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Pavel Drábek

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RECENT RESULTS ON QUASILINEAR DIFFERENTIAL EQUATIONS. II

Pavel Drábek

Abstract. This lecture follows joint result of the speaker, Petr Girg, Peter Takáč and Michael Ulm. We concentrate on the Fredholm alternative for the p-Laplacian at the first eigenvalue. In contrast with the linear case ($p = 2$), the nonlinear case ($p \neq 2$) appears to be completely different not only concerning the methods (which cannot benefit from the linear structure of the problem and the Hilbert structure of the function spaces) but also from the point of view of the results which seem to be rather surprizing. In particular, the difference between the cases $1 < p < 2$ and $p > 2$ is quite interesting. The main tool to prove existence and multiplicity results is "the bifurcation from infinity" argument.

1. Introduction

These lecture notes are "copy and paste" of selected parts of the joint paper of the speaker and P. Girg, P. Takáč and M. Ulm [13]. We refer the reader to that paper for the proofs which are omitted here for the brevity of this text. Let us note that in this text we updated some parts of above mentioned paper [13]. In particular, we use some facts proved in the paper by H. Lou [21] which was published after [13]. Thanks to the results from [21] we could simplify some technically complicated assumptions from [13].

In the past few years, nonlinear eigenvalue problems for degenerate or singular elliptic boundary value problems have attracted considerable attention. Related to them is the following problem for the Dirichlet p-Laplacian Δ_p in a bounded smooth domain $\Omega \subset \mathbb{R}^N$ ($N \geq 1$):

$$\begin{aligned} -\Delta_p u &= \lambda |u|^{p-2} u + f(x) && \text{in } \Omega, \\ u &= 0 && \text{on } \partial\Omega. \end{aligned} \tag{1.1}$$

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Here, $\Delta_p u \stackrel{\text{def}}{=} \operatorname{div}(|\nabla u|^{p-2}\nabla u)$, where $p \in (1,\infty)$ is a fixed number, $f \in L^\infty(\Omega)$ is a given function, and $\lambda \in \mathbb{R}$ stands for a spectral parameter. One looks for a weak solution $u\colon \Omega \to \mathbb{R}$ to problem (1.1) in the Sobolev space $W_0^{1,p}(\Omega)$.

In this lecture we focus on the solvability of problem (1.1) for parameter values λ near λ_1, where λ_1 stands for the first (smallest) eigenvalue of the positive Dirichlet p-Laplacian $-\Delta_p$ in Ω. If $\lambda < \lambda_1$, existence can be obtained by a standard minimization argument applied to the energy functional

$$\mathcal{J}_\lambda(u) \stackrel{\text{def}}{=} \frac{1}{p}\int_\Omega |\nabla u|^p \,\mathrm{d}x - \frac{\lambda}{p}\int_\Omega |u|^p \,\mathrm{d}x - \int_\Omega f u \,\mathrm{d}x \tag{1.2}$$

on $W_0^{1,p}(\Omega)$. Unfortunately, for $\lambda \geq \lambda_1$, this functional is no longer coercive, unless an additional hypothesis is imposed on the function f (see e.g. Drábek [12] and Takáč [23]). For $\lambda \leq 0$, the strict convexity of $\mathcal{J}_\lambda$ guarantees uniqueness. In contrast, for $0 < \lambda < \lambda_1$, multiple solutions (in space dimension one) have been constructed in del Pino, Elgueta and Manásevich [8] (for $2 < p < \infty$) and Fleckinger et al. [16] (for $1 < p < 2$). However, if $f \geq 0$ in Ω, uniqueness still holds for every $\lambda < \lambda_1$ (Díaz and Saa [9]).

Unlike in [12], [14], [23], [24], where mostly variational and degree-theoretical arguments are used, in the work reported here we make extensive use of topological methods with bifurcations from infinity based on general facts from [11, Chapt. 5]. What is essentially needed to treat the general case $N \geq 1$ are (rather precise) asymptotic estimates of large solutions to problem (1.1) as $\lambda \to \lambda_1$ developed in [23], [24].

We first motivate our results by considering the linear boundary value problem

$$\begin{aligned} -\Delta u - \lambda u &= f \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{1.3}$$

which corresponds to $p = 2$ in (1.1). Let $f \in L^\infty(\Omega)$ be given, $f \not\equiv 0$. Then the set of all pairs $(\lambda, u) \in (-\infty, \lambda_2) \times W_0^{1,2}(\Omega)$ that satisfy (1.3) can be interpreted by means of a bifurcation diagram in $\mathbb{R} \times W_0^{1,2}(\Omega)$. Namely, let us write $u = c\varphi_1 + u^\top$ with $\int_\Omega u^\top \varphi_1 \,\mathrm{d}x = 0$. As usual, φ_1 is the eigenfunction of the negative Dirichlet Laplacian $-\Delta$ associated with the (simple) eigenvalue λ_1 that is normalized by $\varphi_1 > 0$ in Ω and $\int_\Omega \varphi_1^2 \,\mathrm{d}x = 1$, and λ_2 stands for the second eigenvalue of $-\Delta$. Then problem (1.3) is equivalent to

$$\begin{aligned} -\Delta u^\top - \lambda u^\top + (\lambda_1 - \lambda) c \varphi_1 &= f^\top + a\varphi_1 \quad \text{in } \Omega, \\ u^\top &= 0 \quad \text{on } \partial\Omega, \end{aligned}$$

where $\int_\Omega f^\top \varphi_1 \,\mathrm{d}x = 0$ and $a = \int_\Omega f\varphi_1 \,\mathrm{d}x$. Clearly, $(\lambda_1 - \lambda)c = a$. The linear Fredholm alternative implies that the problem

$$\begin{aligned} -\Delta u^\top - \lambda u^\top &= f^\top \quad \text{in } \Omega, \\ u^\top &= 0 \qquad \text{on } \partial\Omega, \end{aligned}$$

has a unique solution $u^\top \in W_0^{1,2}(\Omega)$ with $\int_\Omega u^\top \varphi_1 \,\mathrm{d}x = 0$. We have the following two different cases:

(i) If $\int_\Omega f\varphi_1 \,\mathrm{d}x = 0$ then
  (a) for any $\lambda \in (-\infty, \lambda_1) \cup (\lambda_1, \lambda_2)$, problem (1.3) has a unique solution $u_\lambda = u^\top$;
  (b) for $\lambda = \lambda_1$, all solutions of problem (1.3) can be written in the form $u_{\lambda_1} = c\varphi_1 + u^\top$ with $c \in \mathbb{R}$ arbitrary.

(ii) If $\int_\Omega f\varphi_1 \,\mathrm{d}x \neq 0$ then
  (a) there is no solution of (1.3) for $\lambda = \lambda_1$;
  (b) for any $\lambda \in (-\infty, \lambda_1) \cup (\lambda_1, \lambda_2)$ there is a unique solution of (1.3) expressed by $u_\lambda = c\varphi_1 + u^\top$, where

$$c = (\lambda_1 - \lambda)^{-1} \int_\Omega f\varphi_1 \,\mathrm{d}x.$$

The solution pairs $(\lambda, u) \in \mathbb{R} \times W_0^{1,2}(\Omega)$ of (1.3) can thus be sketched in the bifurcation diagrams indicated in Figure 1.

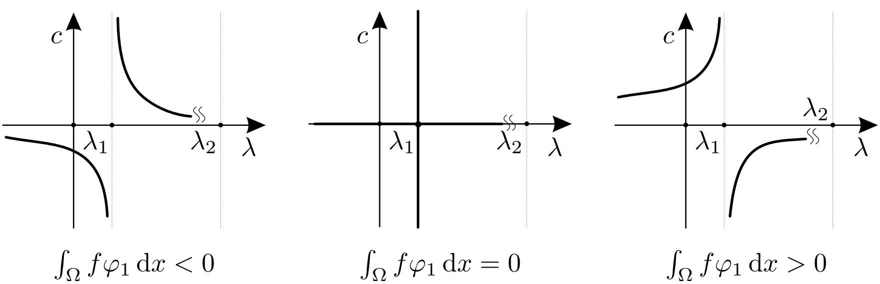

Figure 1: Bifurcations from infinity of solutions to (1.3), $c \stackrel{\text{def}}{=} \int_\Omega u\varphi_1 \,\mathrm{d}x$.

Motivated by this picture of the solution set of (1.3), we have decided to study the nonlinear problem (1.1) for $p \neq 2$ and to investigate the solution pairs $(\lambda, u) \in \mathbb{R} \times W_0^{1,p}(\Omega)$ for λ near λ_1. Again, φ_1 is the eigenfunction of the

p-Laplacian associated with λ_1 and normalized by $\varphi_1 > 0$ and $\int_\Omega \varphi_1^p \, \mathrm{d}x = 1$ (cf. (2.1) below). Notice that $a = (\int_\Omega \varphi_1^2 \, \mathrm{d}x)^{-1} \int_\Omega f\varphi_1 \, \mathrm{d}x$.

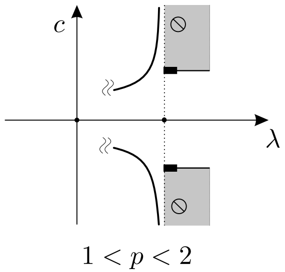

Figure 2: A priori bounds and bifurcations from infinity of solutions to (1.1) for $p > 1$, $p \neq 2$ and $a = 0$. There is no solution in the shaded regions (owing to a priori bounds).

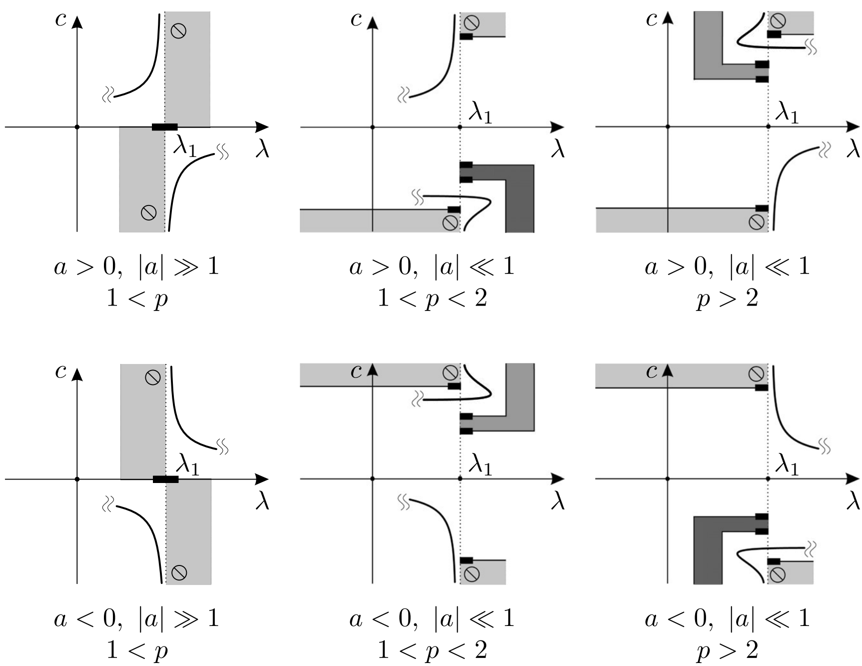

Figure 3: A priori bounds and bifurcations from infinity of solutions to (1.1) for $a \neq 0$, $1 < p < 2$ and/or $p > 2$.

The main results concerning the asymptotic behavior of the solution set to (1.1) as $\lambda \to \lambda_1$ are sketched in Figures 2 and 3. We assume that $f^\top \in L^\infty(\Omega)$ is a given function satisfying $\int_\Omega f^\top \varphi_1 \,\mathrm{d}x = 0$ and $f^\top \not\equiv 0$. In (1.1) we write $f = a\varphi_1 + f^\top$, $a \in \mathbb{R}$, and split the solution as $u = c\varphi_1 + u^\top$. Note, that there are no solutions in the shaded regions (we have a priori bounds) while there may be many other solutions in the nonshaded regions.

We emphasize that Figures 2 and 3 depict the situation for $|\lambda - \lambda_1| \ll 1$ and $|\int_\Omega f\varphi_1 \,\mathrm{d}x| \ll 1$. Precise statements of these asymptotic results can be found in Section 5. Let us mention only some of their important consequences. Although a number of these results have already been known, our approach provides new proofs. Below we list them briefly.

In order to formulate our existence and multiplicity results, we rewrite problem (1.1) as follows, with $f = f^\top + a\varphi_1$:

$$\begin{aligned} -\Delta_p u - \lambda |u|^{p-2} u &= f^\top + a\varphi_1 \quad &&\text{in } \Omega, \\ u &= 0 &&\text{on } \partial\Omega. \end{aligned} \tag{1.4}$$

Here, $f^\top \in L^\infty(\Omega)$ is a given function, with $\int_\Omega f^\top \varphi_1 \,\mathrm{d}x = 0$ and $f^\top \not\equiv 0$, and $\lambda, a \in \mathbb{R}$ are real parameters.

