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COAREA INTEGRATION IN METRIC SPACES

Jan Malý

Abstract. Let X be a metric space with a doubling measure, Y be a boundedly compact metric space and $u : X \to Y$ be a Lebesgue precise mapping whose upper gradient g belongs to the Lorentz space $L_{m,1}$, $m \geq 1$. Let $E \subset X$ be a set of measure zero. Then $\widehat{\mathcal{H}}_m(E \cap u^{-1}(y)) = 0$ for $\mathcal{H}_m$-a.e. $y \in Y$, where $\mathcal{H}_m$ is the m-dimensional Hausdorff measure and $\widehat{\mathcal{H}}_m$ is the m-codimensional Hausdorff measure. This property is closely related to the coarea formula and implies a version of the Eilenberg inequality. The result relies on estimates of Hausdorff content of level sets of mappings between metric spaces and analysis of their Lebesgue points. Adapted versions of the Frostman lemma and of the Muckenhoupt-Wheeden inequality appear as essential tools.

Contents

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1. Introduction

The Federer's coarea formula is a common generalization of the formula on change of variables in integral and of the Fubini theorem. Suppose that we integrate a non-negative measurable function ω on an open set $\Omega \subset \mathbb{R}^n$ through a transformation of variables represented by a mapping $u : \Omega \to \mathbb{R}^d$. The formula gives us a chance to integrate first over the level sets $u^{-1}(y)$, $y \in \mathbb{R}^d$, and then conclude the operation by integration over y. If either $n = d$ or the range of u is m-dimensional, then under some assumptions on the quality of u we can expect the coarea formula in the form

$$\int_{\mathbb{R}^d} \left(\int_{u^{-1}(y)} \omega(x) \, d\mathcal{H}_{n-m}(x) \right) d\mathcal{H}_m(y) = \int_\Omega \omega(x) |J_m u(x)| \, dx. \tag{1.1}$$

Here $J_m u$ is the $\left[\binom{n}{m}\binom{d}{m}\right]$-tuple of all $m \times m$ minors of the Jacobi matrix of u and $\mathcal{H}_s$ is the s-dimensional Hausdorff measure.

Recently, some new results on coarea formula for Sobolev transformations and fine properties of Sobolev functions have been obtained by J. Malý, D. Swanson and W. P. Ziemer in [MSZ1], [MSZ2] and [M3]. The main goal of this article is to present these results and to show a generalization to metric spaces.

In Section 2, we discuss the coarea formula and the Eilenberg inequality for mappings between Euclidean spaces. The main result there, Theorem 2.6, is reduced to verification of the so-called coarea property. This is done in the remaining sections. Starting from Section 3, all is done in the generality of metric spaces equipped with a doubling measure. With the aid of a version of the Frostman lemma (Section 6), we estimate the Hausdorff content of level sets of potentials of Riesz type (Section 7). This is, in fact, a version of the relationship between Hausdorff content and Sobolev-Lorentz capacity. In Sections 8–13, properties of functions with integrable upper gradients are studied. We prove a kind of the inequality between the Hausdorff content and $W^{1,1}$-capacity (Section 10), existence of Lebesgue points outside a set of null Hausdorff measure (Section 11), and the coarea property needed in the proof of the coarea formula (Section 12). Although the statement (1.1) of the coarea formula does not seem to give a sense in the very generality of metric spaces, the Eilenberg-type inequality in Section 13 shows that also in this abstract setting some interesting results can be achieved.

2. Results in the Euclidean setting

In this section we review some results on area and coarea for mappings between Euclidean spaces. Some notions used already here (like Hausdorff measure, Lorentz spaces) are explained in Section 3 in the setting of metric spaces. The n-dimensional Lebesgue measure is denoted by $\mathcal{L}_n$.

First, we list H. FEDERER's result for Lipschitz transformations. The case $m = n$ of (1.1) is known as the *area formula* [F, 3.2.3]:

Theorem 2.1. *Let $\Omega \subset \mathbb{R}^n$ be an open set, $E \subset \Omega$ be a measurable set and $u : \Omega \to \mathbb{R}^d$ be a Lipschitz function. Let $\omega : \Omega \to \mathbb{R}$ be a measurable function. Then*

$$\int_{\mathbb{R}^d} \Bigl(\sum_{E \cap u^{-1}(y)} \omega(x) \Bigr) d\mathcal{H}_n(y) = \int_E \omega(x) |J_n u(x)| \, dx,$$

provided the integral on the right makes sense.

Another important case, namely the *coarea formula* in the narrow sense, is $m = d$, see [F, 3.2.12].

Theorem 2.2. *Let $\Omega \subset \mathbb{R}^n$ be an open set, $E \subset \Omega$ be a measurable set and $u : \Omega \to \mathbb{R}^d$ be a Lipschitz function. Let $\omega : \Omega \to \mathbb{R}$ be a measurable function. Then*

$$\int_{\mathbb{R}^d} \Bigl(\int_{E \cap u^{-1}(y)} \omega(x) \, d\mathcal{H}_{n-d}(x) \Bigr) dy = \int_E \omega(x) |J_d u(x)| \, dx,$$

provided the integral on the right makes sense.

If $m < \min\{n, d\}$, then the formula (1.1) breaks down. If, for example, $n = d$ and u is the identity mapping, then the point preimages are single points, so that the integral on the left-hand side is zero. On the right-hand side we integrate $\binom{n}{m} \omega(x)$. One might think that the effect is due to the fact that the rank of the Jacobi matrix is bigger than m. Therefore we present a bit more sophisticated example.

