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From a Ramsey-Type Theorem to Independence

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The aim of this paper is to investigate some natural conditions on a partial orderings (conditions on a family of sets) and their consequences. It will be proved that the special kind of families having binary property, inclusion property (I) and so on, contains independent subfamilies. A Ramsey-type theorem will be used as a main tool in proofs of theorems included in this paper.

The problem concerning mappings onto generalized Cantor discontinua was studied by several authors. We can equivalently think about the Boolean algebra of clopen subsets in zero-dimensional compact space. For such a space there exists a continuous function onto generalized Cantor discontinua of weight τ if and only if Boolean algebra of clopen subsets of the space contains an independent subset of cardinality τ.

Partial positive solutions under additional set-theoretical assumptions where done, among others, by Efimov [3], Koppelberg [5], Monk [8], and Błaszczyk [2]. A short historical survey of this problern can be found in [2].

The fundamental paper for this theory was published in 1982 by Balcar and Franěk, (see [1]). They presented a proof (without any set-theoretical assumptions), that in each infinite complete Boolean algebra B there is an independent family $\mathscr{T} \subset B$ such that $|\mathscr{T}| = |B|$.

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The aim of this paper is to investigate some natural conditions on a partial orderings (conditions on a family of sets) and their consequences. It will be proved that the special kind of families having binary property, inclusion property (I) and so on, contains independent subfamilies. We will prove (without any set-theoretical assumptions) that compact zero-dimensional space can be mapped onto the Cantor cube D^{κ} if and only if it possesses a binary family of clopen subsets $\mathcal{S}$, (where $|\mathcal{S}| \geq \kappa > c(\mathcal{S})$) closed with respect to the complements and inclusion property (I). A Ramsey-type theorem will be used as a main tool in proofs of theorems included in this paper. The most general version of Ramsey-type theorem is presented below

Theorem ([6]) *For each family of sets of cardinality greater than or equal to $(2^{\lambda})^{+}$, closed under finite intersections, either there exists a subfamily of λ^{+} pairwise disjoint subsets or there exists a centered subfamily of cardinality $(2^{\lambda})^{+}$.*

Let $(\mathcal{P}, \preccurlyeq)$ be a partially ordered set.

We say that $a, b \in \mathcal{P}$ are *compatible* if there exists $c \in \mathcal{P}$ such that

$$c \preccurlyeq a \quad \text{and} \quad c \preccurlyeq b.$$

By $a \perp b$ we will denote that a and b are not compatible (are incompatible).

We say that $A \subset \mathcal{P}$ is *linked* if each of two elements in A are compatible.

Definition 1 *We say that $A \subset \mathcal{P}$ fulfills condition (I) if for all $a_0, a_1, a_2 \in A$, if $a_0 \perp a_1$ and $a_0 \perp a_2$ then either $a_1 \perp a_2$ or $a_1 \preccurlyeq a_2$ or $a_2 \preccurlyeq a_1$.*

Definition 2 *We say that $A \subset \mathcal{P}$ fulfills condition $(T(\kappa))$ if for each $a \in A$ there is*

$$|\{x \in A : x \preccurlyeq a\}| < \kappa .$$

The following Ramsey-type theorem will be proved.

Theorem 1 *Let κ be a regular uncountable cardinal number. Let $(\mathcal{P}, \preccurlyeq)$ be a partially ordered set of cardinality κ and let $A \subset \mathcal{P}$ be a set of cardinality κ which fulfills conditions (I) and $(T(\kappa))$. Then either there exists a linked set $Z \subset A$ of cardinality κ or there exists a set $Z \subset A$ of cardinality κ consisting of pairwise incompatible elements.*

Proof. If there exists a linked subset $Z \subset A$ of cardinality κ, then the proof is complete. Suppose that each linked subset $Z \subset A$ has cardinality less than κ. Let $H_1 \subset A$ be an arbitrary maximal linked subset of A. Then $|H_1| < \kappa$. Let $z_1 \in H_1$ be an arbitrary element. Then the set $\{z_1\} \cup H_1$ is linked. Let us choose an arbitrary element $z_2 \in A \setminus H_1$. According to maximality of H_1 we have that $\{z_2\} \cup H_1$ is not linked. Let us choose an arbitrary maximal linked subset $H_2 \subset A \setminus H_1$ such that $z_2 \in H_2$. Then the set $\{z_2\} \cup H_2$ is linked.