We have the following existence and multiplicity results:

(E1) For $\lambda = \lambda_1$, $a = 0$, problem (1.4) has at least one solution; all possible solutions of (1.4) are a priori bounded in $C^{1,\beta}(\overline{\Omega})$, $0 < \beta < 1$, by a constant which depends on $f^\top$.

(E2) There exist $a_0 = a_0(f^\top) > 0$ and $\delta = \delta(f^\top) > 0$ such that
- if either $\lambda \in (\lambda_1 - \delta, \lambda_1)$ and $a \ge a_0$, or else $\lambda \in (\lambda_1, \lambda_1 + \delta)$ and $a \le -a_0$, then problem (1.4) can have only positive solutions;
- if either $\lambda \in (\lambda_1 - \delta, \lambda_1)$ and $a \le -a_0$, or else $\lambda \in (\lambda_1, \lambda_1 + \delta)$ and $a \ge a_0$, then problem (1.4) can have only negative solutions.

(M1) There exists $\eta = \eta(f^\top) > 0$ such that for $a = 0$ problem (1.4) has at least three distinct solutions (among them at least one positive and one negative) provided either $1 < p < 2$ and $\lambda \in (\lambda_1 - \eta, \lambda_1)$, or $p > 2$ and $\lambda \in (\lambda_1, \lambda_1 + \eta)$.

(M2) There exists $\varepsilon > 0$ with the following properties:
- for every $\varepsilon' \in (0, \varepsilon)$, there is $\eta = \eta(f^\top, \varepsilon, \varepsilon') > 0$ such that $\varepsilon' < |a| < \varepsilon$ and $\lambda \in (\lambda_1 - \eta, \lambda_1) \cup (\lambda_1, \lambda_1 + \eta)$ imply that problem (1.4) has at least three distinct solutions, of which at least one is positive and at least one is negative;
- $\lambda = \lambda_1$ and $0 < |a| < \varepsilon$ imply that problem (1.4) has at least two distinct solutions, of which at least one is negative if $(p-2)a < 0$, and at least one is positive if $(p-2)a > 0$.

2. Preliminaries

We will write $\langle \cdot\,,\cdot \rangle$ for the inner product and $|\cdot|$ for the induced norm in the Euclidean space $\mathbb{R}^N$. We reserve the dot "$\cdot$" for stressing multiplication in complicated expressions. The closure, interior and boundary of a set $S \subset \mathbb{R}^N$ are denoted by $\overline{S}$, $\operatorname{int}(S)$ and ∂S, respectively, and the characteristic function of S by $\chi_S\colon \Omega \to \{0,1\}$. We write $\operatorname{meas}(S) \stackrel{\text{def}}{=} \int_{\mathbb{R}^N} \chi_S(x)\,\mathrm{d}x$ if $S \subset \mathbb{R}^N$ is Lebesgue measurable.

All Banach and Hilbert spaces used in this text are real. We work with the standard inner product in $L^2(\Omega)$ defined by $(u,v)_{L^2(\Omega)} \stackrel{\text{def}}{=} \int_\Omega uv\,\mathrm{d}x$ for $u,v \in L^2(\Omega)$. The orthogonal complement in $L^2(\Omega)$ of a set $\mathcal{M} \subset L^2(\Omega)$ is denoted by $\mathcal{M}^{\perp,L^2}$,

$$\mathcal{M}^{\perp,L^2} \stackrel{\text{def}}{=} \{u \in L^2(\Omega) : (u,v)_{L^2(\Omega)} = 0 \text{ for all } v \in \mathcal{M}\}.$$

The inner product $(\cdot\,,\cdot)_{L^2(\Omega)}$ in $L^2(\Omega)$ induces a duality between the Lebesgue spaces $L^p(\Omega)$ and $L^{p'}(\Omega)$, where $1 \le p, p' \le \infty$ with $\frac{1}{p} + \frac{1}{p'} = 1$, and between the Sobolev space $W_0^{1,p}(\Omega)$ and its dual $W^{-1,p'}(\Omega)$, as well. We keep the same notation also for the duality between the Cartesian products $[L^p(\Omega)]^N$ and $[L^{p'}(\Omega)]^N$. Similarly, if X is a Banach space that is continuously and densely embedded in $L^2(\Omega)$, we denote by X' its dual space, so that $X \hookrightarrow L^2(\Omega) \hookrightarrow X'$.

For simplicity we assume that $\Omega \subset \mathbb{R}^N$ is a bounded domain with C^2-boundary.

The variational formula

$$\lambda_1 = \inf\Big\{ \int_\Omega |\nabla u|^p\,\mathrm{d}x : u \in W_0^{1,p}(\Omega) \text{ with } \int_\Omega |u|^p\,\mathrm{d}x = 1 \Big\}$$

gives the first (smallest) eigenvalue of the positive Dirichlet p-Laplacian for $1 < p < \infty$, that is,

$$\begin{aligned} -\Delta_p \varphi_1 &= \lambda_1 |\varphi_1|^{p-2}\varphi_1 && \text{in } \Omega,\\ \varphi_1 &= 0 && \text{on } \partial\Omega, \end{aligned} \tag{2.1}$$

holds with a nontrivial eigenfunction $\varphi_1 \in W_0^{1,p}(\Omega)$. The first eigenvalue λ_1 is simple and the eigenfunction φ_1 associated with λ_1 can be normalized by $\varphi_1 > 0$ in Ω and $\|\varphi_1\|_{L^p(\Omega)} = 1$, by a result due to Anane [2, Théorème 1, p. 727] and later generalized in Lindqvist [20, Theorem 1.3, p. 157]. We have $\varphi_1 \in L^\infty(\Omega)$ by another result of Anane [3, Théorème A.1, p. 96]. Consequently, recalling the smoothness of $\partial\Omega$, we get even $\varphi_1 \in C^{1,\beta}(\overline{\Omega})$ for some $\beta \in (0,1)$, by a regularity result which is due to DiBenedetto [10, Theorem 2, p. 829] and Tolksdorf [27, Theorem 1, p. 127] (interior regularity, shown independently), and to Lieberman [19, Theorem 1, p. 1203] (regularity near the boundary). The constant β depends solely on N and p. We keep the meaning of the constant β throughout the entire text and denote by β' an arbitrary, but fixed number such that $0 < \beta' < \beta < 1$. Finally, the Hopf maximum principle [25, Prop. 3.2.1 and 3.2.2, p. 801] or [28, Theorem 5, p. 200] can be applied to obtain

$$\varphi_1 > 0 \text{ in } \Omega \quad \text{and} \quad \frac{\partial \varphi_1}{\partial \nu} < 0 \text{ on } \partial\Omega. \tag{2.2}$$

As usual, $\partial/\partial\nu$ denotes the outer normal derivative on $\partial\Omega$. We set

$$U \stackrel{\text{def}}{=} \{x \in \Omega : \nabla\varphi_1(x) \neq \mathbf{0}\} \text{ and } U' \stackrel{\text{def}}{=} \Omega \setminus U = \{x \in \Omega : \nabla\varphi_1(x) = \mathbf{0}\},$$

and observe that U' is a compact subset of Ω, by (2.2). Moreover, it follows from H. Lou [21] that $\operatorname{meas} U' = 0$.

Often, a function $u \in L^1(\Omega)$ will be decomposed as the orthogonal sum

$$u = c\varphi_1 + u^\top, \text{ where } c = \|\varphi_1\|^{-2}_{L^2(\Omega)}(u, \varphi_1)_{L^2(\Omega)} \text{ and } (u^\top, \varphi_1)_{L^2(\Omega)} = 0.$$

Given a set $\mathcal{M} \subset L^1(\Omega)$, we write

$$\mathcal{M}^\top \stackrel{\text{def}}{=} \big\{u^\top : u = c\varphi_1 + u^\top \in \mathcal{M} \text{ for some } c \in \mathbb{R} \text{ and } (u^\top, \varphi_1)_{L^2(\Omega)} = 0\big\}.$$

In particular, if $\mathcal{M}$ is a linear subspace of $L^1(\Omega)$ with $\varphi_1 \in \mathcal{M}$, then we have

$$\mathcal{M}^\top = \{u \in \mathcal{M} : (u, \varphi_1)_{L^2(\Omega)} = 0\}.$$

Throughout the entire article we often need to compare weak solutions u_i $(i = 1, 2)$ of two different boundary value problems of the weak form

$$\int_\Omega |\nabla u_i|^{p-2}\langle\nabla u_i, \nabla\phi\rangle \,\mathrm{d}x - \lambda \int_\Omega |u_i|^{p-2} u_i \phi \,\mathrm{d}x = \int_\Omega f_i \phi \,\mathrm{d}x$$

for all $\phi \in W_0^{1,p}(\Omega)$ or for all ϕ in some similar test space, where $f_i \in L^\infty(\Omega)$ for $i = 1, 2$. A standard way of doing this is to subtract the two equations from one another and then use the integral form of the first order Taylor formula for the terms

$$|\nabla u_1|^{p-2}\nabla u_1 - |\nabla u_2|^{p-2}\nabla u_2 \quad \text{and} \quad |u_1|^{p-2}u_1 - |u_2|^{p-2}u_2.$$

This procedure yields

$$\begin{aligned}
&\int_\Omega \Big\langle \Big[\int_0^1 \mathbf{A}\big((1-s)\nabla u_1 + s\nabla u_2\big)\,\mathrm{d}s\Big](\nabla u_1 - \nabla u_2), \nabla\phi \Big\rangle\,\mathrm{d}x \\
&\qquad - (p-1)\lambda \int_\Omega \Big[\int_0^1 |(1-s)u_1 + su_2|^{p-2}\,\mathrm{d}s\Big](u_1-u_2)\phi\,\mathrm{d}x \\
&\quad = \int_\Omega (f_1 - f_2)\phi\,\mathrm{d}x,
\end{aligned}$$

where we have introduced the abbreviation

$$\mathbf{A}(\mathbf{a}) \stackrel{\text{def}}{=} |\mathbf{a}|^{p-2}\left(\mathbf{I} + (p-2)\frac{\mathbf{a}\otimes\mathbf{a}}{|\mathbf{a}|^2}\right) \quad \text{for } \mathbf{a}\in\mathbb{R}^N,\ \mathbf{a}\neq\mathbf{0}\in\mathbb{R}^N. \tag{2.3}$$

If $2 < p < \infty$, we also set $\mathbf{A}(\mathbf{0}) \stackrel{\text{def}}{=} \mathbf{0} \in \mathbb{R}^{N\times N}$. In fact, this will turn out to be a useful convention also for $1 < p < 2$. For $\mathbf{a}\neq\mathbf{0}$, $\mathbf{A}(\mathbf{a})$ is a positive definite, symmetric matrix. The "elliptic" degeneracy of the matrix $\mathbf{A}(\mathbf{a})$ is expressed by the inequalities

$$\min\{1, p-1\} \le \frac{\langle \mathbf{A}(\mathbf{a})\mathbf{v}, \mathbf{v}\rangle}{|\mathbf{a}|^{p-2}|\mathbf{v}|^2} \le \max\{1, p-1\} \quad \text{for all } \mathbf{a},\mathbf{v}\in\mathbb{R}^N\setminus\{\mathbf{0}\}.$$

In what follows we frequently use the notation $\mathbf{A}_{\varphi_1} = \mathbf{A}(\nabla\varphi_1)$.

Now we need to distinguish between the cases $p > 2$ and $1 < p < 2$, the former one being somewhat easier.

2.1. The degenerate case $2 < p < \infty$. We introduce a new norm on $W_0^{1,p}(\Omega)$ by

$$\|v\|_{\mathcal{D}_{\varphi_1}} \stackrel{\text{def}}{=} \left(\int_\Omega |\nabla\varphi_1|^{p-2}|\nabla v|^2\,\mathrm{d}x\right)^{1/2} \quad \text{for } v\in W_0^{1,p}(\Omega), \tag{2.4}$$

and denote by $\mathcal{D}_{\varphi_1}$ the completion of $W_0^{1,p}(\Omega)$ with respect to this norm. The Hilbert space $\mathcal{D}_{\varphi_1}$ is compactly embedded in the Lebesgue space $L^2(\Omega)$; see [23, Lemma 4.2] and Lemma A.1 below. There, it is also shown that the seminorm (2.4) is in fact a norm on $W_0^{1,p}(\Omega)$.

