, 423–447. Zbl 0930.46029, MR1776418.

[61] M. Krbec and H.-J. Schmeisser: *On extrapolation of Sobolev and Morrey type embeddings.* In: H. Hudzik and L. Skrzypczak (eds.), Function spaces, 297–322. Proceedings of the 5th international conference, Poznań, Poland, August 28–September 3, 1998. Lect. Notes Pure Appl. Math. **213**, 297–321. New York, NY: Marcel Dekker, 2000. Zbl 0966.46016, MR1772133.

[62] M. Krbec and H.-J. Schmeisser: *Imbeddings of Brézis-Wainger type. The case of missing derivatives.* Proc. Roy. Soc. Edinburgh Sect. A **131** (2001), no. 3, 667–700. Zbl 0985.46018, MR1838506.

[63] M. Krbec and H.-J. Schmeisser: *Refined limiting imbeddings for Sobolev spaces of vector-valued functions.* J. Funct. Anal. **227** (2005), no. 2, 372–388. Zbl 1090.46027, MR2168079.

[64] M. Krbec and H.-J. Schmeisser: *Imbeddings of Brézis-Wainger type. The case of missing derivatives.* Preprint **130**, Math. Inst. Czech Acad. Sci., Prague, 1998.

[65] M. Krbec and H.J. Schmeisser: *Critical imbeddings with multivariate rearrangements.* Studia Math. **181** (2007), no. 3, 255–284. Zbl 1131.46026, MR2322578.

[66] A. Kufner, O. John and S. Fučík.: *Function spaces.* Monographs and Textbooks on Mechanics of Solids and Fluids; Mechanics: Analysis. Leyden: Noordhoff International Publishing. Prague: Academia, Publishing House of the Czechoslovak Academy of Sciences, 1977. Zbl 0364.46022, MR0482102.

[67] Th. Kühn: *Compact embeddings of Besov spaces in exponential Orlicz spaces.* J. London Math. Soc. (2) **67** (2003), no. 1, 235–244. Zbl 1052.46022, MR1942423.

[68] Th. Kühn and T. Schonbek: *Compact embeddings of Besov spaces into Orlicz and Lorentz-Zygmund spaces.* Houston J. Math. **31** (2005), no. 4, 1221–1243 (electronic). Zbl 1107.47015, MR2175433.

[69] V. G. Maz'ya: *Sobolev spaces.* Springer Series in Soviet Mathematics. Berlin: Springer, 1985. Transl. from the Russian by T. O. Shaposhnikova. Zbl 0692.46023, MR0817985.

[70] V. G. Maz'ya: *Conductor and capacitary inequalities for functions on topological spaces and their applications to Sobolev type imbeddings.* J. Funct. Anal. **224** (2005), no. 2, 408–430. Zbl 1105.46017, MR2146047.

[71] M. Milman: *Extrapolation and optimal decomposition.* Lecture Notes in Mathematics **1580**. Berlin: Springer-Verlag, 1994. Zbl 0852.46059, MR1301774.

[72] M. Milman: *A note on extrapolation theory.* J. Math. Anal. Appl. **282** (2003), no. 1, 26–47. Zbl 1031.46085, MR2000327.

[73] J. Moser: *A sharp form of an inequality by N. Trudinger.* Indiana Univ. Math. J. **20** (1970/71), 1077–1092. Zbl 0213.13001, MR0301504.

[74] J. S. Neves: *Extrapolation results on general Besov-Hölder-Lipschitz spaces.* Math. Nachr. **230** (2001), 117–141. Zbl 1033.26004, MR1854881.

[75] J. S. Neves: *On decompositions in generalised Lorentz-Zygmund spaces.* Boll. Unione Mat. Ital. Sez. B Artic. Ric. Mat. (8) **4** (2001), no. 1, 239–267. Zbl 1178.46029, MR1821406.

[76] J. S. Neves: *Lorentz-Karamata spaces, Bessel and Riesz potentials and embeddings.* Dissertationes Math. (Rozprawy Mat.) **405** (2002), 46 pp. Zbl 1017.46021, MR1927106.

[77] B. Opic: *Embeddings of Bessel-potential-type spaces.* Preprint **169**, Math. Inst. Czech Acad. Sci., Prague, 2007.

[78] B. Opic and L. Pick: *On generalized Lorentz-Zygmund spaces.* Math. Inequal. Appl. **2** (1999), no. 3, 391–467. Zbl 0956.46020, MR1698383.

[79] B. Opic and W. Trebels: *Bessel potentials with logarithmic components and Sobolev-type embeddings.* Anal. Math. **26** (2000), no. 4, 299–319. Zbl 0973.46021, MR1805506.

[80] B. Opic and W. Trebels: *Sharp embeddings of Bessel potential spaces with logarithmic smoothness.* Math. Proc. Cambridge Philos. Soc. **134** (2003), no. 2, 347–384. Zbl 1036.46027, MR1972143.

[81] J. Peetre: *Espaces d'interpolation et théorème de Soboleff.* Ann. Inst. Fourier (Grenoble) **16** (1966), no. 1, 279–317. Zbl 0151.17903, MR0221282.

[82] L. Pick: *Optimal Sobolev embeddings.* In: M. Krbec, (ed.) et al., Nonlinear analysis, function spaces and applications. Vol. 6. Proceedings of the spring school held in

Prague, Czech Republic, May 31–June 6, 1998, 156–199. Prague: Czech Academy of Sciences, Mathematical Institute, 1999. Zbl 0964.46012, MR1777715.

