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On Relations Among Metric Derived Numbers

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Let X be a metric space and $f : \mathbb{R} \to X$ a mapping. We prove that for a continuous mapping f the set A_f of points $x \in \mathbb{R}$ at which the unilateral lower metric derived numbers of f differ is meager. We construct a continuous mapping f such that A_f has full Lebesgue measure. We further prove that for an arbitrary mapping f the set of points $x \in \mathbb{R}$ at which f is Lipschitz from the right and not Lipschitz from the left is σ-strongly porous.

1. Introduction

In this paper we study some relations among metric derived numbers that are analogues of the usual notions of Dini derivatives (also called derived numbers) which hold for mappings with values in metric spaces. Notions of metric derivatives and metric differentiability in $\mathbb{R}^n$ were introduced by B. Kirchheim in [5]. Later, these notions have been further studied by several other authors.

There is a well established theory of Dini derivatives of real-valued functions $f : \mathbb{R} \to \mathbb{R}$ (see e.g. [2]). Several of classical theorems concerning relationships among the Dini derivatives were modified to the context of metric derived numbers by J. Duda and O. Maleva in [3]. In particular, they proved that for any mapping with values in a metric space, the set of points where its unilateral upper metric derived numbers differ is a σ-porous set. Similarly they proved that the set of points where its lower metric derived numbers differ is also a σ-porous set, provided this mapping is pointwise Lipschitz.

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In Theorem 3.2, we show that for any continuous mapping with values in a metric space, the set of points where its unilateral lower metric derived numbers differ is a meager set. Moreover, in Example 3.1, we construct such a continuous real function that this set has full Lebesgue measure. In Theorem 3.3, we further show that for an arbitrary mapping with values in a metric space, the sets of points where one unilateral upper metric derived number is finite and the other one is infinite are unilaterally σ-strongly porous.

2. Preliminaries

We start this section by basic definitions of porosity and unilateral strong porosity for subsets of the real line. The distance from a point $x \in \mathbb{R}$ to an open interval $I \subset \mathbb{R}$ is denoted by $\operatorname{dist}(x, I)$ and the length of the open interval I is denoted by $|I|$.

If $\{(a_n, b_n)\}_{n=1}^{\infty}$ is a sequence of open intervals in $\mathbb{R}$ and $x \in \mathbb{R}$, we say that $\{(a_n, b_n)\}_{n=1}^{\infty}$ *converges to* x if $a_n \to x$ and $b_n \to x$. If moreover $a_n \geq x$ (resp. $b_n \leq x$) for every $n \in \mathbb{N}$, we say that $\{(a_n, b_n)\}_{n=1}^{\infty}$ *converges to* x *from the right* (resp. *left*).

Definition 2.1 Let $A \subset \mathbb{R}$ and $x \in \mathbb{R}$.

The set A is *porous at the point* x if there exist $c > 0$ and a sequence of open intervals $\{I_n\}_{n=1}^{\infty}$ converging to x such that $I_n \cap A = \emptyset$ and $\frac{|I_n|}{\operatorname{dist}(x,I_n)} \geq c$ for every $n \in \mathbb{N}$.

The set A is *strongly right* (resp. *left*) *porous at the point* x if there exists a sequence of open intervals $\{I_n\}_{n=1}^{\infty}$ converging to x from the right (resp. left) such that $I_n \cap A = \emptyset$ for every $n \in \mathbb{N}$ and $\frac{|I_n|}{\operatorname{dist}(x,I_n)} \to +\infty$.

The set A is *porous* [1] (resp. *strongly right* (*left*) *porous*) if it is porous (resp. strongly right (left) porous) at every point $x \in A$.

The set A is *σ-porous* (resp. *σ-strongly right* (*left*) *porous*) if it is a countable union of porous (resp. strongly right (left) porous) sets.

Besides the already defined notions of porosity, we will also use the notion of porosity index from [6, Definition A2.12]. We put $\sup \emptyset = 0$.