Suppose that for some $\alpha < \kappa$ sequences $(\{z_\gamma\})_{\gamma<\alpha}$ and $(H_\gamma)_{\gamma<\alpha}$ and have been defined. Since κ is regular and $\alpha < \kappa$ hence

$$A \setminus \bigcup \{H_\gamma : \gamma < \alpha\} \neq \emptyset .$$

Let $z_\alpha \in A \setminus \bigcup \{H_\gamma : \gamma < \alpha\}$ be an arbitrary element. Obviously for each $\gamma < \alpha$ the set $\{z_\alpha\} \cup H_\gamma$ is not linked. Let us choose an arbitrary maximal linked subset $H_\alpha \subset A \setminus \bigcup \{H_\gamma : \gamma < \alpha\}$ such that $z_\alpha \in H_\alpha$. Thus the sequences $(\{z_\alpha\})_{\alpha<\kappa}$ and $(H_\alpha)_{\alpha<\kappa}$ have been defined. According to above construction the sets $\{z_\gamma\} \cup H_1$ are not linked for each $\gamma < \kappa$, i.e. there exists an element $a_\gamma \in H_1$ such that $a_\gamma \perp z_\gamma$. According to our assumption $|H_1| < \kappa$. Since κ is regular, there exists an element $a_{\gamma 0} \in H_1$ such that the set

$$W = \{z_\gamma : z_\gamma \perp a_{\gamma 0}\}$$

has cardinality κ. Since A fulfills condition (I) and $W \subset A$ hence W fulfills condition (I) too, (i.e. for each $z_\gamma, z_\delta \in W$ we have $z_\gamma \perp a_{\gamma 0}$, and from this it follows $z_\gamma \perp z_\delta$ or $z_\gamma \preccurlyeq z_\delta$ or $z_\delta \preccurlyeq z_\gamma$). Each linked set in A has cardinality less than κ, thus each chain in A, as a special kind of a linked set, has cardinality less than κ. Let $z(0)$ be an arbitrary element in W. Let $\mathscr{L}_0$ be a maximal chain in W such that $z(0) \in \mathscr{L}_0$. Consider a set

$$C_0 = \{s \in W\text{: there exists } z \in \mathscr{L}_0 \text{ such that } s \preccurlyeq z\}.$$

According to condition $(T(\kappa))$ and assumption that $|\mathscr{L}_0| < \kappa$ there is $|\mathscr{C}_0| < \kappa$ and $|W \setminus C_0| = \kappa$. For each $z \in W \setminus C_0$ there exists $s \in C_0$ such that $z \perp s$. Let us choose an arbitrary element $s_0 \in \mathscr{L}_0$. Let us observe that $z \perp s_0$ for each $z \in W \setminus C_0$.

If not, then according to condition (I) there is $z \preccurlyeq s_0$ or $s_0 \preccurlyeq z$. If $z \preccurlyeq s_0$, then $z \in C_0$. Contradiction. If $s_0 \preccurlyeq z$, then $z \in \mathscr{L}_0$ and then $z \in C_0$. Contradiction. Denote the set $W \setminus C_0$ by W_0.

Suppose that for some $\alpha < \kappa$ the sets $(C_\gamma)_{\gamma<\alpha}$, $(W_\gamma)_{\gamma<\alpha}$, $(\mathscr{L}_\gamma)_{\gamma<\alpha}$ and elements $(s_\gamma)_{\gamma<\alpha}$ such that

(1) $W_\gamma = W \setminus \bigcup_{\beta \le \gamma} C_\beta$
(2) $\mathscr{L}_\gamma \subset W_\gamma$,
(3) $s_\gamma \in \mathscr{L}_\gamma$,
(4) $z \perp s_\gamma$ for each $z \in W_\gamma$

have been defined.

Obviously, $|\bigcup_{\gamma<\alpha} C_\gamma| < \kappa$ (because $|C_\gamma| < \kappa$ for each $\gamma < \alpha$).
Hence $|W \setminus \bigcup_{\gamma<\alpha} C_\gamma| = \kappa$.