The second order Taylor expansion for the energy functional

$$\begin{aligned}\frac{1}{p}\int_\Omega |\nabla(\varphi_1+\phi)|^p\,\mathrm{d}x - \frac{\lambda_1}{p}\int_\Omega |\varphi_1+\phi|^p\,\mathrm{d}x \\ &= \int_0^1\int_\Omega |\nabla(\varphi_1+s\phi)|^{p-2}\langle\nabla(\varphi_1+s\phi),\nabla\phi\rangle\,\mathrm{d}x\,\mathrm{d}s \\ &\quad - \lambda_1\int_0^1\int_\Omega |\varphi_1+s\phi|^{p-2}(\varphi_1+s\phi)\phi\,\mathrm{d}x\,\mathrm{d}s \\ &\equiv \mathcal{Q}_\phi(\phi,\phi),\end{aligned}$$

associated with (1.1) where $f\equiv 0$, computed near φ_1 and in an arbitrary direction $\phi\in W_0^{1,p}(\Omega)$, is given by the symmetric bilinear form $\mathcal{Q}_\phi$ on the Cartesian product $[W_0^{1,p}(\Omega)]^2$ defined as follows, using the matrix abbreviation (2.3):

$$\begin{aligned}\mathcal{Q}_\phi(v,w) &\stackrel{\text{def}}{=} \int_\Omega \Big\langle \Big[\int_0^1 \mathbf{A}(\nabla(\varphi_1+s\phi))(1-s)\,\mathrm{d}s\Big]\nabla v,\nabla w\Big\rangle\,\mathrm{d}x \\ &\quad - \lambda_1(p-1)\int_\Omega\Big[\int_0^1 |\varphi_1+s\phi|^{p-2}(1-s)\,\mathrm{d}s\Big]vw\,\mathrm{d}x\end{aligned}$$

for $v,w\in W_0^{1,p}(\Omega)$. In particular, one has

$$2\cdot\mathcal{Q}_0(v,v) = \int_\Omega \langle \mathbf{A}_{\varphi_1}\nabla v,\nabla v\rangle\,\mathrm{d}x - \lambda_1(p-1)\int_\Omega \varphi_1^{p-2}v^2\,\mathrm{d}x.$$

The quadratic form $\mathcal{Q}_0$ is positive semidefinite, i.e., $\mathcal{Q}_0(v,v)\geq 0$ for all $v\in W_0^{1,p}(\Omega)$. Furthermore, $\mathcal{Q}_0$ is closable in $L^2(\Omega)$, the domain of its closure being equal to $\mathcal{D}_{\varphi_1}$. Finally, one has $\mathcal{Q}_0(u,u)=0$ if and only if $u=\kappa\varphi_1$ for some $\kappa\in\mathbb{R}$, due to $\operatorname{meas} U'=0$ (see [23, Prop. 4.4] and [21]).

2.2. The singular case $1<p<2$. The Hilbert space $\mathcal{D}_{\varphi_1}$, endowed with the norm (2.4) for $p>2$, needs to be redefined for $1<p<2$ as follows. We define $v\in\mathcal{D}_{\varphi_1}$ if and only if $v\in W_0^{1,2}(\Omega)$ and

$$\|v\|_{\mathcal{D}_{\varphi_1}} \stackrel{\text{def}}{=} \Big(\int_U |\nabla\varphi_1|^{p-2}|\nabla v|^2\,\mathrm{d}x\Big)^{1/2} < \infty. \tag{2.5}$$

Consequently, $\mathcal{D}_{\varphi_1}$ endowed with the norm $\|\cdot\|_{\mathcal{D}_{\varphi_1}}$ is continuously embedded into $W_0^{1,2}(\Omega)$.

A good way of understanding the definition of $\mathcal{D}_{\varphi_1}$ is to first identify $W_0^{1,2}(\Omega)$ with a closed linear subspace of the Cartesian product $[L^2(\Omega)]^{N+1}$ by means of the isometric isomorphism $v \mapsto (v, \nabla v)$, and then define $v \in \mathcal{D}_{\varphi_1}$ by requiring $\nabla v(x) = \mathbf{0}$ for $x \in U'$, together with (2.5) in $U = \Omega \setminus U'$.

It follows from $\operatorname{meas} U^{'} = 0$ that $\mathcal{D}_{\varphi_1}$ is dense in $L^2(\Omega)$ (see [23]).

Let us define another norm on $W_0^{1,2}(\Omega)$ by

$$\|v\|_{\mathcal{H}_{\varphi_1}} \stackrel{\text{def}}{=} \left(\int_\Omega \varphi_1^{p-2} v^2 \, \mathrm{d}x \right)^{1/2} \quad \text{for } v \in W_0^{1,2}(\Omega),$$

and denote by $\mathcal{H}_{\varphi_1}$ the completion of $W_0^{1,2}(\Omega)$ with respect to this norm. The Sobolev space $W_0^{1,2}(\Omega)$ is compactly embedded in the Hilbert space $\mathcal{H}_{\varphi_1}$, by Hardy's inequality; see [23, Lemma 8.2] and Lemma A.2 below. Notice that $\mathcal{H}_{\varphi_1} = L^2(\Omega; d(x)^{p-2} \, \mathrm{d}x)$ both, algebraically and topologically, where the function

$$d(x) \stackrel{\text{def}}{=} \operatorname{dist}(x, \partial\Omega) = \inf_{x_0 \in \partial\Omega} |x - x_0|, \quad x \in \overline{\Omega},$$

denotes the distance from x to $\partial\Omega$. It is easy to see that

$$\mathcal{H}'_{\varphi_1} = L^2(\Omega; d(x)^{2-p} \, \mathrm{d}x)$$

is the dual space of $\mathcal{H}_{\varphi_1}$ when endowed with the dual norm

$$\|w\|_{\mathcal{H}'_{\varphi_1}} \stackrel{\text{def}}{=} \left(\int_\Omega \varphi_1^{2-p} w^2 \, \mathrm{d}x \right)^{1/2} \quad \text{for } w \in L^2(\Omega; d(x)^{2-p} \, \mathrm{d}x).$$

3. A global bifurcation result

Under a *solution* of (1.1) we understand a pair (λ, u) of $\lambda \in \mathbb{R}$ and $u \in W_0^{1,p}(\Omega)$ satisfying the integral equality

$$\int_\Omega |\nabla u|^{p-2} \langle \nabla u, \nabla \phi \rangle \, \mathrm{d}x - \lambda \int_\Omega |u|^{p-2} u \phi \, \mathrm{d}x = \int_\Omega f \phi \, \mathrm{d}x \tag{3.1}$$

for every $\phi \in W_0^{1,p}(\Omega)$.

Let $X = W_0^{1,p}(\Omega)$ and let X' stand for its dual space, i.e., $X' = W^{-1,p'}(\Omega)$. Then (3.1) is equivalent to the abstract operator equation

$$\mathcal{I}(u) - \lambda \mathcal{S}(u) = F, \tag{3.2}$$

where $\mathcal{I}, \mathcal{S} \colon X \to X'$ and $F \in X'$ are defined as follows, for any $u, \phi \in X$:

$$\begin{aligned} (\mathcal{I}(u), \phi)_X &= \int_\Omega |\nabla u|^{p-2} \langle \nabla u, \nabla \phi \rangle \, \mathrm{d}x, \\ (\mathcal{S}(u), \phi)_X &= \int_\Omega |u|^{p-2} u \phi \, \mathrm{d}x, \\ (F, \phi)_X &= \int_\Omega f \phi \, \mathrm{d}x. \end{aligned}$$

Here, $(\,\cdot\,,\cdot\,)_X$ denotes the duality pairing between X and X'.

It is proved in [11, Chapter 5] that the operator $\mathcal{I} - \lambda\mathcal{S}$ satisfies condition $\alpha(X)$ from [22] (which is nothing else but condition (S_+) from [6]) and so its (topological) degree can be defined.

Definition 3.1. Let $\mu_0 \in \mathbb{R}$. We say that (μ_0, ∞) is an *asymptotic bifurcation point* for (3.2) if there exists a sequence of pairs $\{(\mu_n, u_n)\}_{n=1}^\infty \subset \mathbb{R} \times X$ such that (3.2) holds with $(\lambda, u) = (\mu_n, u_n)$, $n = 1, 2, 3, \ldots$, and $(\mu_n, \|u_n\|_X) \to (\mu_0, \infty)$.

For $u \in X$, $u \neq 0$, set $v = u/\|u\|_X^2$. Then (3.2) is equivalent to

$$\mathcal{I}(v) - \lambda\mathcal{S}(v) = \|v\|_X^{2(p-1)} F,$$

and so the term

$$\mathcal{G}(\lambda, v) \stackrel{\text{def}}{=} \begin{cases} \|v\|_X^{2(p-1)} F & \text{if } v \neq 0, \\ 0 & \text{if } v = 0 \end{cases}$$

for $\lambda \in \mathbb{R}$, represents a compact perturbation "of higher order" in the variable v in the equation

$$\mathcal{I}(v) - \lambda\mathcal{S}(v) = \mathcal{G}(\lambda, v). \tag{3.3}$$

It follows immediately from this transformation that the pair (μ_0, ∞) is an asymptotic bifurcation point for (3.2) if and only if $(\mu_0, 0)$ is a bifurcation point (from the set of trivial solutions) for (3.3). For $\mathcal{C} \subset \mathbb{R} \times X$ we define (the set) $\widetilde{\mathbb{C}}$ to be the closure in $\mathbb{R} \times X$ of the set of all pairs $(\mu, v) \in \mathbb{R} \times X$ such that $v \neq 0$ and $(\mu, v/\|v\|X^2) \in \mathbb{C}$.

In [11, Theorem 14.18], it was proved that $(\lambda_1, 0)$ is a bifurcation point for (3.3). Let us reformulate this result in terms of problem (3.2).

Proposition 3.2. *Let $F \in X'$, $F \neq 0$. Then the pair (λ_1, ∞) is an asymptotic bifurcation point for* (3.2). *Moreover, there exists a maximal (in the ordering by set inclusion) closed set $\mathcal{C} \subset \mathbb{R} \times X$, such that $\widetilde{\mathcal{C}}$ is connected in $\mathbb{R} \times X$ and the following properties hold:*

(i) *there exists a sequence $\{(\mu_n, u_n)\}_{n=1}^{\infty} \subset \mathbb{C}$ such that $(\mu_n, \|u_n\|_X) \to (\lambda_1, \infty)$;*

(ii) *either $\mathcal{C}$ is unbounded in the λ-direction, or else there exists an eigenvalue λ_0 of $-\Delta_p$ such that $\lambda_0 > \lambda_1$ and there is a sequence $\{(\mu_n, u_n)\}_{m=1}^{\infty} \subset \mathcal{C}$ satisfying $(\mu_n, \|u_n\|_X) \to (\lambda_0, \infty)$.*

Remark 3.3. The assumption $F \neq 0$ (which corresponds to $f \not\equiv 0$ in (1.1)) implies that (3.2) cannot have the trivial solution $u = 0$. Consequently, $\mathcal{C}$ contains no sequence of pairs (μ_k, u_k) with $(\mu_k, \|u_k\|_X) \to (\hat{\mu}, 0)$. Hence, the statement of Proposition 3.2 follows directly from [11, Theorem 14.18] using the transformation $u \mapsto v = u/\|u\|_X^2$.

4. A priori asymptotic estimates

In this section we will establish an asymptotic estimate that plays the key role in the study of the structure of the solution set to (1.1). We assume $1 < p < \infty$, $p \neq 2$, throughout the entire section. From now on, we denote by λ_2 ($\lambda_2 > \lambda_1$) the second eigenvalue of the positive Dirichlet p-Laplacian $-\Delta_p$. We use only the well-known fact from [4] that there is no eigenvalue of $-\Delta_p$ in the open interval (λ_1, λ_2), by a variational characterization of λ_2. The following theorem is the key to the results of this lecture. We skip the proofs and indicate only the main steps. The details can be found in the paper [13].