E x a m p l e 2.3. There exists a $\mathcal{C}^1$ function $u : \mathbb{R}^2 \to \mathbb{R}^2$ and a measurable set $E \subset \mathbb{R}^2$ of positive measure such that $J_2 u = 0$ on E, $J_1 u = 1$ on E and u is one-to-one on E. Hence

$$\int_{\mathbb{R}^2} \mathcal{H}_1\bigl(E \cap u^{-1}(y)\bigr) \, dy = 0 < \mathcal{L}_2(E) = \int_E |J_1 u(x)| \, dx.$$

C o n s t r u c t i o n. Let D be a discontinuum of positive measure in $[0, 1]$, $E = D \times [0, 1]$ and $g : \mathbb{R} \to \mathbb{R}$ be a continuous function which is strictly positive on $(0, 1) \setminus D$ and vanishing elsewhere. Let

$$v(t) = \int_{-\infty}^t g(s) \, ds, \quad u(x) = \bigl(v(x_1), x_2\bigr).$$

Then u is one-to-one on E and thus the point preimages are points, which makes the left-hand side to be 0. For the right-hand side, we notice that $J_1 u = 1$ on E.

The coarea formula (1.1) remains valid if the range of u is, in some suitable sense, m-dimensional, e.g. if $u(\Omega)$ is $\mathcal{H}_m$-rectifiable, see [F, 3.2.22]. We shall not pursue this direction. We shall concentrate on the inequality which is preserved even in the case when any of m-dimensionality on the image fails. If we replace $|J_m u|$ by $|\nabla u|^m$, we even do not need m to be an integer. The next theorem is a version of Eilenberg's inequality [E]. The general case is due to H. Federer, [F, 2.10.25, 2.10.26]. We write

$$\boldsymbol{\alpha}(s) = \frac{\pi^{s/2}}{\Gamma(\frac{s}{2}+1)}.$$

Theorem 2.4. *Let $\Omega \subset \mathbb{R}^n$ be an open set, $E \subset \Omega$ be a measurable set and $u : \Omega \to \mathbb{R}^d$ be a Lipschitz function. Let $\omega : \Omega \to \mathbb{R}$ be a non-negative measurable function. Suppose that $1 \le m \le n$ is a real number. Then*

$$\begin{aligned}\int_{\mathbb{R}^d} \left(\int_{E \cap u^{-1}(y)} \omega(x) \mathcal{H}_{n-m}(x) \right) d\mathcal{H}_m(y) \\ \le \frac{\boldsymbol{\alpha}(n-m)\boldsymbol{\alpha}(m)}{\boldsymbol{\alpha}(n)} \int_E \omega(x) |\nabla u(x)|^m \, dx.\end{aligned} \tag{2.1}$$

In this lecture we are interested in validity of (1.1) or (2.1) for Sobolev transformations of variables. It is well known that the problem can be reduced to analysis of Lebesgue null sets E.

For $m = n$ this leads to the so-called Lusin N-property: if $\mathcal{L}_n(E) = 0$, then $\mathcal{H}_n(u(E)) = 0$. We shall consider a more general version, which fits also to $m < n$.

Let m be a real number, $1 \le m \le n$. We say that a Sobolev mapping $u : \Omega \to \mathbb{R}^d$ satisfies the *m-coarea property* in Ω if for every Lebesgue null set $E \subset \Omega$ and $\mathcal{H}_m$-almost every $y \in \mathbb{R}^d$ we have $\mathcal{H}_{n-m}(E \cap u^{-1}(y)) = 0$.

The following theorem is still an easy consequence of results in [F] although it is not explicitly written there. See also [Ha].

Theorem 2.5. *Let $\Omega \subset \mathbb{R}^n$ be an open set, $E \subset \Omega$ be a measurable set and $u : \Omega \to \mathbb{R}^d$ be a Sobolev function satisfying the m-coarea property. Let $1 \le m \le n$. Let $\omega : \Omega \to \mathbb{R}$ be a non-negative measurable function. Then the following assertions are true.*

(a) *The Eilenberg-type inequality* (2.1) *is valid.*
(b) *If $m = \min\{n, d\}$, then the area or coarea (respectively) formula* (1.1) *is valid.*

Proof. It is enough to investigate the case when ω is a characteristic function. By [F, Thm. 3.1.8], there is a sequence u_j of Lipschitz mappings of $\mathbb{R}^n$ to $\mathbb{R}^d$ such that $u_j = u$ and $\nabla u_j = \nabla u$ a.e. in Ω. Since each formula holds for Lipschitz mappings, it remains to consider sets of measure zero. However, the claim for $\omega = \chi_E$ with Lebesgue null set E is exactly the m-coarea property. $\square$

Theorem 2.5, though very general, does not give the final answer to the question of validity of coarea formula and other results on change of variables for Sobolev functions. It only converts the problem to verification of the most delicate case, namely that of null sets. Now, let us discuss the final question, namely, if u is a well represented Sobolev function and ∇u is in a function space in consideration, when this implies that the formulas on change of variables hold.

We really need a restriction on representatives, since even very regular Sobolev functions may loose their good properties for change of variables if we modify them on a null set. Therefore we assume that they are as well represented as possible. We say that a measurable function $u : \Omega \to \mathbb{R}^d$ is *Lebesgue represented* if it does not have "removable singularities" of the type that a point would become Lebesgue when correcting the value at it. In the following discussion we tacitly assume that functions are Lebesgue represented; this, for example, implies that elements of $W^{1,p}$ are continuous for $p > n$.