Let $z(\alpha)$ be an arbitrary element in $W \setminus \bigcup_{\gamma<\alpha} C_\gamma$. Let us observe that $z(\alpha) \perp s_\gamma$ for each $\gamma < \beta$. Let $\mathscr{L}_\alpha$ be a maximal chain in $W \setminus \bigcup_{\gamma<\alpha} C_\gamma$ such that $z(\alpha) \in \mathscr{L}_\alpha$. Consider a set

$$C_\alpha = \{s \in W \setminus \bigcup_{\gamma<\alpha} C_\gamma\text{: there exists } z \in \mathscr{L}_\alpha \text{ such that } s \preccurlyeq z\}.$$

According to condition $(T(\kappa))$ and assumption that $|\mathscr{L}_\alpha| < \kappa$ there is $|C_\alpha| < \kappa$. Hence $|W \setminus C_\alpha| = \kappa$. For each $z \in W \setminus C_\alpha$ there exists $s \in C_\alpha$ such that $z \perp s$.
Let us choose an arbitrary element $s_\alpha \in \mathscr{L}_\alpha$. Let us observe that $z \perp s_\alpha$ for each $z \in W \setminus \bigcup_{\gamma \leq \alpha} C_\gamma$
If not, then according to condition (I) there is $z \preccurlyeq s_\alpha$ or $s_\alpha \preccurlyeq z$. If $z \preccurlyeq s_\alpha$, then $z \in C_\alpha$. Contradiction. If $s_\alpha \preccurlyeq z$, then $z \in \mathscr{L}_\alpha$ and then $z \in C_\alpha$. Contradiction.

This way we defined

$$Z = \{s_\alpha : \alpha < \kappa\}$$

such that $s_\alpha \perp s_\beta$ for all $\alpha, \beta < \kappa$ with $a \neq \beta$. □

Definition 3 *We say that a family of sets $\mathscr{S}$ is linked if for all $A, B \in \mathscr{S}$ there is $A \cap B \neq \emptyset$.*

Definition 4 *We say that a family of sets $\mathscr{S}$ fulfills condition (I) if for all $S_0, S_1, S_2 \in \mathscr{S}$, if $S_0 \cap S_1 = \emptyset$ and $S_0 \cap S_2 = \emptyset$ then either $S_1 \cap S_2 = \emptyset$ or $S_1 \subset S_2$ or $S_2 \subset S_1$.*

Definition 5 *We say that a family of sets $\mathscr{S}$ fulfills condition $(T(\kappa))$ if for each set $U \in \mathscr{S}$ there is*

$$|\{V \in \mathscr{S} : V \subset U\}| < \kappa$$

Denote by $\wedge \mathscr{S}$ a family consisting of all finite intersections of subfamilies in $\mathscr{S}$.

Theorem 2 *Let κ be an uncountable regular cardinal number. Let $\mathscr{S}$ be a family of sets of cardinality greater than or equal to κ which fulfills conditions (I) and $(T(\kappa))$. Then either there exists a linked family $Z \subset \mathscr{S}$ of cardinality κ or there exists a family $Z \subset \mathscr{S}$ of cardinality κ consisting of pairwise disjoint sets.*

Proof. Let $\mathscr{S}$ be a family of sets which fulfills assumptions of our theorem. Without the loss of generality we can assume that $|\mathscr{S}| = \kappa$.
Let $\{A_\gamma : \gamma < \kappa\}$ well-order $\wedge \mathscr{S}$.
Since each of two sets are disjoint or have nonempty intersection we can define a partial ordering on a set

$$\mathscr{P} = \{\gamma < \kappa : A_\gamma \in \wedge \mathscr{S}\}.$$

by inclusion between elements of $\wedge \mathscr{S}$. Then

$$\gamma \text{ and } \beta \text{ are compatible iff } A_\gamma \cap A_\beta \neq \emptyset$$

and by analogy

$$\gamma \text{ and } \beta \text{ are not compatible iff } A_\gamma \cap A_\beta = \emptyset.$$