Theorem 4.1. *Let $\{\mu_n\}_{n=1}^{\infty} \subset \mathbb{R}$, $\{f_n\}_{n=1}^{\infty} \subset L^\infty(\Omega)$, $\{u_n\}_{n=1}^{\infty} \subset W_0^{1,p}(\Omega)$ be sequences, and let $\delta > 0$ be such that*

(i) *$\lambda_1 + \mu_n < \lambda_2 - \delta$ for all $n \in \mathbb{N}$;*

(ii) *$f_n \overset{*}{\rightharpoonup} f$ weakly-star in $L^\infty(\Omega)$;*

(iii) *$\|u_n\|_{W_0^{1,p}(\Omega)} \to \infty$ as $n \to \infty$;*

(iv) *in addition, assume that for all $n \in \mathbb{N}$ and $\phi \in W_0^{1,p}(\Omega)$,*

$$\int_\Omega |\nabla u_n|^{p-2} \langle \nabla u_n, \nabla \phi \rangle \, \mathrm{d}x = (\lambda_1 + \mu_n) \int_\Omega |u_n|^{p-2} u_n \phi \, \mathrm{d}x + \int_\Omega f_n \phi \, \mathrm{d}x. \quad (4.1)$$

Then $\mu_n \to 0$ and, writing $u_n = t_n^{-1}(\varphi_1 + v_n^\top)$ with $t_n \in \mathbb{R}$, $t_n \neq 0$, and $v_n^\top \in W_0^{1,p}(\Omega)^\top$, we have $t_n \to 0$, $|t_n|^{-p} t_n v_n^\top \to V^\top$ strongly in $\mathcal{D}_{\varphi_1}$ if $p > 2$

and in $W_0^{1,2}(\Omega)$ *if* $1 < p < 2$, *and*

$$\begin{aligned} \mu_n = &-|t_n|^{p-2} t_n \int_\Omega f_n \varphi_1 \,\mathrm{d}x + (p-2)|t_n|^{2(p-1)} \mathcal{Q}_0(V^\top, V^\top) \\ &+ (p-1)|t_n|^{2(p-1)} \Big(\int_\Omega f \varphi_1 \,\mathrm{d}x \Big) \Big(\int_\Omega \varphi_1^{p-1} V^\top \,\mathrm{d}x \Big) + o(|t_n|^{2(p-1)}). \end{aligned} \tag{4.2}$$

In particular, if $\int_\Omega f_n \varphi_1 \,\mathrm{d}x = 0$ *for all* $n \in \mathbb{N}$, *then*

$$\mu_n = (p-2)|t_n|^{2(p-1)} \mathcal{Q}_0(V^\top, V^\top) + o(|t_n|^{2(p-1)}).$$

Moreover, $V^\top \in \mathcal{D}_{\varphi_1} \cap \{\varphi_1\}^{\perp, L^2}$ *is the (unique) solution to*

$$2 \cdot \mathcal{Q}_0(V^\top, \phi) = \int_\Omega f^\dagger \phi \,\mathrm{d}x \quad \text{for all } \phi \in \mathcal{D}_{\varphi_1}, \tag{4.3}$$

where we have denoted

$$2 \cdot \mathcal{Q}_0(V^\top, \phi) = \int_\Omega \langle \mathbf{A}_{\varphi_1} \nabla V^\top, \nabla \phi \rangle \,\mathrm{d}x - \lambda_1 (p-1) \int_\Omega \varphi_1^{p-2} V^\top \phi \,\mathrm{d}x$$

and $f^\dagger = f - (\int_\Omega f \varphi_1 \,\mathrm{d}x) \varphi_1^{p-1}$.

Remark 4.2. The linear equation (4.3) represents the weak form of the "limiting" Dirichlet boundary value problem for the limit function $|t_n|^{-p} t_n v_n^\top \to V^\top$ in the approximation scheme with $u_n = t_n^{-1}(\varphi_1 + v_n^\top)$. This is a resonant problem to which a standard version of the Fredholm alternative for a selfadjoint linear operator in a Hilbert space applies. More precisely, given a function $f \in L^2(\Omega)$, a weak solution $V \in \mathcal{D}_{\varphi_1}$ to the equation

$$2 \cdot \mathcal{Q}_0(V, \phi) = \int_\Omega f \phi \,\mathrm{d}x \quad \text{for all } \phi \in \mathcal{D}_{\varphi_1}, \tag{4.4}$$

exists in $\mathcal{D}_{\varphi_1}$ if and only if $\int_\Omega f \varphi_1 \,\mathrm{d}x = 0$. Such a solution is always unique under the orthogonality condition $\int_\Omega V \varphi_1 \,\mathrm{d}x = 0$.

Consequently, given $f^\top \in \{\varphi_1\}^{\perp, L^2} \subset L^2(\Omega)$, we denote by

$$V^\top \equiv V^\top(f^\top) \in \mathcal{D}_{\varphi_1} \cap \{\varphi_1\}^{\perp, L^2}$$

the unique weak solution to problem (4.4) with $f^\top$ in place of f. It is easy to see that $f^\top \mapsto V^\top : \{\varphi_1\}^{\perp, L^2} \to \mathcal{D}_{\varphi_1}$ is a compact linear mapping. Clearly, this mapping is linear and bounded. To show that it is compact, let $\{f_n\}_{n=1}^\infty \subset \{\varphi_1\}^{\perp,L^2}$ be any weakly convergent sequence, $f_n \rightharpoonup f$ in $L^2(\Omega)$ as $n \to \infty$. Hence, $\{V^\top(f_n)\}_{n=1}^\infty$ is a weakly convergent sequence, $V^\top(f_n) \rightharpoonup V^\top(f)$ in $\mathcal{D}_{\varphi_1}$ as $n \to \infty$. The embedding $\mathcal{D}_{\varphi_1} \hookrightarrow L^2(\Omega)$ being compact, we have also $V^\top(f_n) \to V^\top(f)$ strongly in $L^2(\Omega)$, and

$$\int_\Omega f_n \phi \,\mathrm{d}x \longrightarrow \int_\Omega f\phi \,\mathrm{d}x$$

uniformly for $\phi \in \mathcal{D}_{\varphi_1}$ with $\|\phi\|_{\mathcal{D}_{\varphi_1}} \le 1$. Inserting these results into equation (4.4) we deduce

$$\int_\Omega \langle \mathbf{A}_{\varphi_1} \nabla V^\top(f_n), \nabla\phi\rangle \,\mathrm{d}x \longrightarrow \int_\Omega \langle \mathbf{A}_{\varphi_1} \nabla V^\top(f), \nabla\phi\rangle \,\mathrm{d}x$$

uniformly for $\phi \in \mathcal{D}_{\varphi_1}$ with $\|\phi\|_{\mathcal{D}_{\varphi_1}} \le 1$. We have shown $V^\top(f_n) \to V^\top(f)$ strongly in $\mathcal{D}_{\varphi_1}$, and thus the desired compactness.

Main steps of the proof of Theorem 4.1.

Step 1. Hypothesis (iii) is equivalent to (iii'): $\|u_n\|_{L^\infty(\Omega)} \to \infty$ as $n \to \infty$.

Step 2. We prove that μ_n is bounded also below.

Step 3. We prove that $\mu_n \to 0$.

Step 4. Boundedness of $v_n^\top/|t_n|^{p-1}$ in $L^2(\Omega)$ if $p > 2$, in $\mathcal{H}_{\varphi_1}$ if $1 < p < 2$, and of $\mu_n/|t_n|^{p-1}$ in $\mathbb{R}$.

Step 5. We prove that $v_n^\top/(|t_n|^{p-2}t_n) \to V^\top$ strongly in $\mathcal{D}_{\varphi_1}$ if $p > 2$ and in $W_0^{1,2}(\Omega)$ if $1 < p < 2$.

Step 6. First order asymptotic estimate for μ_n.

Step 7. Second order asymptotic estimate for μ_n. □

5. Refined global bifurcation results

In this section we make use of the asymptotic estimate from Section 4 in order to extend the results obtained in Section 3. We use the notation introduced in Section 3. The following nonexistence result is a consequence of Theorem 4.1.

Proposition 5.1. *Let $f \in L^\infty(\Omega)$, $f \not\equiv 0$. Then there exists a constant $R > 0$ such that every weak solution $u \in W_0^{1,p}(\Omega)$ of the problem*

$$\begin{aligned} -\Delta_p u - \lambda_1 |u|^{p-2}u &= f(x) \quad && \text{in } \Omega, \\ u &= 0 && \text{on } \partial\Omega \end{aligned} \tag{5.1}$$

satisfies the a priori bound $\|u\|_{W_0^{1,p}(\Omega)} \le R$.

Proof. On the contrary, assume that, for each $n \in \mathbb{N}$, there exists $u_n \in W_0^{1,p}(\Omega)$ such that $\|u_n\|_{W_0^{1,p}(\Omega)} \geq n$ and u_n verifies (5.1). Then also

$$\int_\Omega |\nabla u_n|^p \,\mathrm{d}x - \lambda_1 \int_\Omega |u_n|^p \,\mathrm{d}x = \int_\Omega f u_n \,\mathrm{d}x.$$

Denoting $v_n = u_n / \|u_n\|_{W_0^{1,p}(\Omega)}$, we get $\|v_n\|_{W_0^{1,p}(\Omega)} = 1$ and

$$\int_\Omega |\nabla v_n|^p \,\mathrm{d}x - \lambda_1 \int_\Omega |v_n|^p \,\mathrm{d}x = \frac{1}{\|u_n\|^{p-1}_{W_0^{1,p}(\Omega)}} \int_\Omega f v_n \,\mathrm{d}x. \tag{5.2}$$

Passing to a subsequence if necessary, we may assume that $v_n \rightharpoonup v_0$ weakly in $W_0^{1,p}(\Omega)$ and $v_n \to v_0$ strongly in $L^p(\Omega)$. Then it follows from (5.2) and $\|u_n\|_{W_0^{1,p}(\Omega)} \to \infty$ that

$$1 = \|v_n\|^p_{W_0^{1,p}(\Omega)} \to \lambda_1 \int_\Omega |v_0|^p \,\mathrm{d}x \tag{5.3}$$

and

$$\int_\Omega |\nabla v_0|^p \,\mathrm{d}x - \lambda_1 \int_\Omega |v_0|^p \,\mathrm{d}x \leq 0. \tag{5.4}$$

The variational characterization of λ_1, (5.3) and (5.4) imply that $v_0 = k\varphi_1$ for some $k \in \mathbb{R} \setminus \{0\}$. Assume $k > 0$ (the case $k < 0$ is analogous). We apply Theorem 4.1 with $f_n = f$ and $\mu_n = 0$ for every $n \in \mathbb{N}$ large enough; hence $u_n = t_n^{-1}(\varphi_1 + v_n^\top)$, $t_n > 0$. If $\int_\Omega f\varphi_1 \,\mathrm{d}x = 0$ then

$$0 = (p-2) t_n^{2(p-1)} \cdot \mathcal{Q}_0(V^\top, V^\top) + o(t_n^{2(p-1)}),$$

which forces $\mathcal{Q}_0(V^\top, V^\top) = 0$, a contradiction to $\mathcal{Q}_0(V^\top, V^\top) > 0$. If $\int_\Omega f\varphi_1 \,\mathrm{d}x = a \neq 0$ then $0 = -a t_n^{p-1} + o(t_n^{p-1})$, a contradiction again. □

Recall that X and X' together with $\mathcal{I}(u)$, $\mathcal{S}(u)$ and F have been specified in Section 3.

Theorem 5.2. *Let $F \in X'$, $F \neq 0$. Then there is a pair of maximal closed sets $\mathcal{C}^+, \mathcal{C}^- \subset \mathbb{R} \times X$ of solutions of* (3.2) *such that both sets $\widetilde{\mathbb{C}}^+$ and $\widetilde{\mathbb{C}}^-$ are connected in $\mathbb{R} \times X$, where $\widetilde{\mathbb{C}}^\pm$ are the closures in $\mathbb{R} \times X$ of the respective sets of all pairs $(\mu, v) \in \mathbb{R} \times X$ such that $v \neq 0$ and $(\mu, v/\|v\|_X^2) \in \mathcal{C}^\pm$, and the following properties hold:*

(a) *there exist sequences of pairs $(\mu_n, u_n) \in \mathcal{C}^+$ and $(\mu_n', u_n') \in \mathbb{C}^-$ such that $\mu_n \to \lambda_1$, $\mu_n' \to \lambda_1$, $\|u_n\|_X \to \infty$ and $\|u_n'\|_X \to \infty$, together with $u_n/\|u_n\|_X \to \varphi_1/\|\varphi_1\|_X$ and $u_n'/\|u_n'\|_X \to -\varphi_1/\|\varphi_1\|_X$ strongly in X;*

(b) *either both $\mathcal{C}^+$ and $\mathcal{C}^-$ are unbounded in the λ-direction, or else $\widetilde{\mathbb{C}}^+ \cap \widetilde{\mathbb{C}}^-$ contains a point other than $\{(\lambda_1, 0)\}$.*

Proof. After the transformation $v_n = u_n/\|u_n\|_X^2$, the statement of our Theorem 5.2 follows directly from Proposition 5.1 and [11, Theorem 14.20]. $\square$

Remark 5.3. Let us point out that it follows from Theorem 5.2 combined with regularity results [19] that the convergence above $u_n/\|u_n\|_X \to \varphi_1/\|\varphi_1\|_X$, and $u_n'/\|u_n'\|_X \to -\varphi_1/\|\varphi_1\|_X$ as well, occurs strongly not only in $X = W_0^{1,p}(\Omega)$ but even in $C^{1,\beta'}(\overline{\Omega})$. Thus, $\|u_n\|_X \to \infty$ is equivalent to $\|u_n\|_{L^\infty(\Omega)} \to \infty$, and also to $\|u_n\|_{C^{1,\beta'}(\overline{\Omega})} \to \infty$, for $(\lambda_n, u_n) \in \mathcal{C}^+ \cup \mathcal{C}^-$.

Remark 5.4. Our aim is to study the local behaviour of the bifurcation branches $\mathcal{C}^\pm$ near λ_1. For this reason we restrict our attention to $\lambda \in (-\infty, \Lambda)$ with some $\Lambda \in (\lambda_1, \lambda_2)$. Recall that λ_2 is the second eigenvalue of the negative Dirichlet p-Laplacian $-\Delta_p$ (cf. Theorem 4.1).