Definition 2.2 Let $A \subset \mathbb{R}$ and $x \in \mathbb{R}$. The *right* (resp. *left*) *porosity index* of A at x, denoted by $PI^{+}(A, x)$ (resp. $PI^{-}(A, x)$), is the supremum of all nonnegative numbers r for which a sequence of open intervals $\{I_n\}_{n=1}^{\infty}$ converging to x from the right (resp. left) may be found such that $I_n \cap A = \emptyset$ and $r < \frac{|I_n|}{d(x,I_n)}$ for every $n \in \mathbb{N}$.

In the proof of Theorem 3.3 we will need the following elementary lemma concerning the porosity index which is implicitly used in the proof of [6, Theorem 73.2]. For the sake of completeness we provide the proof of this lemma.

[1] In the terminology of [7], our definition of porous sets corresponds to the notion of upper porous sets.

Lemma 2.3 *Let $E \subset \mathbb{R}$ and $x \in E$. If $t < +\infty$ and $PI^-(E,x) < t$ then there exists a sequence of points $\{x_k\}_{k=1}^\infty$ in E such that $x_k \to x$, $x_k < x_{k+1}$ and $x_{k+1} - x_k < t(x - x_{k+1})$ for every $k \in \mathbb{N}$.*

Proof. There exists $t' < t$ such that $PI^-(E,x) < t'$. By the definition of $PI^-(E,x)$ we may choose a left punctured neighborhood V of x such that whenever $I \subset V$ is a nonempty open interval with $I \cap E = \emptyset$ then $\frac{|I|}{d(x,I)} < t'$.

Since $PI^-(E,x) < +\infty$, the set $V \cap E$ is nonempty. We choose $x_1 \in V \cap E$ arbitrarily.

Suppose that for a natural number $n \geq 1$ we have already constructed $x_1, ..., x_n \in V \cap E$ so that $x_i < x_{i+1}$ and $x_{i+1} - x_i < t(x - x_{i+1})$ for every $i \in \{1, ..., n-1\}$ if $n > 1$. We find points $z_n \in (x_n, x)$ and $y_n \in (x_n, z_n)$ so that

$$\frac{z_n - x_n}{x - z_n} = t \text{ and } \frac{z_n - y_n}{x - z_n} = t'. \tag{1}$$

Since $(y_n, z_n) \subset V$ and for every nonempty open interval $I \subset V$ with $I \cap E = \emptyset$ it holds that $\frac{|I|}{d(x,I)} < t'$, the set $(y_n, z_n) \cap E$ is nonempty. We choose $x_{n+1} \in (y_n, z_n) \cap E$ arbitrarily. Then $x_n < x_{n+1}$ and

$$\frac{x_{n+1} - x_n}{x - x_{n+1}} < \frac{z_n - x_n}{x - z_n} = t.$$

Thus we have constructed the increasing sequence $\{x_k\}_{k=1}^\infty \subset E$ satisfying $x_{k+1} - x_k < t(x - x_{k+1})$ for every $k \in \mathbb{N}$. It only suffices to prove that $x_k \to x$.

Clearly, $\{x_k\}_{k=1}^\infty$ converges to some $y \leq x$. By (1) and the obvious inequality $x_{n+1} > y_n$ we get

$$x_{n+1} - x_n > \frac{t - t'}{1 + t}(x - x_n) \tag{2}$$

for every $n \in \mathbb{N}$.

Passing to the limit in (2), we obtain

$$0 \geq \frac{t - t'}{1 + t}(x - y).$$

Since $\frac{t-t'}{1+t} > 0$ and $x \geq y$ it follows that $x = y$ and the lemma is proved. □

Let (X, ϱ) be a metric space. Every metric space can be isometrically embedded in some Banach space (see [1, Lemma 1.1]). In the sequel, we always suppose that X is a real Banach space and that the distance in X is generated by a norm $\|\cdot\|$.