Let

$$A = \{\gamma \in \mathscr{P} : A_\gamma \in \mathscr{S}\}.$$

Since the family $\mathscr{S}$ fulfills conditions (I) and $(T(\kappa))$ hence A fulfills conditions (I) and $(T(\kappa))$ too. According to theorem 1 we receive our claim. □

Let $\mathscr{S}$ be a family of sets. Then

$$c(\mathscr{S}) = sup\{|A|: A \text{ is a cellular family in } \mathscr{S}\} + \omega .$$

Corollary 1 *Let κ be an uncountable regular cardinal number. Let $\mathscr{S}$ be a family of sets of cardinality greater than or equal to κ which fulfills conditions (I) and $(T(\kappa))$. If $c(\wedge \mathscr{S}) < \kappa$, then for each family $\mathscr{A} \subset \mathscr{S}$ of regular cardinality $|A| > c(\wedge \mathscr{S})$ there exists $\mathscr{L} \subset \mathscr{A}$ which is linked and $|\mathscr{L}| = |\mathscr{A}|$.*

Proof. Let $\mathscr{S}$ be a family of sets which fulfills conditions (I) and $(T(\kappa))$ and let $\mathscr{A} \subset \mathscr{S}$ be a subfamily of $\mathscr{S}$. Then $\mathscr{A}$ fulfills conditions (I) and $(T(\kappa))$ too. Hence by theorem 2 there exists a linked subfamily $\mathscr{L} \subset \mathscr{A}$ such that $|\mathscr{L}| = |\mathscr{A}|$. □

Now we will be considering weakly independent families (compare [7]). Their existence is determined by a mapping from topological space onto the Cantor cube.

Definition 6 *A family $\{(A^0_\xi, A^1_\xi) : \xi < \alpha\}$ of ordered pairs of subsets of X such that $A^0_\xi \cap A^1_\xi = \emptyset$ for $\xi < \alpha$ is called a weakly independent family (of length α) if for each $\xi, \zeta < \alpha$ with $\xi \neq \zeta$ we have $A^i_\xi \cap A^j_\zeta \neq \emptyset$,* where $i,j \in \{0,1\}$.

Theorem 3 *Let $\mathscr{S}$ be a family of sets which has the following properties:*
(i) $\mathscr{S}$ fulfills condition (I);
(ii) $\mathscr{S}$ fulfills condition $(T(\kappa))$;
(iii) for each $U \in \mathscr{S}$ there is $X \setminus U \in \mathscr{S}$.
Then for each regular cardinal number κ such that $|\mathscr{S}| \geq \kappa > c(\mathscr{S})$ there exists a weakly independent family in $\mathscr{S}$ of cardinality κ.

Proof. (compare [9]). Let $\mathscr{S}$ be a family of sets which has the properties *(i)–(iii)*. According to theorem 2 there exists a linked family $\mathscr{B} \subset \mathscr{S}$ such that $|\mathscr{B}| = \kappa$. Consider a family

$$\mathscr{C} = \{X \setminus B \in \mathscr{S} : B \in \mathscr{B}).$$

Notice that the family $\mathscr{C}$ has cardinality κ (because $|\mathscr{B}| = \kappa$). According to theorem 2 there exists a linked subfamily $\mathscr{C}' \subset \mathscr{C}$ of cardinality κ. Denote $\mathscr{A}_1 = \{B \in \mathscr{B} : X \setminus B \in \mathscr{C}'\}$ and $\mathscr{B}_1 = \mathscr{C}'$. Let order the family $\mathscr{A}_1$. This way the family $\mathscr{B}_1$ is ordered and

(1) $A_\gamma \cap B_\gamma = \emptyset$ for all $A_\gamma \in \mathscr{A}_1$ and $B_\gamma \in \mathscr{B}_1$;

(2) $B_\gamma = X \setminus A_\gamma$ for all $\gamma < \kappa$.

Let (A_1, B_1) be the first pair of a weakly independent family.