5.1. Case $\int_\Omega f\varphi_1 \,\mathrm{d}x \neq 0$.

We will now establish a priori bounds that allow us to detect whether λ belongs to the left or the right neighborhood of λ_1 provided $(\lambda, u) \in \mathcal{C}^\pm$ and the norm $\|u\|_{L^\infty(\Omega)}$ is large enough.

Theorem 5.5. *Let $\Lambda \in (\lambda_1, \lambda_2)$. For every $f \in L^\infty(\Omega)$ with $\int_\Omega f\varphi_1 \,\mathrm{d}x \neq 0$, there exists a constant $M > 0$ such that the following statements and implications hold.*

(i) *If $\int_\Omega f\varphi_1 \,\mathrm{d}x < 0$ then every solution (λ, u) to* (1.1) *satisfies*
  - (a) *$u(\hat{x}) < 0$ for some $\hat{x} \in \Omega$ and $\lambda_1 \le \lambda \le \Lambda \Longrightarrow \|u\|_{L^\infty(\Omega)} \le M$;*
  - (b) *$u(\hat{x}) > 0$ for some $\hat{x} \in \Omega$ and $\lambda \le \lambda_1 \Longrightarrow \|u\|_{L^\infty(\Omega)} \le M$.*

(ii) *If $\int_\Omega f\varphi_1 \,\mathrm{d}x > 0$ then every solution (λ, u) to* (1.1) *satisfies*
  - (a) *$u(\hat{x}) < 0$ for some $\hat{x} \in \Omega$ and $\lambda \le \lambda_1 \Longrightarrow \|u\|_{L^\infty(\Omega)} \le M$;*
  - (b) *$u(\hat{x}) > 0$ for some $\hat{x} \in \Omega$ and $\lambda_1 \le \lambda \le \Lambda \Longrightarrow \|u\|_{L^\infty(\Omega)} \le M$.*

Proof. Case (i) (a). Assume by contradiction that there exists $f \in L^\infty(\Omega)$, $\int_\Omega f\varphi_1 \,\mathrm{d}x < 0$, such that for each $n \in \mathbb{N}$ there exist $\lambda = \lambda_1 + \mu_n \ge \lambda_1$ and $u_n \in W_0^{1,p}(\Omega)$ with $u_n(\hat{x}_n) < 0$ for some $\hat{x}_n \in \Omega$, such that $\|u_n\|_{L^\infty(\Omega)} \ge n$ and (λ, u_n) is a solution of (4.1) with $\lambda = \lambda_1 + \mu_n$. Let us write $u_n = t_n^{-1}(\varphi_1 + v_n^\top)$ with some $t_n \in \mathbb{R} \setminus \{0\}$. Using the proof of Theorem 4.1, Step 4, we have $t_n \to 0$ as $n \to \infty$. Moreover, if $n \in \mathbb{N}$ is large enough, then also $t_n < 0$, because $u_n(\hat{x}_n) < 0$ for some $\hat{x}_n \in \Omega$. Then, by Theorem 4.1, we find that $\mu_n \to 0$ and (4.2) yields

$$0 \le \mu_n = -t_n|t_n|^{p-2} \int_\Omega f\varphi_1 \,\mathrm{d}x + o(t_n^{p-1})$$

for all $n \in \mathbb{N}$ large enough. Since $\int_\Omega f\varphi_1 \,\mathrm{d}x < 0$ and $t_n < 0$, the last inequality is absurd and (i) (a) holds.

Cases (i) (b), (ii) (a), (ii) (b) are proved analogously. $\square$

Corollary 5.6. *Let $\Lambda \in (\lambda_1, \lambda_2)$ and $f \in L^\infty(\Omega)$, $\int_\Omega f\varphi_1 \,\mathrm{d}x \neq 0$, be given. Moreover, let $\mathcal{C}^\pm$ be as in Theorem 5.2. Then there exists a constant $\widehat{M} \geq M$ (M being the constant from Theorem 5.5) such that, for every $u \in W_0^{1,p}(\Omega)$ with $\|u\|_{L^\infty(\Omega)} > \widehat{M}$ and written as $u = t^{-1}(\varphi_1 + v^\top)$, we have:*

(i) (a) $\int_\Omega f\varphi_1 \,\mathrm{d}x < 0$, $(\lambda, u) \in \mathbb{C}^- \cap ((-\infty, \Lambda] \times W_0^{1,p}(\Omega))$ *and* $t < 0$ *implies* $u < 0$ *in* Ω *and* $\lambda < \lambda_1$.

(b) $\int_\Omega f\varphi_1 \,\mathrm{d}x < 0$, $(\lambda, u) \in \mathbb{C}^+ \cap ((-\infty, \Lambda] \times W_0^{1,p}(\Omega))$ *and* $t > 0$ *implies* $u > 0$ *in* Ω *and* $\lambda > \lambda_1$.

(ii) (a) $\int_\Omega f\varphi_1 \,\mathrm{d}x > 0$, $(\lambda, u) \in \mathbb{C}^- \cap ((-\infty, \Lambda] \times W_0^{1,p}(\Omega))$ *and* $t < 0$ *implies* $u < 0$ *in* Ω *and* $\lambda > \lambda_1$.

(b) $\int_\Omega f\varphi_1 \,\mathrm{d}x > 0$, $(\lambda, u) \in \mathbb{C}^+ \cap ((-\infty, \Lambda] \times W_0^{1,p}(\Omega))$ *and* $t > 0$ *implies* $u > 0$ *in* Ω *and* $\lambda < \lambda_1$.

Proof. Let us prove (i) (a), the other cases being similar. There is $\widehat{M} > 0$ such that for $(\lambda, u) \in \mathcal{C}^- \cap ((-\infty, \Lambda] \times W_0^{1,p}(\Omega))$ and $t < 0$ we have $u < 0$ in Ω. This follows from Remark 5.3. Taking $\widehat{M} \geq M$ (M from Theorem 5.5), we must have $\lambda < \lambda_1$ owing to Theorem 5.5, Case (i) (a). $\square$

A sketch of the bifurcation diagram that corresponds to Corollary 5.6 is depicted in Figure 3.

5.2. Case $\int_\Omega f\varphi_1 \,\mathrm{d}x = 0$. In this subsection we will distinguish between the cases $1 < p < 2$ and $p > 2$.

Theorem 5.7. *Let $\Lambda \in (\lambda_1, \lambda_2)$. For every $f \in L^\infty(\Omega)$, $\int_\Omega f\varphi_1 \,\mathrm{d}x = 0$, $f \not\equiv 0$, there exists a constant $M > 0$ such that every solution (λ, u) to (1.1) satisfies:*

(i) *if $1 < p < 2$, then $\lambda_1 \leq \lambda \leq \Lambda$ implies $\|u\|_{L^\infty(\Omega)} \leq M$;*

(ii) *if $p > 2$, then $\lambda \leq \lambda_1$ implies $\|u\|_{L^\infty(\Omega)} \leq M$.*

Proof. Part (i). Assume on the contrary that there exists $f \in L^\infty(\Omega)$, $\int_\Omega f\varphi_1 \,\mathrm{d}x = 0$, $f \not\equiv 0$, such that for any $n \in \mathbb{N}$ there exist $\lambda_n \geq \lambda_1$ and $u_n \in W_0^{1,p}(\Omega)$, $\|u_n\|_{W_0^{1,p}(\Omega)} \geq n$, such that (λ_n, u_n) is a solution to (4.1). Let us write $\lambda_n = \lambda_1 + \mu_n$ and $u_n = t_n^{-1}(\varphi_1 + v_n^\top)$ with $t_n \to 0$. Then, by Theorem 4.1, we find that $\mu_n \to 0$, and so (4.2) yields

$$0 \leq \mu_n = (p-2)|t_n|^{2(p-1)}\mathcal{Q}_0(V^\top, V^\top) + o(|t_n|^{2(p-1)})$$

for all $n \in \mathbb{N}$. Since $1 < p < 2$ and $\mathcal{Q}_0(V^\top, V^\top) > 0$ for all $f^\top \in L^\infty(\Omega)$ with $\int_\Omega f^\top \varphi_1 \,\mathrm{d}x = 0$ and $f^\top \not\equiv 0$ (see Remark 4.2), the last inequality is absurd. Hence, Part (i) is proved. Part (ii) is proved analogously. $\square$

Corollary 5.8. *Let $\Lambda \in (\lambda_1, \lambda_2)$ and $f \in L^\infty(\Omega)$, $\int_\Omega f\varphi_1 \,\mathrm{d}x = 0$, $f \not\equiv 0$, be given. Let $\mathcal{C}^\pm$ be as in Theorem 5.2. Then there exists a constant $\widehat{M} \geq M$ (M being the constant from Theorem 5.5) such that, for every $u \in W_0^{1,p}(\Omega)$ with $\|u\|_{L^\infty(\Omega)} > \widehat{M}$ and written as $u = t^{-1}(\varphi_1 + v^\top)$, we have:*

- (i) $1 < p < 2$, $(\lambda, u) \in (\mathcal{C}^- \cup \mathcal{C}^+) \cap ((-\infty, \Lambda] \times W_0^{1,p}(\Omega))$ *and* $t > 0$ *($t < 0$, respectively) implies* $u > 0$ *($u < 0$) in* Ω *and* $\lambda < \lambda_1$*;*
- (ii) $p > 2$, $(\lambda, u) \in (\mathcal{C}^- \cup \mathcal{C}^+) \cap ((-\infty, \Lambda] \times W_0^{1,p}(\Omega))$ *and* $t > 0$ *($t < 0$, respectively) implies* $u > 0$ *($u < 0$) in* Ω *and* $\lambda > \lambda_1$*.*

Proof. Let us prove Part (i), proof of Part (ii) being similar. There is $\widehat{M} > 0$ such that for $(\lambda, u) \in (\mathcal{C}^- \cup \mathcal{C}^+) \cap ((-\infty, \Lambda] \times W_0^{1,p}(\Omega))$, u does not change sign in Ω (i.e., u is either positive or negative in Ω). This follows from Remark 5.3. Taking $\widehat{M} \geq M$ (M being the constant from Theorem 5.7), we must have $\lambda > \lambda_1$ due to Theorem 5.7. Thus, Part (i) is proved. $\square$

A sketch of the bifurcation diagram which corresponds to Corollary 5.8 is depicted in Figure 2.

5.3. Case $\int_\Omega f\varphi_1 \,\mathrm{d}x \neq 0$ revisited. In this subsection we consider perturbations f of functions $f^\top \in L^\infty(\Omega)$, $\int_\Omega f^\top \varphi_1 \,\mathrm{d}x = 0$, of the form $f = f^\top + a\varphi_1$ with $a \in \mathbb{R}$ small enough. Therefore, we need to distinguish between the cases $1 < p < 2$ and $p > 2$ again.

Theorem 5.9. *Let $f^\top \in L^\infty(\Omega)$ be fixed, $f^\top \not\equiv 0$ and $\int_\Omega f^\top \varphi_1 \,\mathrm{d}x = 0$. Given $a \in \mathbb{R}$, let $\mathcal{C}_a^\pm$ denote continua of solutions $(\lambda, u) \in \mathbb{R} \times X$ (in the sense of Theorem 5.2) to*

$$\begin{aligned} -\Delta_p u - \lambda |u|^{p-2} u &= f^\top + a\varphi_1 \quad \text{in } \Omega, \\ u &= 0 \qquad\qquad\quad \text{on } \partial\Omega. \end{aligned} \tag{5.5}$$

Let every solution u of (5.5) be written in the form $u = t^{-1}(\varphi_1 + v^\top)$, $t \in \mathbb{R}$, $v^\top \in C^{1,\beta'}(\overline{\Omega})$. Then there exist $\varepsilon > 0$, $\underline{a} = \underline{a}(f^\top, p)$ and $\overline{a} = \overline{a}(f^\top, p)$, such that $\underline{a} < 0 < \overline{a}$ and for $a \in (\underline{a}, 0) \cup (0, \overline{a})$ we have $u > 0$ in Ω if $t \in (0, \varepsilon)$ and $u < 0$ if $t \in (-\varepsilon, 0)$. Furthermore,

- (i) *if* $1 < p < 2$ *then*
  - (a) *for any* $a \in (\underline{a}, 0)$ *there exists* $t_0 \in (0, \varepsilon)$ *such that* $u = t_0^{-1}(\varphi_1 + v^\top)$ *and* $(\lambda, u) \in \mathcal{C}_a^+$ *imply* $\lambda < \lambda_1$*;*
  - (b) *for any* $a \in (0, \overline{a})$ *there exists* $t_0 \in (-\varepsilon, 0)$ *such that* $u = t_0^{-1}(\varphi_1 + v^\top)$ *and* $(\lambda, u) \in \mathcal{C}_a^-$ *imply* $\lambda < \lambda_1$*;*

(ii) *if* $p > 2$ *then*

(a) *for any* $a \in (\underline{a}, 0)$ *there exists* $t_0 \in (-\varepsilon, 0)$ *such that* $u = t_0^{-1}(\varphi_1 + v^\top)$ *and* $(\lambda, u) \in \mathcal{C}_a^-$ *imply* $\lambda > \lambda_1$;

(b) *for any* $a \in (0, \overline{a})$ *there exists* $t_0 \in (0, \varepsilon)$ *such that* $u = t_0^{-1}(\varphi_1 + v^\top)$ *and* $(\lambda, u) \in \mathcal{C}_a^+$ *imply* $\lambda > \lambda_1$.