Definition 2.4 Let $f : \mathbb{R} \to X$ and $x \in \mathbb{R}$. We define the *unilateral upper* (resp. *lower*) *metric derived numbers* of the mapping f at the point x by

$$mD^\pm(f,x) = \limsup_{t \to 0_+} \frac{\|f(x \pm t) - f(x)\|}{t}$$

and

$$mD_\pm(f,x) = \liminf_{t \to 0_+} \frac{\|f(x \pm t) - f(x)\|}{t}.$$

In the previous definition we also allow the value $+\infty$. If the right (resp. left) upper and lower metric derived numbers of f at x are equal, their common value is called the *right* (resp. *left*) *metric derivative* of f at x and denoted by $md^+(f,x)$ (resp. $md^-(f,x)$). If all four metric derived numbers of a mapping f at a point x are equal, their common value is called the *metric derivative* of f at x and denoted by $md(f,x)$.

It is useful to mention the connection among metric derived numbers and the classical Dini derivatives in the special case of $X = \mathbb{R}$. The upper and lower unilateral Dini derivatives of a function $f : \mathbb{R} \to \mathbb{R}$ at a point $x \in \mathbb{R}$ will be denoted by $D^\pm f(x)$ and $D_\pm f(x)$. The connection among upper metric derived numbers and Dini derivatives is obvious. However in the case of lower metric derived numbers it is somewhat more complicated and we establish this connection only for continuous functions $f : \mathbb{R} \to \mathbb{R}$ since it will suffice for our application in the next section.

Observation 2.5

- Let $f : \mathbb{R} \to \mathbb{R}$ be arbitrary. For any $x \in \mathbb{R}$ the upper metric derived numbers can be expressed by

$$mD^\pm(f,x) = \max(|D^\pm f(x)|, |D_\pm f(x)|).$$

- Let $f : \mathbb{R} \to \mathbb{R}$ be continuous.
  If $\operatorname{sign} D^\pm f(x) = \operatorname{sign} D_\pm f(x)$ then

$$mD_\pm(f,x) = \min(|D^\pm f(x)|, |D_\pm f(x)|),$$

  otherwise

$$mD_\pm(f,x) = 0.$$

We will further need the following version of classical Denjoy-Young-Saks theorem for Dini derivatives (its other version where the exceptional Lebesgue null set is replaced by a σ-porous set can be found in [6, §73]).

Theorem 2.6 [6, §70] *For an arbitrary function $f : \mathbb{R} \to \mathbb{R}$, almost every point $x \in \mathbb{R}$ must fall into one of the following sets:*

$$\{x \in \mathbb{R} : f'(x) \text{ exists and is finite}\},$$

$$\{x \in \mathbb{R} : D^+ f(x) = D_- f(x) \text{ are finite}, D_+ f(x) = -\infty, D^- f(x) = +\infty\},$$

$$\{x \in \mathbb{R} : D_+ f(x) = D^- f(x) \text{ are finite}, D^+ f(x) = +\infty, D_- f(x) = -\infty\}$$

and

$$\{x \in \mathbb{R} : D^+ f(x) = D^- f(x) = +\infty, D_+ f(x) = D_- f(x) = -\infty\}.$$

Let us briefly mention definitions of other notions that will be used later in this paper.

Definition 2.7 Let $I \subset \mathbb{R}$ be an interval and $f : I \to \mathbb{R}$ be an arbitrary function.

The function f is said to be *nowhere differentiable* if it has nowhere a finite or infinite derivative on I.

Let $x \in I$. The function f has *symmetrical Dini derivatives* at x if the Dini derivatives of f at x satisfy the relations

$$D^+ f(x) = D^- f(x) \text{ and } D_+ f(x) = D_- f(x).$$

Otherwise we say that f has *asymmetrical Dini derivatives* at x.

When investigating the behavior of Besicovitch's function, K. M. Garg proved the following result.

Theorem 2.8 [4, Theorem 1] *There exist nowhere differentiable functions defined on* $[0, 1]$ *which have asymmetrical Dini derivatives almost everywhere.*

Now we remind the notion of pointwise Lipschitz continuity that will occur in the next theorem.