Consider the sets

$$U_1 = \{B_\alpha \in \mathscr{B}_1 : A_1 \cap B_\alpha = \emptyset \text{ and } \alpha \neq 1\} = \{B_\alpha \in \mathscr{B}_1 : B_\alpha \subset X \setminus A_1 \text{ and } \alpha \neq 1\}$$

and

$$V_1 = \{A_\alpha \in \mathscr{A}_1 : A_\alpha \cap B_1 = \emptyset \text{ and } \alpha \neq 1\} = \{A_\alpha \in \mathscr{A}_1 : A_\alpha \subset X \setminus B_1 \text{ and } \alpha \neq 1\}.$$

Denote $U_1' = \{\alpha < \kappa : B_\alpha \in U_1\}$ and $V_1' = \{\alpha < \kappa : A_\alpha \in V_1\}$. According to condition $(T(\kappa))$ there is $|U_1| < \kappa$ and $|V_1| < \kappa$. Thus $|U_1'| < \kappa$ and $|V_1'| < \kappa$.

Suppose that for some $\beta < \kappa$ a weakly independent family

$$\{(A_\alpha, B_\alpha) : \alpha < \beta\}$$

and the families $\mathscr{A}_\alpha, \mathscr{B}_\alpha$ have been defined. For each $\alpha < \beta$ and for each selector i_α defined on the weakly independent family $\{(A_\xi, B_\xi) : \xi < \alpha\}$ families

$$i_\alpha\{(A_\xi, B_\xi) : \xi < \alpha\} \cup \mathscr{A}_\alpha$$

and

$$i_\alpha\{(A_\xi, B_\xi) : \xi < \alpha\} \cup \mathscr{B}_\alpha$$

are linked.

For each $\alpha < \beta$ consider the sets

$$U_\alpha = \{B_\xi \in \mathscr{B}_\alpha : A_\alpha \cap B_\xi = \emptyset \text{ and } \xi \neq \alpha\} = \{B_\xi \in \mathscr{B}_\alpha : B_\xi \subset X \setminus A_\alpha \text{ and } \xi \neq \alpha\}$$

and

$$V_\alpha = \{A_\xi \in \mathscr{A}_\alpha : A_\xi \cap B_\alpha = \emptyset \text{ and } \xi \neq \alpha\} = \{A_\xi \in \mathscr{A}_\alpha : A_\xi \subset X \setminus B_\alpha \text{ and } \xi \neq \alpha\}.$$

Denote $U_\alpha' = \{\xi < \kappa : B_\xi \in U_\alpha\}$ and $V_\alpha' = \{\xi < \kappa : A_\xi \in V_\alpha\}$.

Consider a set

$$T = \kappa \setminus \bigcup\{U_\alpha' \cup V_\alpha' : \alpha < \beta\}.$$

Since $|U_\alpha'| < \kappa$ and $|V_\alpha'| < \kappa$ for each $\alpha < \beta$, hence

$$|\bigcup\{U_\alpha' \cup V_\alpha' : \alpha < \beta\}| < \kappa$$

and then

$$|T| = |\kappa \setminus \bigcup\{U_\alpha' \cup V_\alpha' : \alpha < \beta\}| = \kappa.$$

Consider families

$$\mathscr{A}_\beta = \{A_\alpha \in \mathscr{A}_1 : \alpha \in T\}$$

and

$$\mathscr{B}_\beta = \{B_\alpha \in \mathscr{B}_1 : \alpha \in T\}.$$

Let take the smallest $\alpha \in T$; name it β. Then the pair (A_β, B_β) is the next pair of a weakly independent family. The proof is complete. □

Definition 7 *A family $\{(A_\xi, B_\xi) : \xi < \alpha\}$ of ordered pairs of subsets of X, such that $A_\xi \cap B_\xi = \emptyset$ for $\xi < \alpha$ is called an independent family (of length α) if for each finite subset $F \subset \alpha$ and each function i: $F \to \{-1, +1\}$ we have*

$$\bigcap\{i(\xi) A_\xi : \xi \in F\} \neq \emptyset$$

(where $(+1)A_\xi = A_\xi, (-1)A_\xi = B_\xi$). (compare [7]).

Definition 8 *A family of sets $\mathcal{S}$ is said to be binary if for each finite subfamily $\mathcal{M} \subset \mathcal{S}$ with $\bigcap \mathcal{M} = \emptyset$ there exist $A, B \in \mathcal{M}$ such that $A \cap B = \emptyset$.* (compare [4]).