The area formula for Sobolev spaces $W^{1,p}$, $p > n$, was established by M. Marcus and V. J. Mizel [MMi]. We cannot pass to the borderline case $p = n$. The counterexample is due to L. Cesari [Ce], his example was further adapted and generalized by O. Martio and J. Malý [MM], J. Kauhanen, P. Koskela and J. Malý [KKM] and P. Hajłasz [Ha] to demonstrate sharpness of all further discussed results on area and coarea formulas. It is shown in [KKM] that the area formula holds for Sobolev mappings with gradient in the Lorentz space $L_{n,1}$. This result is the best possible in the class of rearrangement invariant spaces, see also [M1].

The coarea formula for scalar $W^{1,1}$-functions is due to H. Federer [F, 4.5.9 (14)]. In $W^{1,p}$ spaces, $p > n$, it was obtained by R. Van der Putten [VP]. The correct borderline exponent is, however, m. The range $p > m$ has been reached by P. Hajłasz [Ha] for measuring level sets by the integral geometric measure, and the final statement with the Hausdorff measure and the Lorentz space $L_{m,1}$ (for the gradient) has been established by J. Malý, D. Swanson and W. P. Ziemer [MSZ1]. The "Eilenberg part" has been added in [M3]. The presentation of the following theorem in this paper is self-contained in the sense that the proof, based on the results of later sections (namely on the m-coarea property in Section 12), is given here. The development here is not the mere translation of previous proofs to metric spaces, the methods used now are essentially different.

Theorem 2.6. *Let* $1 \le m \le n$. *Let* $\Omega \subset \mathbb{R}^n$ *be an open set,* $E \subset \Omega$ *be a measurable set and* $u : \Omega \to \mathbb{R}^d$ *be a Lebesgue represented Sobolev function with* $\nabla u \in L_{m,1}$. *Let* $\omega : \Omega \to \mathbb{R}$ *be a non-negative measurable function. Then the following assertions are true.*

(a) *The Eilenberg-type inequality* (2.1) *is valid.*
(b) *If* $m = \min\{n, d\}$, *then the area or coarea (respectively) formula* (1.1) *is valid.*

Proof. From Theorem 2.5 we see that it is enough to verify the m-coarea property. This is done below in Theorem 12.3. □

We can prove the m-coarea property with all consequences for a mapping $u \in W^{1,m}$ if u is Hölder continuous. Such a result was first proved in the area case by J. Malý and O. Martio [MM], the coarea case based on estimates by S. Hencl and J. Malý [HM] has been established by J. Malý, D. Swanson and W. P. Ziemer [MSZ1]. See also [M3].

Finally, let us note that another approach to the coarea formula is based on the BV theory. A classical version is due to W. H. Fleming and R. Rishel [FR], for new developments see R. L. Jerrard and H. M. Soner [JS]. Following this direction, we obtain a weak formulation of the result for a very general class of transformations.

3. Setting in metric spaces: preliminaries

The main goal of the rest of this paper is to establish estimates of level sets and coarea properties in the setting of metric spaces. This enables applications to weighted spaces, manifolds, Carnot-Carathéodory spaces etc. and simultaneously demonstrates that the structure of metric spaces with doubling measures is the only requirement for this type of results.

We refer to [HaK], [GGKK], [He], [AT] for introduction to the analysis on metric spaces and historical remarks.

We consider a metric space (X, d_X) with a doubling measure $\boldsymbol{m}$, i.e., we assume that $\boldsymbol{m}$ is a Borel measure and there is a constant D such that

$$\boldsymbol{m}(B(x,2r)) \le D\,\boldsymbol{m}(B(x,r)) \tag{3.1}$$

for every ball $B(x,r)$ in X. A *measure* here means an outer Borel-regular measure, if it is in addition locally finite, we call it a Borel measure.

Notice that any doubling measure is sigma-finite and forces the space to be separable.

We shall work with *Riesz potentials of measures*,

$$I_\alpha^R \mu(x) = \int_0^R \frac{\mu(B(x,r))}{\boldsymbol{m}(B(x,r))}\, dr^\alpha.$$

The Riesz potential of a measure with density g with respect to $\boldsymbol{m}$ (i.e. $d\mu = g\, d\boldsymbol{m}$) is labeled as $I_\alpha g$.

We also consider the *fractional maximal operator*

$$M_\alpha^R \mu(x) = \sup_{0<r<R} \frac{r^\alpha \mu(B(x,r))}{\boldsymbol{m}(B(x,r))}.$$

The *m-dimensional Hausdorff measure* $\mathcal{H}_m$ on a metric space Y is defined by

$$\mathcal{H}_m(E) = \lim_{\delta\to 0} \mathcal{H}_m^\delta(E),$$

where

$$\mathcal{H}_m^\delta(E) = 2^{-m}\boldsymbol{\alpha}_m \inf\Big\{\sum \operatorname{diam}^m(E_j) : \operatorname{diam}(E_j) \le \delta,\ E \subset \bigcup_j E_j\Big\},$$

$$\boldsymbol{\alpha}_m = \frac{\pi^{1/2}}{\Gamma(\frac{m}{2}+1)}.$$

We shall also use the *spherical Hausdorff measure of codimension q* on X:

$$\widehat{\mathcal{H}}_q^\delta(E) = \inf\Big\{\sum r_j^{-q}\boldsymbol{m}(B(x_j,r_j)) : r_j \le \delta,\ E \subset \bigcup_j B(x_j,r_j)\Big\}.$$

Again, the limiting process $\delta \to 0$ gives a Borel measure labeled as $\widehat{\mathcal{H}}_q$.