Definition 2.9 Let $f : \mathbb{R} \to X$. We say that f is *pointwise Lipschitz* if

$$\limsup_{y \to x} \frac{\|f(y) - f(x)\|}{|y - x|} < +\infty$$

for every $x \in \mathbb{R}$.

In [3], the following results concerning metric derived numbers are proved:

Theorem 2.10 [3, Theorems 3.2–3.4] *Let* $f : \mathbb{R} \to X$. *Then the following statements hold:*

(i) *The set*

$$\{x \in \mathbb{R} : mD_-(f, x) > mD^+(f, x) \text{ or } mD_+(f, x) > mD^-(f, x)\}$$

is countable.

(ii) *The set*

$$\{x \in \mathbb{R} : mD^+(f, x) \neq mD^-(f, x)\}$$

is σ*-porous.*

(iii) *If* f *is pointwise Lipschitz, then the set*

$$\{x \in \mathbb{R} : mD_+(f, x) \neq mD_-(f, x)\}$$

is σ*-porous.*

3. Results

Although the assumption that f is pointwise Lipschitz in statement (iii) of Theorem 2.10 may seem rather artificial, it cannot be omitted. In the case of a merely continuous function $f : \mathbb{R} \to \mathbb{R}$, the conclusion of this statement is wrong as the following example shows since every porous set has Lebesgue measure zero (see e.g. [8, Theorem 2.8]).

Example 3.1 There exists a continuous function $f : \mathbb{R} \to \mathbb{R}$ such that the set

$$\{x \in \mathbb{R} : mD_+(f,x) = mD_-(f,x)\}$$

has Lebesgue measure zero.

Proof. Denote by g the continuous function defined on $[0,1]$ which is nowhere differentiable and has asymmetrical Dini derivatives almost everywhere. The existence of such a function is assured by Theorem 2.8. Put

$$\widetilde{f}(x) = g\left(\frac{1}{\pi}\operatorname{arctg} x + \frac{1}{2}\right)$$

for $x \in \mathbb{R}$. Then $\widetilde{f}$ is continuous, nowhere differentiable on $\mathbb{R}$ and has asymmetrical Dini derivatives almost everywhere. By Denjoy-Young-Saks theorem (Theorem 2.6), at almost every point $x \in \mathbb{R}$ one of the following two cases occurs:
either

$$D^+\widetilde{f}(x) = D_-\widetilde{f}(x) \text{ are finite, } D_+\widetilde{f}(x) = -\infty \text{ and } D^-\widetilde{f}(x) = +\infty,$$

or

$$D_+\widetilde{f}(x) = D^-\widetilde{f}(x) \text{ are finite, } D^+\widetilde{f}(x) = +\infty \text{ and } D_-\widetilde{f}(x) = -\infty.$$

For a function $h : \mathbb{R} \to \mathbb{R}$ and $c \in \mathbb{R}$, denote

$$A_c(h) = \{x \in \mathbb{R} : D^+h(x) = D_-h(x) = c, D_+h(x) = -\infty, D^-h(x) = +\infty\}$$

and

$$B_c(h) = \{x \in \mathbb{R} : D_+h(x) = D^-h(x) = c, D^+h(x) = +\infty, D_-h(x) = -\infty\}.$$

It easily follows from [2, Theorem 2.2] that Dini derivatives of arbitrary continuous functions are Borel measurable. Hence the sets $A_c(h)$ and $B_c(h)$ are Borel measurable for every continuous function h and every $c \in \mathbb{R}$.

The set

$$\{c \in \mathbb{R} : A_c(\widetilde{f}) \text{ or } B_c(\widetilde{f}) \text{ has positive Lebesgue measure}\}$$

is at most countable. Hence we may choose $\widetilde{c} \in \mathbb{R}$ and put $f(x) = \widetilde{f}(x) + \widetilde{c}x$ for $x \in \mathbb{R}$ so that both sets $A_0(f)$ and $B_0(f)$ have Lebesgue measure zero.