Corollary 2 *Let X be a compact zero-dimensional space. Let $\mathcal{S}$ be a family consisting of clopen sets which has the following properties:*
(i) $\mathcal{S}$ is a binary family;
(ii) $\mathcal{S}$ fulfills condition (I);
(iii) $\mathcal{S}$ fulfills condition $(T(\kappa))$;
(iv) for each $U \in \mathcal{S}$ the set $X \setminus U \in \mathcal{S}$.
Then for each regular cardinal number κ such that $|\mathcal{S}| \geq \kappa > c(\mathcal{S})$ there exists an independent family in $\mathcal{S}$ of cardinality κ.

Proof. Let us notice that if in theorem 3 we assume that the family $\mathcal{S}$ is moreover binary we obtain a family of cardinality κ which is independent family in the sense of definition 7. □

Let $\{X_\alpha : \alpha \in J\}$ be a family of topological spaces with subbases $S(X_\alpha) \subset \mathcal{P}(X_\alpha)$; $\alpha \in J$. Then by

$$X = \prod \{X_\alpha : \alpha \in J\}$$

we denote the Cartesian product of topological spaces X_α; $\alpha \in J$.

Let $\pi_\alpha : X \to X_\alpha$ be a projection on α-axis. Let

$$\{\pi_\alpha^{-1}(U) : U \in S(X_\alpha);\ \alpha \in J\}$$

be a canonical subbase of X.

The proof of theorem below is obvious.

Theorem 4 *Let $\{X_\alpha : \alpha \in J\}$ be a family of topological spaces and let $S(X_\alpha) \subset \mathcal{P}(X_\alpha)$ be subbases of X_α for $\alpha \in J$ which fulfill conditions (I) and $(T(\kappa))$. Then the canonical subbase*

$$S(X) = \{\pi_\alpha^{-1}(U) : U \in S(X_\alpha);\ \alpha \in J\}$$

of the Cartesian product $X = \prod \{X_\alpha : \alpha \in J\}$ fulfills conditions (I) and $(T(\kappa))$.

Let $\{0,1\}$ be a two-point set with a subbase consisting of two one-point sets $\{0\}$ and $\{1\}$. Obviously the subbase fulfills conditions (I) and $(T(\kappa))$.
Let T be an infinite set. Denote *the Cantor cube* by

$$D^T = \{p : p : T \to \{0,1\}\}.$$

For $s \subset T$, $i : s \to \{0,1\}$ we will use the following notation

$$H_s^i = \{p \in D^T : p|s = i\}.$$

The family

$$\mathcal{B} = \{H_s^i : s \subset T,\ |s| < \omega \text{ and } i \in \{0,1\}^s\}$$

is a canonical base for the Cantor cube and the family

$$\mathscr{P} = \{H_s^i : s \subset T, |s| = 1 \text{ and } i \in \{0,1\}^s\}$$

is a canonical subbase for topology on the Cantor cube.

Corollary 3 *A canonical subbase of the Cantor cube D^κ fulfills conditions (I) and $(T(\kappa))$.*

Proof. According to theorem 4 and previous remark we receive our claim. □

Let observe that

Theorem 5 *Let X be a space for which there exists a surjection*

$$f : X \to D^\kappa$$

where D^κ is the Cantor cube. Let $\mathscr{P}$ be the canonical subbase of D^κ and let

$$\mathscr{H} = \{f^{-1}(H_s^i) : H_s^i \in \mathscr{P}\}.$$

Then the family $\mathscr{H}$ fulfills conditions (I) and $(T(\kappa))$.
Moreover the family $\mathscr{H}$ has the following properties:
(i) $\mathscr{H}$ is a binary family;
(ii) for each $U \in \mathscr{H}$ the set $X \setminus U \in \mathscr{H}$

Now the main theorem is an easy consequence of the previous theorems.

Corollary 4 *A compact zero-dimensional space X can be mapped onto the generalized Cantor discontinuum D^κ if and only if there exists a binary family $\mathscr{S}$, (where $|\mathscr{S}| \geq \kappa > c(\mathscr{S})$) consisting of clopen subsets of X closed with respect to complements which fulfills conditions (I) and $(T(\kappa))$.*

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