[83] S. Pohozhaev: *On eigenfunctions of the equation* $\Delta u + \lambda f(u) = 0$. Sov. Math., Dokl. **6** (1965), 1408–1411; transl. from Dokl. Akad. Nauk SSSR **165** (1965), 36–39. Zbl 0141.30202.

[84] H.-J. Schmeisser: *Recent developments in the theory of function spaces with dominating mixed smoothness.* In: NAFSA 8 – Nonlinear analysis, function spaces and applications. Vol. 8, 144–204. Prague: Czech. Acad. Sci., 2007. MR2657119.

[85] H.-J. Schmeisser and H. Triebel: *Topics in Fourier analysis and function spaces.* A Wiley-Interscience Publication. Chichester: John Wiley & Sons, 1987. Zbl 0661.46025, MR0891189.

[86] C. Schneider: *On dilation operators in Besov spaces.* Rev. Mat. Complut. **22** (2009), no. 1, 111–128. Zbl 1175.46024, MR2499328.

[87] S. L. Sobolev: *Sur un théorème d'analyse fonctionnelle.* Mat. Sb., N. Ser. **4** (1938), 471–497. Engl. transl.: Amer. Math. Soc., Transl., II. Ser. **34** (1963), 39–68. Zbl 0131.11501

[88] E. M. Stein: *Singular integrals and differentiability properties of functions.* Princeton Mathematical Series, **30**. Princeton, N.J.: Princeton Univ. Press, 1970. Zbl 0207.13501, MR0290095.

[89] R. S. Strichartz: *A note on Trudingers extension of Sobolev's inequalities.* Indiana Univ. Math. J. **21** (1971/72), 841–842. Zbl 0241.46028, MR0293389.

[90] E. C. Titchmarsh: *On conjugate functions.* Proc. London Math. Soc. **29** (1928), 49–80. JFM 54.0307.01.

[91] E. C. Titchmarsh: *Additional note on conjugate functions.* J. London Math. Soc. **4** (1929), 204–206.

[92] A. Torchinsky: *Real-variable methods in harmonic analysis.* Pure and Appl. Mathematics, **123**. Orlando, FL: Academic Press, Inc., 1986. Zbl 0621.42001, MR0869816.

[93] H. Triebel: *Theory of function spaces.* Monographs in Mathematics, **78**. Basel: Birkhäuser, 1983. Zbl 0546.46027, MR0781540.

[94] H. Triebel: *Approximation numbers and entropy numbers of embeddings of fractional Besov-Sobolev spaces in Orlicz spaces.* Proc. London Math. Soc. **66** (1993), no. 3, 589–618. Zbl 0792.46024, MR1207550.

[95] H. Triebel: *The structure of functions.* Monographs in Mathematics **97**. Basel: Birkhäuser, 2001. Zbl 0984.46021, MR1851996.

[96] H. Triebel: *Besov-Zygmund spaces and logarithmic Besov spaces.* Preprint, 2004.

[97] H. Triebel: *Sampling numbers and embedding constants.* Trudy Mat. Inst. Steklov **248** (2005), 275–284. Engl. transl.: Proc. Steklov Inst. Math. **248** (2005), no. 1, 268–277. Zbl 1129.46028, MR2165934.

[98] H. Triebel: *Theory of function spaces. III.* Monographs in Mathematics, **100**. Basel: Birkhäuser, 2006. Zbl 1104.46001, MR2250142.

[99] H. Triebel: *Comments on extrapolation spaces.* Preprint, 2007.

[100] H. Triebel: *Function Spaces and wavelets on domains.* EMS Tracts in Mathematics, **7**. Zürich: European Mathematical Society (EMS), 2008. Zbl 1158.46002, MR2455724.

[101] H. Triebel: *Wavelet bases in Lorentz and Zygmund spaces.* Georgian Math. J. **15** (2008), no. 2, 389–402. Zbl 1156.46028, MR2428479.

[102] N. S. Trudinger: *On imbeddings into Orlicz spaces and some applications.* J. Math. Mech. **17** (1967), 473–483. Zbl 0163.36402, MR0216286.

[103] Sh. Yano: *Notes on Fourier analysis. XXIX. An extrapolation theorem.* J. Math. Soc. Japan **3** (1951), 296–305. Zbl 0045.17901, MR0048619.

[104] V. I. Yudovich: *Some estimates connected with integral operators and with solutions of elliptic equations.* Sov. Math. Dokl. **2** (1961), 746–749; transl. from Dokl. Akad. Nauk SSSR **138** (1961), 805–808. Zbl 0144.14501.

[105] A. Zygmund: *Sur les fonctions conjuguées.* C. R. Acad. Sci. Paris, Sér. I Math. **187** (1928), 1025–1026. JFM 54.0307.02.

[106] A. Zygmund: *Sur les fonctions conjuguées.* Fundamenta Math. **13** (1929), 284–303. JFM 55.0751.02.

[107] A. Zygmund: *Smooth functions.* Duke Math. J. **12** (1945), 47–76. Zbl 0060.13806, MR0012691

[108] A. Zygmund: *Trigonometric series. Vol. I and II.* 2nd reprint of the 2nd ed. Cambridge: Cambridge Univ. Press, 2nd edition, 1977. Zbl 0367.42001, MR0617944.