Proof of Theorem 5.9. Let $a \in J$, where J is a bounded interval. According to Step 4 of the proof of Theorem 4.1, there exists $\varepsilon > 0$ small enough such that for any solution $u = t^{-1}(\varphi_1 + v^\top)$ of (5.5) with $t \in (0, \varepsilon)$ we have $\varphi_1(x) + v^\top(x) > 0$, $x \in \Omega$, i.e., $u > 0$ in Ω. Analogously we prove that $u < 0$ in Ω if $t \in (-\varepsilon, 0)$. Fix such a number $\varepsilon > 0$.

Let us prove Case (ii) (b). The proofs of the remaining three cases are analogous. We proceed via contradiction. Assume that there exist $a_n > 0$, $a_n \to 0$, and for any $t \in (0, \varepsilon)$, $u = t^{-1}(\varphi_1 + v^\top)$, $(\lambda, u) \in \mathcal{C}_{a_n}^+$, we have $\lambda \le \lambda_1$. Recall that the set $\mathcal{C}_{a_n}^+$ is connected for any fixed $n \in \mathbb{N}$. Hence, taking n large enough, we can pick $t_n \in (0, \varepsilon)$, $t_n = a_n^{\frac{1}{3(p-1)}}$, and $\lambda_n = \lambda_1 + \mu_n$ with $\mu_n \le 0$, such that $u_n = t_n^{-1}(\varphi_1 + v_n^\top) > 0$ in Ω and $(\lambda_n, u_n) \in \mathcal{C}_{a_n}^+$. This choice guarantees that (4.1) holds with $f_n = f^\top + a_n\varphi_1 \overset{*}{\rightharpoonup} f = f^\top$ weakly-star in $L^\infty(\Omega)$. Applying Theorem 4.1 we obtain

$$\mu_n = -t_n^{p-1} a_n \|\varphi_1\|_{L^2(\Omega)}^2 + t_n^{2(p-1)}(p-2) \cdot \mathcal{Q}_0(V^\top, V^\top) + o(t_n^{2(p-1)}). \quad (5.6)$$

It follows from our choice of t_n that $-t_n^{p-1} a_n = -t_n^{4(p-1)}$, whence (5.6) is equivalent to

$$\mu_n = t_n^{2(p-1)}(p-2) \cdot \mathcal{Q}_0(V^\top, V^\top) + o(t_n^{2(p-1)}). \quad (5.7)$$

This is a contradiction because $\mu_n \le 0$ and the right-hand side of (5.7) is positive for n large enough due to $f^\top \not\equiv 0$. $\square$

Corollary 5.10. *Let* $f^\top \in L^\infty(\Omega)$, $\mathcal{C}_a^\pm$, ε, $\underline{a}$, $\overline{a}$ *and* a *be as in Theorem 5.9 above. Then*

(i) *there exists* $t_a \in (0, \varepsilon)$ *and* $u = t_a^{-1}(\varphi_1 + v^\top) > 0$ *in* Ω *such that* $(\lambda_1, u) \in \mathbb{C}_a^+$;

(ii) *there exists* $t_a \in (-\varepsilon, 0)$ *and* $u = t_a^{-1}(\varphi_1 + v^\top) < 0$ *in* Ω *such that* $(\lambda_1, u) \in \mathbb{C}_a^-$.

Proof. The proof follows from Theorems 5.7 and 5.9 combined with the fact that the sets $\widetilde{\mathbb{C}}_a^\pm$ are connected for each $a \in (\underline{a}, 0) \cup (0, \overline{a})$. $\square$

For a given $a \in (\underline{a}, 0) \cup (0, \overline{a})$, let us denote $\widehat{t}_a = \inf |t_a|$, where the infimum is taken over all t_a ($0 < |t_a| < \varepsilon$) associated to a via Corollary 5.10.

Corollary 5.11. *We have* $\lim_{a\to 0} \widehat{t}_a = 0$.

Proof. Let $1 < p < 2$. Assume that there exist $a_n < 0$, $a_n \to 0$ as $n \to \infty$, and $\eta > 0$ such that $\widehat{t}_{a_n} \geq \eta$. Then we can find a sequence $\{t_n\}_{n=1}^{\infty} \subset \mathbb{R} \setminus \{0\}$ such that $t_n^{-1} > (\widehat{t}_{a_n})^{-1}$, $\{t_n^{-1}\}_{n=1}^{\infty}$ is bounded, $u_n = t_n^{-1}(\varphi_1 + v_n^{\top}) > 0$ in Ω, $(\lambda_n, u_n) \in \mathcal{C}^+$ with $\lambda_n > \lambda_1$, and $\|u_n\|_{W_0^{1,p}(\Omega)}$ is bounded as well. At the same time, $\{t_n\}_{n=1}^{\infty}$ can be chosen such that $\|u_n\|_{W_0^{1,p}(\Omega)} \geq M + 1$, where M is associated to $f = f^{\top}$ via Theorem 5.7. Hence,

$$\begin{aligned} -\Delta_p u_n - \lambda_n |u_n|^{p-2} u_n &= f^{\top} + a_n \varphi_1 && \text{in } \Omega, \\ u_n &= 0 && \text{on } \partial\Omega. \end{aligned}$$

Applying a standard compactness argument and passing to a suitable subsequence we obtain $u_n \to u_0$ strongly in $W_0^{1,p}(\Omega)$, $u_0 > 0$ in Ω, $\|u_0\|_{W_0^{1,p}(\Omega)} \geq M + 1 > M$, together with $\lambda_n \to \lambda$, where $\lambda_1 \leq \lambda \leq \Lambda < \lambda_2$, and

$$\begin{aligned} -\Delta_p u_0 - \lambda |u_0|^{p-2} u_0 &= f^{\top} && \text{in } \Omega, \\ u_0 &= 0 && \text{on } \partial\Omega. \end{aligned}$$

This fact contradicts Theorem 5.7. The case $p > 2$ is treated in a similar way. $\square$

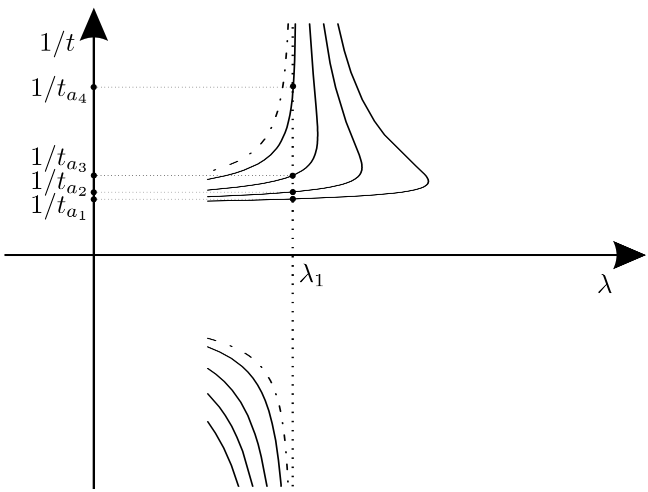

Figure 4: Illustration of Corollary 5.11 for $1 < p < 2$. The dash-dotted curves represent branches $\mathcal{C}^{\pm}$ for $a = 0$. One can see how the transcritical bifurcation for $a \neq 0$ transforms to a subcritical one for $a = 0$.

The statement of Corollary 5.11 is illustrated in Figure 4 (with $a_n \nearrow 0$). Furthermore, in Figure 5, we illustrate how the graphs depend on the value of a in case $1 < p < 2$.

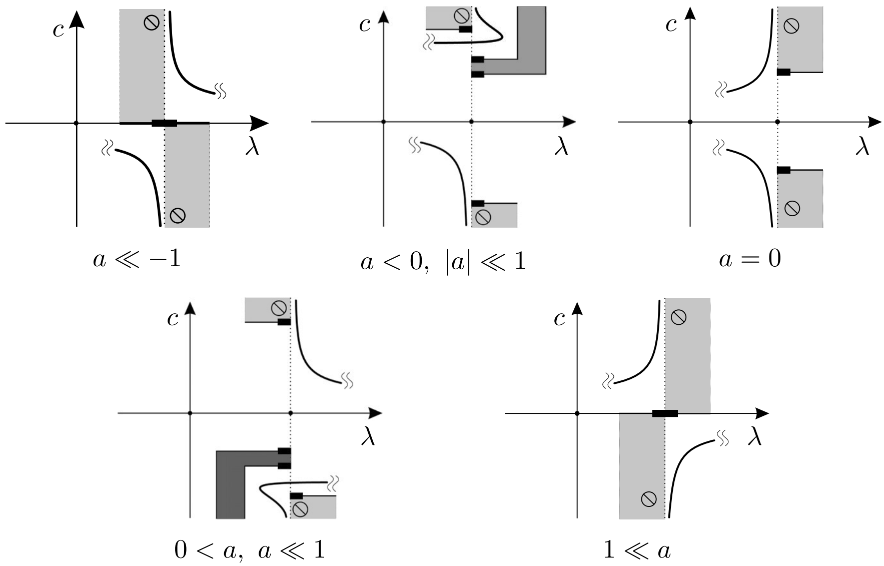

Figure 5: Dependence of a priori bounds and bifurcations from infinity of solutions to (1.1) on $a = \int_\Omega f\varphi_1 \mathrm{d}x$ for $1 < p < 2$. There is no solution in shaded regions.

6. Main Results

In this section we will state some applications and consequences of the asymptotic formulas derived in the previous section. We will assume that $f^\top \in L^\infty(\Omega)$ is a given function which satisfies $\int_\Omega f^\top \varphi_1 \,\mathrm{d}x = 0$, $f^\top \not\equiv 0$. We begin with the following existence and boundedness result.

Theorem 6.1. *Problem* (1.1) *with* $\lambda = \lambda_1$ *and* $f = f^\top$ *has at least one solution* $u \in C^{1,\beta'}(\overline{\Omega})$. *Moreover, there exists a constant* $K = K(f^\top) > 0$ *such that any solution* u *to* (1.1) *satisfies* $\|u\|_{C^{1,\beta'}(\overline{\Omega})} \le K$.

Sketch of the proof. Let $p > 2$ (the case $1 < p < 2$ is treated similarly). Then it follows from a standard degree argument (see e.g. [17]) that there exists a sequence of solutions $(\lambda_n, u_n) \in \mathbb{R} \times W_0^{1,p}(\Omega)$ of (1.1) with $\lambda_n \to \lambda_1-$. By Theorem 5.7 there exists a constant $M > 0$ such that

$\|u_n\|_{L^\infty(\Omega)} \le M$. A regularity argument (up to the boundary, see [19]) implies that $\|u_n\|_{C^{1,\beta}(\overline{\Omega})} \le K$ with some $\beta \in (0,1)$. Now, using compactness of the embedding of $C^{1,\beta}(\overline{\Omega})$ into $C^{1,\beta'}(\overline{\Omega})$, $0 < \beta' < \beta$, we can use a standard limiting argument to prove the existence of the solution $u \in C^{1,\beta'}(\overline{\Omega})$ of (1.1). The a priori estimate $\|u\|_{C^{1,\beta'}(\overline{\Omega})} \le K$ follows from Proposition 5.1 and a regularity result [19]. $\square$

Theorem 6.2. *There exists $\delta = \delta(f^\top, p) > 0$ such that problem (1.1) with $f = f^\top$ has at least one positive solution and at least one negative solution provided one of the following two alternatives occurs:*

(i) $1 < p < 2$ *and* $\lambda \in [\lambda_1 - \delta, \lambda_1)$;

(ii) $p > 2$ *and* $\lambda \in (\lambda_1, \lambda_1 + \delta]$.

This theorem is an immediate consequence of Theorem 5.2 and Corollary 5.8.