Note that a *Vitali-type covering theorem* in metric spaces is available, namely, from a given family of balls with an upper bound for radii covering a set $E \subset X$ we can select a pairwise disjoint subfamily $\{B(x_j,r_j)\}$ such that

$$E \subset \bigcup_j B(x_j, 5r_j),$$

see e.g. [F], [He], [HaK], [Z]. Since we work in separable spaces, the selected subfamily is always countable (= finite or countably infinite).

In what follows, C will denote a generic constant which can change from expression to expression; the dependence of C on various entries will be indicated in statements.

4. Lorentz spaces

The *distribution function* of a measurable function $f : X \to \mathbb{R}$ is

$$s \mapsto \boldsymbol{m}(\{|f| > s\}), \quad s > 0.$$

The "generalized inverse" of the distribution function is called the *non-increasing rearrangement* of f, it is defined by

$$f^*(t) = \inf\{s > 0 : \boldsymbol{m}(\{|f| > s\}) \le t\}, \quad t > 0.$$

Its fundamental property is

$$\mathcal{L}^1(\{f^* > s\}) = \boldsymbol{m}(\{|f| > s\}), \quad s > 0.$$

Notice that

$$\begin{aligned} \Gamma_p(f) &:= \big\{[r,s] \in \mathbb{R}^2_+ : f^*(r^p) < s\big\} \\ &= \big\{[r,s] \in \mathbb{R}^2_+ : \boldsymbol{m}(\{|f| > s\}) < r^p\big\}. \end{aligned} \tag{4.1}$$

We need the Lorentz spaces $L_{p,1}$ and $L_{p,\infty}$ with "norms"

$$\|f\|_{L_{p,1}} = \int_0^\infty f^*(t)\, d(t^{1/p}),$$

and

$$\|f\|_{L_{p,\infty}} = \sup_{t>0} t^{1/p} f^*(t).$$

These expressions can be rewritten as

$$\|f\|_{L_{p,1}} = \int_0^\infty \boldsymbol{m}(\{|f| > s\})^{1/p}\, ds, \tag{4.2}$$

and

$$\|f\|_{L_{p,\infty}} = \sup_{s>0} s\, \boldsymbol{m}(\{|f| > s\})^{1/p}. \tag{4.3}$$

Indeed, (4.3) is the supremum of measures of rectangles contained in $\Gamma_p(f)$ whereas (4.2) is the two-dimensional Lebesgue measure of $\Gamma_p(f)$, and the equivalence of the expressions follows from (4.1).

Whereas $\|\cdot\|_{L_{p,1}}$ is actually a norm, $\|\cdot\|_{L_{p,\infty}}$ is only equivalent to a norm if $p > 1$ (and for $p = 1$ the situation is even worse). The space $L_{p,\infty}$ is also called the weak L^p space or the Marcinkiewicz space. Let us notice that $L_{1,1} = L^1$.

The only fundamental property that we need is the duality

$$\int_X fg\, d\boldsymbol{m} \le \|f\|_{L_{m',\infty}} \|f\|_{L_{m,1}} \tag{4.4}$$

when $m, m' > 1$ are conjugated exponents, i.e. $\frac{1}{m} + \frac{1}{m'} = 1$, see e.g. [BS].

Lemma 4.1. *Suppose that E_j are pairwise disjoint Borel subsets of X and $f \in L_{m,1}(X)$. Then*

$$\sum_j \|f\chi_{E_j}\|^m_{L_{m,1}} \le \|f\|^m_{L_{m,1}}.$$

P r o o f. Let η be the distribution function of f and η_j be the distribution functions of $f\chi_j$. Then

$$\sum_j \eta_j \le \eta.$$

Let $S = \inf\{s > 0 : \eta(s) = 0\}$. ($S = \infty$ if η is strictly positive everywhere.) Hölder's inequality yields

$$\left(\int_0^S \eta_j^{\frac{1}{m}}(s)\,ds\right)^m \le \left(\int_0^S \eta_j(s)\eta^{\frac{1}{m}-1}(s)\,ds\right)\left(\int_0^S \eta^{\frac{1}{m}}(s)\,ds\right)^{m-1}$$

for every $j \in \mathbb{N}$. Summing over j, we obtain

$$\begin{aligned}\sum_j \|f\chi_{E_j}\|^m_{L_{m,1}} &= \sum_j \left(\int_0^S \eta_j^{\frac{1}{m}}(s)\,ds\right)^m \\ &\le \left(\int_0^S \eta^{\frac{1}{m}}(s)\,ds\right)^{m-1} \sum_j \left(\int_0^S \eta_j(s)\eta^{\frac{1}{m}-1}(s)\,ds\right) \\ &\le \left(\int_0^S \eta^{\frac{1}{m}}(s)\,ds\right)^m = \|f\|^m_{L_{m,1}}.\end{aligned}$$

□

5. Riesz potentials

The Riesz potentials studied here are a version of Riesz potentials from [HaK], see also [MMo], [MP].