Clearly, f is continuous. Moreover, except for a set of Lebesgue measure zero, by Observation 2.5 one of the unilateral lower metric derived numbers of f is zero whereas the other one is strictly positive. Thus the set

$$\{x \in \mathbb{R} : mD_+(f,x) = mD_-(f,x)\}$$

has Lebesgue measure zero. □

As the previous example shows, for continuous functions $f : \mathbb{R} \to \mathbb{R}$ the set $\{x \in \mathbb{R} : mD_-(f,x) \neq mD_+(f,x)\}$ can be big in the measure sense and hence cannot be σ-porous in contrast to the pointwise Lipschitz case (Theorem 2.10 (iii)). However, if f is assumed to be continuous only, the positive result concerning the smallness of this set in the sense of its Baire category presented in the next theorem can be proved.

Theorem 3.2 *Let $f : \mathbb{R} \to X$ be continuous. Then the set*

$$\{x \in \mathbb{R} : mD_+(f,x) \neq mD_-(f,x)\}$$

is meager.

Proof. Since the examined set is obviously a subset of the union

$$\{x \in \mathbb{R} : mD_+(f,x) > mD_-(f,x)\} \cup \{x \in \mathbb{R} : mD_+(f,x) < mD_-(f,x)\},$$

by symmetry it suffices to show that the second of these sets is meager. Denote

$$A = \{x \in \mathbb{R} : mD_+(f,x) < mD_-(f,x)\}$$

and for $r \in \mathbb{Q}$ put

$$A_r = \{x \in \mathbb{R} : mD_+(f,x) < r < mD_-(f,x)\}.$$

For fixed $r \in \mathbb{Q}$ put further

$$A_{rj} = \{x \in A_r : \frac{\|f(x) - f(y)\|}{x - y} > r \text{ for every } x - \frac{1}{j} < y < x\}$$

for any $j \in \mathbb{N}$. Clearly, $A = \bigcup_{r \in \mathbb{Q}} \bigcup_{j \in \mathbb{N}} A_{rj}$.

Next, we will show that for arbitrary $r \in \mathbb{Q}$ and $j \in \mathbb{N}$ the set A_{rj} is nowhere dense. Let us therefore fix $r \in \mathbb{Q}$, $j \in \mathbb{N}$ and on the contrary suppose that there exists a nonempty open interval (α,β) such that A_{rj} is dense in (α,β) and $\beta - \alpha < \frac{1}{j}$. Choose two points $\alpha < y < z < \beta$ arbitrarily. Since A_{rj} is dense in (α,β), we may find a sequence $\{x_n\}_{n=1}^{\infty}$ in $A_{rj} \cap (\alpha,\beta)$ such that $x_n \to z$ and $y < x_n$ for every $n \in \mathbb{N}$. Therefore $x_n - \frac{1}{j} < y < x_n$ for every $n \in \mathbb{N}$ and since $x_n \in A_{rj}$ we obtain

$$\frac{\|f(x_n) - f(y)\|}{x_n - y} > r$$

for every $n \in \mathbb{N}$. Using the continuity of f we conclude that

$$\frac{\|f(z) - f(y)\|}{z - y} \geq r \tag{3}$$

and the previous estimate holds for any two points $\alpha < y < z < \beta$. Since A_{rj} is dense in (α,β), we can choose $\tilde{y} \in A_{rj} \cap (\alpha,\beta)$. Thus $mD_+(f,\tilde{y}) < r$. Therefore there exists a point $\tilde{z} \in (\tilde{y},\beta)$ such that

$$\frac{\|f(\tilde{z}) - f(\tilde{y})\|}{\tilde{z} - \tilde{y}} < r$$

which contradicts (3) and completes the proof. □

Rewriting the proof of [6, Theorem 73.2] in the language of metric derived numbers we get the result presented in the next theorem. Both sets examined in this theorem are clearly σ-porous by statement (ii) of Theorem 2.10. However, the theorem concerns the smallness of these sets in a stronger sense.