Theorem 6.3. *There exists a constant $\varepsilon > 0$ (small enough) with the following properties:*

(i) *for* $1 < p < 2$, *problem* (1.1) *has*

   (a) *at least one positive and at least one negative solution provided* $0 < |a| < \varepsilon$ *and* $\lambda_1 - \delta < \lambda < \lambda_1$, *where* $\delta \equiv \delta(\varepsilon) > 0$ *is a constant (small enough);*

   (b) *at least two distinct negative solutions provided* $\varepsilon' < a < \varepsilon$ *and* $\lambda_1 < \lambda < \lambda_1 + \delta$, *where* $\varepsilon' \in (0, \varepsilon)$ *is an arbitrary number and* $\delta \equiv \delta(\varepsilon, \varepsilon') > 0$ *is a constant (small enough);*

   (c) *at least two distinct positive solutions provided* $-\varepsilon < a < -\varepsilon'$ *and* $\lambda_1 < \lambda < \lambda_1 + \delta$, *where* $\varepsilon' \in (0, \varepsilon)$ *is an arbitrary number and* $\delta \equiv \delta(\varepsilon, \varepsilon') > 0$ *is a constant (small enough);*

(ii) *for* $p > 2$, *problem* (1.1) *has*

   (a) *at least one positive and at least one negative solution provided* $0 < |a| < \varepsilon$ *and* $\lambda_1 < \lambda < \lambda_1 + \delta$, *where* $\delta \equiv \delta(\varepsilon) > 0$ *is a constant (small enough);*

   (b) *at least two distinct positive solutions provided* $\varepsilon' < a < \varepsilon$ *and* $\lambda_1 - \delta < \lambda < \lambda_1$, *where* $\varepsilon' \in (0, \varepsilon)$ *is an arbitrary number and* $\delta \equiv \delta(\varepsilon, \varepsilon') > 0$ *is a constant (small enough);*

   (c) *at least two distinct negative solutions provided* $-\varepsilon < a < -\varepsilon'$ *and* $\lambda_1 - \delta < \lambda < \lambda_1$, *where* $\varepsilon' \in (0, \varepsilon)$ *is an arbitrary number and* $\delta \equiv \delta(\varepsilon, \varepsilon') > 0$ *is a constant (small enough).*

This theorem follows immediately from Corollary 5.6 and Theorem 5.9.

Remark 6.4. We would like to emphasize that the existence of multiple solutions in all cases of Theorem 6.3 does not occur in the linear case $p = 2$, where the uniqueness of the solution is guaranteed by the linear Fredholm alternative.

In what follows we show that for $|a|$ sufficiently large, the statements of Theorem 6.3 are no longer valid. To this end we recall the following nonexistence result.

Proposition 6.5. *There exists $a_0 > 0$ such that problem* (1.1) *with $\lambda = \lambda_1$ and $f = f^\top + a\varphi_1$ has no solution whenever $|a| \geq a_0$.*

Sketch of the proof. We argue by contradiction. Assume that there is a sequence $\{a_n\}_{n=1}^\infty \subset \mathbb{R}$, $a_n \to \infty$, such that (1.1) with $\lambda = \lambda_1$ and $f_n = f^\top + a_n\varphi_1$ has a solution u_n. Dividing the equation in (1.1) by a_n and setting $v_n \stackrel{\text{def}}{=} a_n^{-1/(p-1)} u_n$, we obtain

$$\begin{aligned} -\Delta_p v_n - \lambda_1 |v_n|^{p-2} v_n &= a_n^{-1} f_n + \varphi_1 \quad && \text{in } \Omega, \\ v_n &= 0 && \text{on } \partial\Omega. \end{aligned}$$

From Theorem 4.1 with $u_n = t_n^{-1}(\varphi_1 + v_n^\top)$ we infer that $\{t_n^{-1}\}_{n=1}^\infty$ has to be bounded; otherwise we would have a contradiction with the asymptotic estimate (4.1). Hence, v_n is bounded in $L^\infty(\Omega)$. Therefore, combining the compactness of Δ_p^{-1} with a regularity result [19], we find that $v_n \to v_0$ strongly in $C_0^1(\overline{\Omega})$, and v_0 satisfies

$$\begin{aligned} -\Delta_p v_0 - \lambda_1 |v_0|^{p-2} v_0 &= \varphi_1 \quad && \text{in } \Omega, \\ v_0 &= 0 && \text{on } \partial\Omega. \end{aligned}$$

But this contradicts a nonexistence result proved in [1] or [15]. □

Now we have the following counterpart of Theorem 6.3.

Theorem 6.6. *Let $f = f^\top + a\varphi_1$ with $|a| \geq a_0$, where $a_0 > 0$ is the number from Proposition 6.5. Then there exists $\delta > 0$ such that*

(i) *if $a \geq a_0$ then*
   (a) *problem* (1.1) *has only positive solutions provided $\lambda \in (\lambda_1 - \delta, \lambda_1)$;*
   (b) *problem* (1.1) *has only negative solutions provided $\lambda \in (\lambda_1, \lambda_1 + \delta)$;*

(ii) *if $a \leq -a_0$ then*
   (a) *problem* (1.1) *has only negative solutions provided $\lambda \in (\lambda_1 - \delta, \lambda_1)$;*
   (b) *problem* (1.1) *has only positive solutions provided $\lambda \in (\lambda_1, \lambda_1 + \delta)$.*

Remark 6.7. Let us note that for $\lambda < \lambda_1$, problem (1.1) has a solution by coercivity of the functional (1.2), and for $\lambda_1 < \lambda < \lambda_1 + \delta$, (1.1) is solvable by a topological degree argument [11, Theorem 12.26].

Proof of Theorem 6.6. We prove Case (i) (a) only, the proofs of all remaining cases being analogous. The nonexistence of a solution other than a positive one is proved combining Theorems 4.1 and 5.5 with Proposition 6.5 as follows. Assume by contradiction that there is a sequence $\lambda_n \to \lambda_1$, $\lambda_n < \lambda_1$, such that problem (1.1) with $\lambda = \lambda_n$ and $f = f^\top + a\varphi_1$ has a solution $u_n \in W_0^{1,p}(\Omega)$ such that $u_n(x_n) \le 0$ for some $x_n \in \Omega$. If $\|u_n\|_{L^\infty(\Omega)}$ is unbounded, let us write $u_n = t_n^{-1}(\varphi_1 + v_n^\top)$, where $\lambda_n \to \lambda_1$, $\lambda_n < \lambda_1$, implies $t_n \to 0$, $t_n > 0$, and $\|v_n^\top\|_{C_0^1(\Omega)} \to 0$ as $n \to \infty$. According to Theorems 4.1 and 5.5 (Case (ii) (a)), this contradicts our assumption $a \ge a_0 > 0$. Consequently, there is a constant $M > 0$ such that $\|u_n\|_{L^\infty(\Omega)} \le M$ for all $n = 1, 2, \dots$. Using a standard compactness argument we get $u \in W_0^{1,p}(\Omega)$ such that $u_n \to u$ strongly in $W_0^{1,p}(\Omega)$. Passing to the the limit in (1.1) with $\lambda = \lambda_n$ as $n \to \infty$, we find that u is a solution to the problem

$$\begin{aligned} -\Delta_p u - \lambda_1 |u|^{p-2}u = f^\top + a\varphi_1 \quad & \text{in } \Omega, \\ u = 0 \quad & \text{on } \Omega, \end{aligned}$$

which contradicts Proposition 6.5. Hence, Case (i) (a) is proved. □

Remark 6.8. Theorem 6.6 corresponds to the well-known local maximum and anti-maximum principles, cf. [5, Theorem 27].

Now we derive some multiplicity results. To establish them, we need to recall the standard notions of lower and upper solutions to problem (1.1).

Definition 6.9. A function $\overline{u} \in C^1(\overline{\Omega})$ is called an *upper solution* of (1.1) if for all functions $v \in W_0^{1,p}(\Omega)$ with $v \ge 0$ in Ω, we have

$$\begin{gathered} \int_\Omega |\nabla \overline{u}|^{p-2} \langle \nabla \overline{u}, \nabla v \rangle \,\mathrm{d}x - \lambda_1 \int_\Omega |\overline{u}|^{p-2}\overline{u} v \,\mathrm{d}x \ge \int_\Omega f v \,\mathrm{d}x, \\ \overline{u} \ge 0 \quad \text{on } \partial\Omega. \end{gathered}$$

We define a *lower solution* $\underline{u}$ to (1.1) to be a function from $C^1(\overline{\Omega})$ for which the corresponding reversed inequalities hold.

We now need to employ special versions of more general results due to de Coster and Henrard [7].

Proposition 6.10 ([7, Theorem 8.1]). *Let $\underline{u}$ and $\overline{u}$, respectively, be lower and upper solutions of* (1.1) *such that $\underline{u} \le \overline{u}$. Then problem* (1.1) *has at least one weak solution u satisfying*

$$\underline{u} \le u \le \overline{u} \quad \textit{in } \Omega.$$

Proposition 6.11 ([7, Theorem 8.2]). *Let $\underline{u}$ and $\overline{u}$, respectively, be lower and upper solutions of* (1.1) *and assume that there exists $x_0 \in \Omega$ such that $\underline{u}(x_0) > \overline{u}(x_0)$. Then problem* (1.1) *has at least one solution in the closure (with respect to C^1-norm) of the set*

$$\mathcal{S} = \{u \in C_0^1(\overline{\Omega}) : \exists\, x_1, x_2 \in \Omega \textit{ such that } u(x_1) < \underline{u}(x_1) \textit{ and } u(x_2) > \overline{u}(x_2)\}.$$

Our first multiplicity result extends Theorem 6.2.

Theorem 6.12. *There exists $\eta = \eta(f^\top, p) > 0$ such that problem* (1.1) *with $f = f^\top$ has at least three distinct solutions, at least one of them positive and one negative, provided*

(i) *either $1 < p < 2$ and $\lambda \in (\lambda_1 - \eta, \lambda_1)$;*

(ii) *or $p > 2$ and $\lambda \in (\lambda_1, \lambda_1 + \eta)$.*

Proof. First, let $p > 2$. According to Theorem 6.2, problem (1.1) with $\lambda = \lambda_\delta \stackrel{\text{def}}{=} \lambda_1 + \delta$ and $f = f^\top$ has a positive solution $\overline{u}_{\lambda_\delta}$ and a negative solution $\underline{u}_{\lambda_\delta}$. It follows from Theorems 4.1, 5.2 and 5.7 that there exists an $\eta_+ > 0$ such that $u_\lambda > \overline{u}_{\lambda_\delta}$ in Ω whenever $\lambda \in (\lambda_1, \lambda_1 + \eta_+)$ and $(\lambda, u_\lambda) \in \mathcal{C}^+$. Similarly, there is an $\eta_- > 0$ such that $v_\lambda < \underline{u}_{\lambda_\delta}$ in Ω whenever $\lambda \in (\lambda_1, \lambda_1 + \eta_-)$ and $(\lambda, v_\lambda) \in \mathcal{C}^-$. Here, $\mathcal{C}^+$ and $\mathcal{C}^-$ are as in Corollary 5.8. Set $\eta = \min\{\eta_+, \eta_-\}$.

Note that for $\lambda \in (\lambda_1, \lambda_1 + \eta)$ the function $\overline{u}_{\lambda_\delta}$ ($\underline{u}_{\lambda_\delta}$) is an upper (lower) solution of (1.1) with $f = f^\top$. It follows from Proposition 6.10 that (1.1) with $f = f^\top$ has a solution $w \in C^{1,\beta}(\overline{\Omega})$ such that

$$\underline{u}_{\lambda_\delta} \le w \le \overline{u}_{\lambda_\delta} \quad \text{in } \Omega.$$

Hence, problem (1.1) with $f = f^\top$ has at least three distinct solutions $v_\lambda \le w \le u_\lambda$.

If $1 < p < 2$, we proceed in a similar way using Proposition 6.11 in place of Proposition 6.10. $\square$

Remark 6.13. For $1 < p < 2$, Theorem 6.12 extends [23, Theorem 2.7] by the positivity and negativity statements.

Our second multiplicity result extends Theorems 6.1 and 6.3.