Definition 5.1 (Whitney ball, Whitney covering). Let $R > 0$ be fixed. Let $G \subset X$ be an open set. We say that $B = B(z, r)$ is a *Whitney ball* for G constrained by R if

$$r = \min\left\{\frac{1}{2}\operatorname{dist}(z, X \setminus G), R\right\}.$$

A *Whitney covering* of G constrained by R is such a covering $\mathcal{W}$ of G by Whitney balls for G constrained by R that the balls $\{B(z, r/5) : B(z, r) \in \mathcal{W}\}$

are pairwise disjoint. The existence of a Whitney covering follows from the Vitali-type covering theorem. Every Whitney covering has its overlap multiplicity bounded by a constant depending only on the doubling constant of $\boldsymbol{m}$.

Whitney balls are a powerful replacement of Whitney cubes; evidently the latter ones are not available in metric spaces. The idea comes from R. R. Coifman and G. Weiss [CW].

The Riesz kernels in Euclidean spaces are symmetric. In our generality we have the following.

Lemma 5.2. *Suppose that μ and ν are measures on X. Then*

$$\int_X I_\alpha^R \mu(x)\, d\nu(x) \le D \int_X I_\alpha^R \nu(y)\, d\mu(y),$$

where D is the doubling constant of $\boldsymbol{m}$.

Proof. For $y \in B(x,r)$ we have

$$\boldsymbol{m}(B(y,r)) \le \boldsymbol{m}(B(x,2r)) \le D\, \boldsymbol{m}(B(x,r)).$$

Hence we have

$$\begin{aligned}
&\int_X \left(\int_0^R \frac{\mu(B(x,r))}{\boldsymbol{m}(B(x,r))}\, dr^\alpha \right) d\nu(x) \\
&\quad = \int_0^R \left(\int_X \left(\int_{B(x,r)} \frac{d\mu(y)}{\boldsymbol{m}(B(x,r))} \right) d\nu(x) \right) dr^\alpha \\
&\quad \le D \int_0^R \left(\int_X \left(\int_{B(y,r)} \frac{d\nu(x)}{\boldsymbol{m}(B(y,r))} \right) d\mu(y) \right) dr^\alpha \\
&\quad = D \int_X \left(\int_0^R \left(\int_{B(y,r)} \frac{d\nu(x)}{\boldsymbol{m}(B(y,r))}\, dr^\alpha \right) d\mu(y) \right) \\
&\quad = D \int_X I_\alpha^R \nu(y)\, d\mu(y).
\end{aligned}$$

□

The main result of this section is the following "good lambda inequality". In the Euclidean setting, it is due to Muckenhoupt and Wheeden [MW]; in this generality it is done in [Ho]. The method of good lambda inequalities has been invented by D. L. Burkholder and R. F. Gundy [BG].

Theorem 5.3. *Let $\alpha > 0$ and $\varepsilon > 0$. Then there exist $a = a(D, \alpha) > 0$ and $\sigma = \sigma(\varepsilon, D, \alpha) > 0$, where D is the doubling constant from* (3.1)*, such that, for every measure μ on X,*

$$\boldsymbol{m}\big(\{I_\alpha^R \mu \geq a\lambda\}\big) \leq \varepsilon\, \boldsymbol{m}\big(\{I_\alpha^R \mu \geq \lambda\}\big) + \boldsymbol{m}\big(\{M_\alpha^R \mu \geq \sigma\, \lambda\}\big).$$

Proof. Set

$$a = 2^{2-\alpha} D^2. \tag{5.1}$$

Denote

$$G = \{I_\alpha^R \mu > \lambda\}, \quad G^a = \{I_\alpha^R \mu > a\lambda\}.$$

Obviously, G, G^a are open sets. Let $B = B(z, r)$ be a Whitney ball for G constrained by $R/3$. We claim that

$$\boldsymbol{m}\big(B \cap G^a\big) \leq \varepsilon\, \boldsymbol{m}(B) + \boldsymbol{m}\big(B \cap \{M_\alpha^R \mu \geq \sigma(\varepsilon)\lambda\}\big). \tag{5.2}$$

The claim clearly holds if $M_\alpha^R \mu \geq \lambda$ on B. Hence we assume that there is $z' \in B$ such that

$$M_\alpha^R \mu(z') \leq \lambda.$$

Now we choose $\delta = \delta(\varepsilon, D, \alpha) \in (0, 1)$ to be determined later and denote

$$E = B \cap \Big\{I_\alpha^{\delta r} \mu > \frac{a\lambda}{2}\Big\}.$$

Let ν be $\boldsymbol{m}$ restricted to B. Since $B(x, \delta r) \subset B(z', 3r)$ for every $x \in B$, we have

$$I_\alpha^{\delta r} \nu(y) = 0, \quad y \notin B(z', 3r).$$

For $y \in B(z', 3r)$ we have

$$I_\alpha^{\delta r} \nu(y) = \int_0^{\delta r} \frac{\boldsymbol{m}(B(y,t) \cap B(z,r))}{\boldsymbol{m}(B(y,t))}\, dt^\alpha \leq \delta^\alpha r^\alpha.$$

Thus, using Lemma 5.2, we obtain

$$\begin{aligned}
\frac{a\lambda}{2}\, \boldsymbol{m}(E) &\leq \int_B I_\alpha^{\delta r} \mu(x)\, dx \leq D \int_X I_\alpha^{\delta r} \nu(y)\, d\mu(y) \\
&\leq D \int_{B(z',3r)} \delta^\alpha r^\alpha\, d\mu(y) = D\delta^\alpha r^\alpha \mu(B(z', 3r)) \\
&\leq D\delta^\alpha M_\alpha^R \mu(z')\, \boldsymbol{m}(B(z', 3r)) \leq D^3 \delta^\alpha \lambda\, \boldsymbol{m}(B(z, r)).
\end{aligned}$$