Theorem 3.3 *Let $f : \mathbb{R} \to X$. Then the set*

$$\{x \in \mathbb{R} : mD^-(f, x) < +\infty \text{ and } mD^+(f, x) = +\infty\}$$

is σ-strongly right porous and the set

$$\{x \in \mathbb{R} : mD^+(f, x) < +\infty \text{ and } mD^-(f, x) = +\infty\}$$

is σ-strongly left porous.

Proof. Due to the obvious symmetry between both examined sets, we will address only the latter one. Denote

$$A = \{x \in \mathbb{R} : mD^+(f, x) < +\infty \text{ and } mD^-(f, x) = +\infty\}$$

and for $p \in \mathbb{N}$ put

$$A_p = \{x \in \mathbb{R} : mD^+(f, x) < p \text{ and } mD^-(f, x) = +\infty\}.$$

Clearly, $A = \bigcup_{p \in \mathbb{N}} A_p$ and it suffices to prove that each A_p is σ-strongly left porous. Fix $p \in \mathbb{N}$ and for $q \in \mathbb{N}$ define

$$A_{pq} = \{x \in A_p : \frac{\|f(y) - f(x)\|}{y - x} < p \text{ for every } x < y < x + \frac{1}{q}\}.$$

Obviously, we get $A_p = \bigcup_{q \in \mathbb{N}} A_{pq}$. We will show that A_{pq} is strongly left porous for every $q \in \mathbb{N}$. Suppose on the contrary that there exist $q' \in \mathbb{N}$ and $x_0 \in A_{pq'}$ so that $A_{pq'}$ is not strongly left porous at x_0. Then there exists $t < +\infty$ such that the left porosity index of $A_{pq'}$ at x_0 is less than t. Using Lemma 2.3, we easily construct a sequence of points $\{x_k\}_{k=1}^{\infty}$ in $A_{pq'}$ such that $x_1 > x_0 - \frac{1}{q'}$, $x_k \to x_0$, $x_k < x_{k+1}$ and

$$x_{k+1} - x_k < t(x_0 - x_{k+1})$$

for every $k \in \mathbb{N}$.

For every $x_1 < y < x_0$ we can find $k \in \mathbb{N}$ such that $x_k < y \leq x_{k+1}$ and obtain the following inequalities:

$$\|f(y) - f(x_k)\| < p(y - x_k),$$
$$\|f(x_{k+1}) - f(x_k)\| < p(x_{k+1} - x_k)$$

and

$$\|f(x_0) - f(x_{k+1})\| < p(x_0 - x_{k+1}).$$

Putting all these inequalities together we get

$$\begin{aligned}
\|f(x_0) - f(y)\| &\leq \|f(x_0) - f(x_{k+1})\| + \|f(x_{k+1}) - f(x_k)\| + \|f(y) - f(x_k)\| \\
&< p\left[(x_0 - x_{k+1}) + (x_{k+1} - x_k) + (y - x_k)\right] \\
&\leq p\left[(x_0 - y) + 2(x_{k+1} - x_k)\right] \\
&\leq p\,(x_0 - y)\left(1 + 2\frac{x_{k+1} - x_k}{x_0 - x_{k+1}}\right) \\
&< p\,(x_0 - y)(1 + 2t).
\end{aligned}$$

Since the previous estimate holds for any $x_1 < y < x_0$, it provides the upper bound of $p(1 + 2t)$ on $mD^-(f, x_0)$, which contradicts the fact that $mD^-(f, x_0) = +\infty$ and completes the proof. □

Remark 3.4 Finiteness of unilateral upper metric derived numbers is closely related to the notion of unilateral Lipschitz continuity:

Let $f : \mathbb{R} \to X$ and $x \in \mathbb{R}$. Then f is Lipschitz from the right (resp. left) at x if and only if $mD^+(f, x) < +\infty$ (resp. $mD^-(f, x) < +\infty$).

Thus the previous theorem can be restated in the following way:

Let $f : \mathbb{R} \to X$. Then the set of points at which f is Lipschitz from the left and not Lipschitz from the right is σ-strongly right porous and the set of points at which f is Lipschitz from the right and not Lipschitz from the left is σ-strongly left porous.

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