Theorem 6.14. *Let $f = f^\top + a\varphi_1$ with $a \neq 0$. Then there exists $\varepsilon > 0$ (small enough) with the following properties:*

(i) *for every $\varepsilon' \in (0, \varepsilon)$, there is $\eta = \eta(f^\top, \varepsilon, \varepsilon') > 0$ such that $\varepsilon' < |a| < \varepsilon$ and $\lambda \in (\lambda_1 - \eta, \lambda_1) \cup (\lambda_1, \lambda_1 + \eta)$ imply that problem* (1.1) *has at least three distinct solutions of which at least one is positive and at least one is negative;*

(ii) *if $0 < |a| < \varepsilon$ then problem* (1.1) *with $\lambda = \lambda_1$ has at least two distinct solutions of which at least one is negative provided $(p-2)a < 0$, and at least one is positive provided $(p-2)a > 0$.*

Proof. (i) Let $1 < p < 2$. We consider only the case $\lambda < \lambda_1$ and $a > 0$, the remaining cases being analogous. It follows from Theorem 6.3 that for $\varepsilon > 0$ small enough and every $\varepsilon' \in (0, \varepsilon)$, problem (1.1) with $\lambda = \lambda_\delta \stackrel{\text{def}}{=} \lambda_1 - \delta/2$ (recall $\delta \equiv \delta(\varepsilon, \varepsilon') > 0$), $f = f^\top + a\varphi_1$ and $\varepsilon' < a < \varepsilon$ has a positive solution $\underline{u}_{\lambda_\delta}$ and a negative solution $\overline{u}_{\lambda_\delta}$, respectively. For $\lambda \in (\lambda_\delta, \lambda_1]$ the function $\underline{u}_{\lambda_\delta}$ ($\overline{u}_{\lambda_\delta}$) is a lower (upper) solution of (1.1). In this case, lower and upper solutions are unordered, so it follows from Proposition 6.11 that (1.1) has a solution $u_\lambda^{(1)}$ such that $u_\lambda^{(1)}(x_1) < \underline{u}_{\lambda_\delta}(x_1)$ and $u_\lambda^{(1)}(x_2) > \overline{u}_{\lambda_\delta}(x_2)$ at some points $x_1, x_2 \in \Omega$. On the other hand, taking $\eta \leq \delta/2$ sufficiently small, it follows from Theorems 4.1 and 5.5 that problem (1.1) with $\lambda \in (\lambda_1 - \eta, \lambda_1)$ has a positive solution $u_\lambda^{(2)} > \underline{u}_{\lambda_\delta} > 0$ in Ω. Hence, for $\lambda \in (\lambda_1 - \eta, \lambda_1)$ we already have two distinct solutions. To get a third solution of (1.1), we now take $\lambda' > \lambda_1$ with $\lambda' - \lambda_1$ small enough. Then, according to Theorems 4.1 and 5.5, problem (1.1) with $\lambda = \lambda'$ has a negative solution $\underline{v}_{\lambda'} < \overline{u}_{\lambda_\delta} < 0$ in Ω which is simultaneously a lower solution of (1.1) with $\lambda \in (\lambda_1 - \eta, \lambda_1]$. Hence, we may apply Proposition 6.10 to get a negative solution $u_\lambda^{(3)}$ satisfying $\underline{v}_{\lambda'} \leq u_\lambda^{(3)} \leq \overline{u}_{\lambda_\delta} < 0$ in Ω.

(ii) If $\lambda = \lambda_1$, one can proceed in the same way as above to get solutions $u_{\lambda_1}^{(1)}$ and $u_{\lambda_1}^{(3)} < 0$ in Ω.

The case $p > 2$ is analogous. □

Appendix A. Function spaces in the linearized problem in weighted Sobolev spaces

The following two compact embedding results are proved in [23, Lemma 4.2] and [23, Lemma 8.2], respectively. For the purpose of this lecture we include these proofs below.

For $0 < \delta < \infty$, we denote by

$$\Omega_\delta \stackrel{\text{def}}{=} \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) < \delta\}$$

the δ-neighborhood of $\partial\Omega$. Its complement in Ω is denoted by $\Omega'_\delta = \Omega \setminus \Omega_\delta$.

Lemma A.1. *Let $2 < p < \infty$. Then*

(a) *for every $\delta > 0$ small enough, $\|\cdot\|_{\mathcal{D}_{\varphi_1}}$ is an equivalent norm on $W_0^{1,2}(\Omega_\delta)$;*

(b) *the embedding $\mathcal{D}_{\varphi_1} \hookrightarrow L^2(\Omega)$ is compact.*

Proof. Part (a) follows immediately from (2.2).

To prove (b), we start with the proof of continuity of $\mathcal{D}_{\varphi_1} \hookrightarrow L^2(\Omega)$. We take advantage of the Dirichlet boundary value problem (2.1) to compute for every $v \in C_0^1(\Omega)$,

$$\begin{aligned}\lambda_1 \int_\Omega \varphi_1^{p-2} v^2 \,\mathrm{d}x &= \lambda_1 \int_\Omega \varphi_1^{p-1} (v^2 \varphi_1^{-1}) \,\mathrm{d}x \\ &= \int_\Omega |\nabla\varphi_1|^{p-2} \nabla\varphi_1 \cdot \nabla(v^2\varphi_1^{-1}) \,\mathrm{d}x \\ &= 2\int_\Omega |\nabla\varphi_1|^{p-2} (\nabla\varphi_1 \cdot \nabla v) v \varphi_1^{-1} \,\mathrm{d}x - \int_\Omega |\nabla\varphi_1|^p v^2 \varphi_1^{-2} \,\mathrm{d}x.\end{aligned}$$

Adding the last integral and estimating the second last one by the Cauchy-Schwarz inequality, we arrive at

$$\begin{aligned}&\lambda_1 \int_\Omega \varphi_1^{p-2} v^2 \,\mathrm{d}x + \int_\Omega |\nabla\varphi_1|^p v^2 \varphi_1^{-2} \,\mathrm{d}x \\ &\qquad \le 2\Big(\int_\Omega |\nabla\varphi_1|^{p-2} |\nabla v|^2 \,\mathrm{d}x\Big)^{1/2} \Big(\int_\Omega |\nabla\varphi_1|^p v^2 \varphi_1^{-2} \,\mathrm{d}x\Big)^{1/2} \\ &\qquad \le 2\int_\Omega |\nabla\varphi_1|^{p-2} |\nabla v|^2 \,\mathrm{d}x + \frac{1}{2}\int_\Omega |\nabla\varphi_1|^p v^2 \varphi_1^{-2} \,\mathrm{d}x,\end{aligned}$$

and therefore,

$$\lambda_1 \int_\Omega \varphi_1^{p-2} v^2 \,\mathrm{d}x + \frac{1}{2}\int_\Omega |\nabla\varphi_1|^p v^2 \varphi_1^{-2} \,\mathrm{d}x \le 2\|v\|^2_{\mathcal{D}_{\varphi_1}}. \tag{A.1}$$

Since $C_0^1(\Omega)$ is dense in $\mathcal{D}_{\varphi_1}$, the last inequality holds also for every $v \in \mathcal{D}_{\varphi_1}$. Using (2.2) we conclude that the embedding $\mathcal{D}_{\varphi_1} \hookrightarrow L^2(\Omega)$ is continuous.

To prove the compactness of $\mathcal{D}_{\varphi_1} \hookrightarrow L^2(\Omega)$, we take advantage of the Dirichlet boundary value problem (2.1) again to compute for every $v \in \mathcal{D}_{\varphi_1}$,

$$\begin{aligned}\lambda_1 \int_\Omega \varphi_1^{p-1} v^2 \,\mathrm{d}x &= \int_\Omega |\nabla\varphi_1|^{p-2} \nabla\varphi_1 \cdot \nabla(v^2) \,\mathrm{d}x \\ &\le 2\int_\Omega |\nabla\varphi_1|^{p-1} |\nabla v| \cdot |v| \,\mathrm{d}x \qquad \text{(A.2)} \\ &\le 2\|v\|_{\mathcal{D}_{\varphi_1}} \Big(\int_\Omega |\nabla\varphi_1|^p v^2 \,\mathrm{d}x\Big)^{1/2},\end{aligned}$$

by the Cauchy-Schwarz inequality. Let $\{v_n\}_{n=1}^{\infty}$ be any weakly convergent sequence in$\mathcal{D}_{\varphi_1}$; we may assume $v_n \rightharpoonup 0$. Hence,

$$v_n \rightharpoonup 0 \text{ weakly in } L^2(\Omega) \tag{A.3}$$

and

$$|\nabla\varphi_1|^{(p-2)/2}\nabla v_n \rightharpoonup \mathbf{0} \text{ weakly in } [L^2(\Omega)]^N \tag{A.4}$$

as $n \to \infty$. We will show that, indeed, $v_n \to 0$ strongly in $L^2(\Omega)$. Given any $0 < \eta < \infty$ small enough, let us decompose $\Omega = U_\eta \cup U'_\eta$, where

$$U_\eta \stackrel{\text{def}}{=} \{x \in \Omega : |\nabla\varphi_1(x)| > \eta\} \quad \text{and} \quad U'_\eta \stackrel{\text{def}}{=} \{x \in \Omega : |\nabla\varphi_1(x)| \le \eta\}.$$

We deduce from (A.3) and (A.4) that the restrictions $v_n|_{U_\eta}$ of v_n to U_η form a weakly convergent sequence in $W^{1,2}(U_\eta)$. It follows that $\|v_n\|_{L^2(U_\eta)} \to 0$ as $n \to \infty$, by Rellich's theorem.

Next, in (A.2) we replace v by v_n. Owing to (A.4), there is a constant $C > 0$ independent from n such that $\|v_n\|_{\mathcal{D}_{\varphi_1}} \le C\lambda_1/2$, and consequently, (A.2) yields

$$\int_\Omega \varphi_1^{p-1} v_n^2 \,\mathrm{d}x \le C\Big(\int_\Omega |\nabla\varphi_1|^p v_n^2 \,\mathrm{d}x\Big)^{1/2}. \tag{A.5}$$

We split the integral on the right-hand side using $\Omega = U_\eta \cup U'_\eta$. The two integrals are estimated by

$$\int_{U_\eta} |\nabla\varphi_1|^p v_n^2 \,\mathrm{d}x \le \|\nabla\varphi_1\|_\infty^p \int_{U_\eta} v_n^2 \,\mathrm{d}x, \tag{A.6}$$

$$\int_{U'_\eta} |\nabla\varphi_1|^p v_n^2 \,\mathrm{d}x \le \eta^p \int_{U'_\eta} v_n^2 \,\mathrm{d}x \le \eta^p \int_\Omega v_n^2 \,\mathrm{d}x. \tag{A.7}$$

Now choose any $0 < \varepsilon < \infty$. First, fix $\eta_0 > 0$ small enough so that

$$\eta_0^{p/2} \cdot \sup_{n \ge 1} \|v_n\|_{L^2(\Omega)} \le \frac{\varepsilon}{C\sqrt{2}}. \tag{A.8}$$

Second, fix $\eta > 0$ and $\delta > 0$ sufficiently small such that $0 < \eta \le \eta_0$ and $\Omega_\delta \subset U_\eta$. This choice is possible by the Hopf maximum principle (2.2) for φ_1. Third, recalling $\|v_n\|_{L^2(U_\eta)} \to 0$ as $n \to \infty$, fix an integer $n_0 \ge 1$ large enough such that

$$\|\nabla\varphi_1\|_\infty^{p/2} \cdot \|v_n\|_{L^2(U_\eta)} \le \frac{\varepsilon}{C\sqrt{2}} \quad \text{for all } n \ge n_0. \tag{A.9}$$

The numbers η, δ and n_0 being fixed, we first apply (A.8) and (A.9) to (A.6) and (A.7), respectively, and then combine the last two with the inequality (A.5), thus arriving at

$$\int_\Omega \varphi_1^{p-1} v_n^2 \,\mathrm{d}x \le \varepsilon \quad \text{for all } n \ge n_0. \tag{A.10}$$

In particular, setting $\Omega'_\delta = \Omega \setminus \Omega_\delta$, we infer from (2.2) and (A.10) that $\|v_n\|_{L^2(\Omega'_\delta)} \to 0$ as $n \to \infty$.

Finally, we make use of $U_\eta \cup \Omega'_\delta = \Omega$ to conclude that $\|v_n\|_{L^2(\Omega)} \to 0$ as $n \to \infty$. The proof of the lemma is finished. $\square$

Imbeddings that involve $\mathcal{H}_{\varphi_1}$ are established next.

Lemma A.2. *Let* $1 < p < 2$. *Then*

(a) *the embedding* $\mathcal{H}_{\varphi_1} \hookrightarrow L^2(\Omega)$ *is continuous;*

(b) *the embedding* $W_0^{1,2}(\Omega) \hookrightarrow \mathcal{H}_{\varphi_1}$ *is compact.*

Proof. Part (a) follows immediately from (2.2).

To prove (b), first notice that there exist constants $0 < c_1 \le c_2 < \infty$ such that $c_1 \le \varphi_1(x)/d(x) \le c_2$ for all $x \in \Omega$, where the function

$$d(x) \stackrel{\text{def}}{=} \operatorname{dist}(x, \partial\Omega) = \inf_{x_0 \in \partial\Omega} |x - x_0|, \quad x \in \overline{\Omega},$$

denotes the distance from x to $\partial\Omega$. By well-known results taken from Kufner [18, §8.8] or Triebel [27, §3.5.2], or simply by an inequality similar to (A.1), the Sobolev space $W_0^{1,2}(\Omega)$ is continuously embedded into the weighted Lebesgue space $L^2(\Omega; d(x)^{-2}\,\mathrm{d}x)$ endowed with the norm

$$\|v\|_{L^2(\Omega; d(x)^{-2}\,\mathrm{d}x)} \stackrel{\text{def}}{=} \left(\int_\Omega v^2 d(x)^{-2} \,\mathrm{d}x \right)^{1/2} < \infty.$$

Notice that $\mathcal{H}_{\varphi_1} = L^2(\Omega; d(x)^{p-2}\,\mathrm{d}x)$. Consequently, using again the splitting $\Omega = \Omega_\delta \cup \Omega'_\delta$ from the proof of Lemma A.1, we conclude that the embedding $W_0^{1,2}(\Omega) \hookrightarrow \mathcal{H}_{\varphi_1}$ is compact. $\square$

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