This shows
$$\boldsymbol{m}(E) \le \varepsilon\, \boldsymbol{m}(B) \tag{5.3}$$
if $\delta = \delta(\varepsilon, D, \alpha)$ is chosen small enough. Choose $x \in B \cap G^a \setminus E$. Then
$$\begin{aligned} \frac{a\lambda}{2} \le \int_{\delta r}^{3r} \frac{\mu(B(x,t))}{\boldsymbol{m}(B(x,t))}\, dt^\alpha + \int_{3r}^{R/2} \frac{\mu(B(x,t))}{\boldsymbol{m}(B(x,t))}\, dt^\alpha \\ + \int_{R/2}^{R} \frac{\mu(B(x,t))}{\boldsymbol{m}(B(x,t))}\, dt^\alpha. \end{aligned} \tag{5.4}$$
We have
$$\int_{\delta r}^{3r} \frac{\mu(B(x,t))}{\boldsymbol{m}(B(x,t))}\, dt^\alpha \le \alpha M_\alpha^R \mu(x) \int_{\delta r}^{3r} \frac{dt}{t} \le \alpha M_\alpha^R \mu(x) \ln(3/\delta) \tag{5.5}$$
and similarly
$$\int_{R/2}^{R} \frac{\mu(B(x,t))}{\boldsymbol{m}(B(x,t))}\, dt^\alpha \le \alpha \ln 2\, M_\alpha^R \mu(x). \tag{5.6}$$
Finally, we consider the integral from $3r$ to $R/2$. In the non-trivial case we have $3r < R$ and thus from the definition of the Whitney ball there exists $z'' \in B(z, 3r) \setminus G$. Since
$$I_\alpha^R \mu(z'') \le \lambda$$
and
$$B(x,t) \subset B(z'', 2t) \subset B(x, 3t), \quad 3r < t,$$
using (3.1) and (5.1), we obtain
$$\begin{aligned} \int_{3r}^{R/2} \frac{\mu(B(x,t))}{\boldsymbol{m}(B(x,t))}\, dt^\alpha &\le D^2 \int_{3r}^{R/2} \frac{\mu(B(z'',2t))}{\boldsymbol{m}(B(z'',2t))}\, dt^\alpha \\ &= 2^{-\alpha} D^2 \int_{6r}^{R} \frac{\mu(B(z'',s))}{\boldsymbol{m}(B(z'',s))}\, ds^\alpha \\ &\le 2^{-\alpha} D^2 I_\alpha^R \mu(z'') \le 2^{-\alpha} D^2 \lambda = \frac{a\lambda}{4}. \end{aligned} \tag{5.7}$$
From (5.4)–(5.7) and (5.1) we infer that
$$\lambda \le C \ln(3/\delta) M_\alpha^R \mu(x), \quad x \in B \cap G^a \setminus E. \tag{5.8}$$
A combination of (5.3) and (5.8) yields (5.2). Summing over B in a Whitney covering, we obtain the assertion of the theorem. $\square$

Now we may easily estimate norms of operators as in [MW].

Theorem 5.4. *Suppose that μ is a measure on X and $p > 1$. Then*

$$\|I_\alpha^R \mu\|_{L_{p,\infty}} \le C \|M_\alpha^R \mu\|_{L_{p,\infty}},$$

with $C = C(D, \alpha, p)$.

P r o o f. By Theorem 5.3,

$$\begin{aligned}(a\lambda)^p \boldsymbol{m}\big(\{I_\alpha^R \mu \ge a\lambda\}\big) &\le \varepsilon a^p \lambda^p \boldsymbol{m}\big(\{I_\alpha^R \mu \ge \lambda\}\big) + a^p \lambda^p \boldsymbol{m}\big(\{M_\alpha^R \mu \ge \sigma(\varepsilon)\lambda\}\big) \\ &\le \varepsilon a^p \|I_\alpha^R \mu\|_{L_{p,\infty}} + C(\varepsilon) \|M_\alpha^R \mu\|_{L_{p,\infty}}^p.\end{aligned}$$

Taking supremum over λ, we obtain the required inequality provided that ε is chosen such that

$$\varepsilon a^p < 1.$$

□

6. The Frostman measure

We give here a version of the Frostman lemma [Fr] which works for q-codimensional Hausdorff measures. For a general information on the Frostman lemma we refer to P. Mattila [Mt]. Related results in abstract spaces were obtained by S. K. Vodop'yanov [V] and T. Sjödin [Sj].

Theorem 6.1. *Let $q > 0$. Let $G \subset X$ be an open set and $R > 0$. Then there exists a measure μ on G such that*

$$M_q^R \mu \le 1 \quad \text{on } X$$

and

$$\widehat{\mathcal{H}}_q^{10R}(G) \le C \mu(G)$$

with $C = C(D, q)$.

P r o o f. Consider a Whitney covering $\{B(x_j, r_j)\}$ of G constrained by $R/2$. Let N be its overlap multiplicity. For every j let ν_j be the measure $\boldsymbol{m}$ restricted to $B(x_j, r_j)$. We construct recursively $a_j \ge 0$ such that, for every j, a_j is the greatest constant making the maximal function M_q^R of

$$\mu_j := \sum_{i=1}^{j} a_i \nu_i$$

bounded from above by 1. At the end we set

$$\mu = \sum_{i=1}^{\infty} a_i \nu_i.$$

Obviously $M_q^R \mu \le 1$ on X. Fix $j \in \mathbb{N}$. By the maximality property of a_j, there exists $z_j \in G$ and $R_j \in (0, R)$ such that

$$\mu_{j-1}(B(z_j, R_j)) + (a_j + R^{-q})\nu_j(B(z_j, R_j)) > R_j^{-q} \boldsymbol{m}(B(z_j, R_j)).$$

Since

$$\mu_{j-1}(B(z_j, R_j)) + a_j \nu_j(B(z_j, R_j)) = \mu_j(B(z_j, R_j)) \le R_j^{-q} \boldsymbol{m}(B(z_j, R_j)),$$

it follows that $\nu_j(B(z_j, R_j)) > 0$, so that

$$B(z_j, R_j) \cap B(x_j, r_j) \ne \emptyset.$$

We have

$$R_j^{-q} \boldsymbol{m}(B(z_j, R_j)) \le \mu(B(z_j, R_j)) + R^{-q} \boldsymbol{m}(B(z_j, R_j)),$$

and thus

$$R_j^{-q} \boldsymbol{m}(B(z_j, R_j)) \le C \mu(B(z_j, R_j)). \tag{6.1}$$

Now, we distinguish two cases.

Case (a): If $R_j \le \frac{1}{4} r_j$, then

$$B(z_j, R_j) \subset B(x_j, \tfrac{3}{2} r_j).$$

We find $k = k(j)$ such that

$$a_k = \max\{a_i : i \le j,\ B(x_i, r_i) \cap B(x_j, \tfrac{3}{2} r_j) \ne \emptyset\}.$$

Then

$$B(x_j, \tfrac{3}{2} r_j) \subset B(x_k, r_k + 3 r_j).$$

By (6.1),

$$R_j^{-q} \boldsymbol{m}(B(z_j, R_j)) \le C \mu(B(z_j, R_j)) \le C N a_k \boldsymbol{m}(B(z_j, R_j))$$

and thus
$$R_j^{-q} \le C a_k. \tag{6.2}$$
Set
$$y_j = x_k, \quad t_j = r_k + 3r_j \le 2R.$$
Then $B(x_j, r_j) \subset B(y_j, t_j)$. Suppose that $2r_k = \operatorname{dist}(x_k, X \setminus G)$. Then
$$2r_j \le \operatorname{dist}(x_j, X \setminus G) \le d_X(x_j, x_k) + \operatorname{dist}(x_k, X \setminus G) \le \frac{3}{2} r_j + r_k + 2r_k,$$
and thus
$$r_j \le 6r_k. \tag{6.3}$$
If $r_k = R/2$, we have trivially $r_j \le r_k$ and thus (6.3) holds as well. We infer that
$$\boldsymbol{m}(B(y_j, t_j)) = \boldsymbol{m}(B(x_k, r_k + 3r_j)) \le D^5 \boldsymbol{m}(B(x_k, r_k)).$$
Since
$$R_j \le \frac{1}{4} r_j \le t_j,$$
by (6.2),
$$\begin{aligned} t_j^{-q} \boldsymbol{m}(B(y_j, t_j)) &\le D^5 R_j^{-q} \boldsymbol{m}(B(x_k, r_k)) \le C\, a_k \boldsymbol{m}(B(x_k, r_k)) \\ &\le C\, \mu(B(x_k, r_k)) \le C\, \mu(B(y_j, t_j)). \end{aligned}$$

Case (b): Suppose that $R_j > \frac{1}{4} r_j$. We set
$$y_j = z_j, \quad t_j = R_j + 2r_j \le 2R.$$
Then again
$$B(x_j, r_j) \subset B(z_j, R_j + 2r_j) = B(y_j, t_j).$$
We have
$$t_j \le 9R_j.$$
By (6.1) and (3.1),
$$t_j^{-q} \boldsymbol{m}(B(y_j, t_j)) \le D^4 R_j^{-q} \boldsymbol{m}(B(z_j, R_j)) \le C\mu(B(z_j, R_j)) \le C\mu(B(y_j, t_j)).$$
In any case, with each j we have associated $y_j \in X$ and $t_j \in (0, 2R)$ such that
$$B(x_j, r_j) \subset B(y_j, t_j) \quad \text{and} \quad t_j^{-q} \boldsymbol{m}(B(y_j, t_j)) \le C\mu(y_j, t_j).$$

Using the Vitali-type covering theorem, we find a set $A \subset \mathbb{N}$ such that

$$\{B(y_j, t_j) : j \in A\} \text{ is pairwise disjoint}$$

and

$$G \subset \bigcup_{j \in A} B(y_j, 5t_j).$$

Thus

$$\begin{aligned}
\widehat{\mathcal{H}}_q^{10R}(G) &\le \sum_{j \in A} (5t_j)^{-q} \boldsymbol{m}(B(y_j, 5t_j)) \\
&\le 5^{-q} D^3 \sum_{j \in A} (t_j)^{-q} \boldsymbol{m}(B(y_j, t_j)) \\
&\le C \sum_{j \in A} \mu(B(y_j, t_j)) \le C\mu(G).
\end{aligned}$$

□

7. Hausdorff content of level sets

In the following theorem we estimate Hausdorff contents of level sets of Riesz potentials. This is in fact a relation between Hausdorff content and capacity. The linear case ($m = 2$) is essentially due to O. Frostman [Fr]. The nonlinear case (for $W^{1,p}$-capacities) has been obtained by Yu. G. Reshetnyak in [R] and its sharp version by V. G. Maz'ya and V. P. Havin in [MH]. See the exposition by D. R. Adams and L. I. Hedberg [AH]. We consider there the estimate by the "capacity" corresponding to the Sobolev-Lorentz space $W^1(L_{m,1})$. This could be done referring to the relation between Lorentz spaces and Orlicz spaces (see [KKM]), this way has been chosen in [MSZ2]. The estimates for Sobolev-Orlicz capacities have been found by A. Fiorenza and A. Prignet ([FP]). In [MSZ2], another proof of these estimates, based on Orlicz version [M2] of [KM], is shown, see also [M3]. The above mentioned result by T. Kilpeläinen and J. Malý of [KM] consists in estimates of solutions of p-Laplace equations with measure data by Wolff potentials, see also [MZ]. For relations between capacities and Hausdorff measures in metric spaces we refer to S. K. Vodop'yanov [V], T. Sjödin [Sj] and P. Hajłasz and J. Kinnunen [HaKi]. Our approach does not require the knowledge of asymptotics of volumes of small balls. This enables applications to a weighted case, where the idea of codimensional Hausdorff measure comes from. Related results on $W^{1,p}$-capacity and codimensional Hausdorff content in a weighted case have been obtained by E. Nieminen [N], P. Mikkonen [Mi] and B. O. Turesson [T].

Here we present a direct proof of the estimate of the m-codimensional Hausdorff content by the "$W^1(L_{m,1})$-capacity", not passing through Orlicz spaces.

Theorem 7.1. *Let $m > 1$. Suppose that $I^R_\alpha g \ge b > 0$ on G. Then*

$$b^m \widehat{\mathcal{H}}^{10R}_{\alpha m}(G) \le C \|g\|^m_{L_{m,1}}, \tag{7.1}$$

with $C = C(D, \alpha, m)$.

Proof. Let μ be the Frostman measure for $\widehat{\mathcal{H}}^{10R}_{\alpha m}(G)$, see Theorem 6.1. Let $\lambda > 0$ and $x \in \{M^R_\alpha \mu > \lambda\}$. Then, for every $t \in (0, R)$,

$$\mu(B(x,t)) \le t^{-\alpha m} \boldsymbol{m}(B(x,t)), \tag{7.2}$$

and there exists $t_x \in (0, R)$ such that

$$\lambda t_x^{-\alpha} \boldsymbol{m}(B(x,t_x)) \le \mu(B(x,t_x)). \tag{7.3}$$

Multiplying (7.2) with the m-th power of (7.3), we obtain

$$\lambda^m \big(\boldsymbol{m}(B(x,t_x))\big)^{m-1} \le \big(\mu(B(x,t_x))\big)^{m-1}.$$

By the Vitali-type theorem, there exists a sequence $\{B(x_j, r_j)\}$ of pairwise disjoint balls such that $x_j \in \{M^R_\alpha \mu > \lambda\}$, $r_j = t_{x_j}$ and

$$\{M^R_\alpha \mu > \lambda\} \subset \bigcup_j B(x_j, 5r_j).$$

Using (3.1), we obtain

$$\begin{aligned}
\lambda^{m'} \boldsymbol{m}\big(\{M^R_\alpha \mu > \lambda\}\big) &\le C\lambda^{m'} \sum_j \boldsymbol{m}(B(x_j, 5r_j)) \\
&\le C\lambda^{m'} \sum_j \boldsymbol{m}(B(x_j, r_j)) \\
&\le C \sum_j \mu(B(x_j, r_j)) \\
&\le C\mu(X).
\end{aligned}$$

It follows that

$$\|M^R_\alpha \mu\|^{m'}_{L_{m',\infty}} \le C\mu(X).$$

By Theorem 5.4, we have also

$$\|I^R_\alpha \mu\|^{m'}_{L_{m',\infty}} \le C\mu(X).$$

Using the duality (4.4) between $L_{m,1}$ and $L_{m',\infty}$ and Lemma 5.2, we obtain

$$\begin{aligned} b\mu(X) &\le C \int_X I^R_\alpha g \, d\mu \le C \int_X g I^R_\alpha \mu \, dx \\ &\le C\|g\|_{L_{m,1}} \|I^R_\alpha \mu\|_{L_{m',\infty}} \\ &\le C\|g\|_{L_{m,1}} \bigl(\mu(X)\bigr)^{1/m'}. \end{aligned}$$

Hence, by the properties of the Frostman measure,

$$b^m \widehat{\mathcal{H}}^{10r}_{\alpha m} \le C b^m \mu(X) \le C\|g\|^m_{L_{m,1}}.$$

□

E x a m p l e 7.2. The estimate (7.1) does not hold for $m = 1$. Indeed, if μ is the $(n-1)$-dimensional Hausdorff measure in $\mathbb{R}^n$ restricted to

$$F := \{x \in \mathbb{R}^n : x_n = 0, \, |x| \le 1\}$$

and $R > 0$, then $I^R_1 \mu \equiv +\infty$ on F, but $\widehat{\mathcal{H}}^R_1 > 0$. In this example μ is not absolutely continuous with respect to the Lebesgue measure. However, if g_j are mollifications of μ with radii $\delta_j \to 0$, then (7.1) cannot hold with $m = 1$ and C independent of j.

In fact, the only estimate that we can have for $m = 1$ is the following Cartan lemma. It can be obtained by the method of proving the Hardy-Littlewood maximal theorem, see [BZ] (for the Euclidean case) and [HaK] or [He] (for the maximal theorem in metric spaces).

Lemma 7.3. *Let $R > 0$. Suppose that $g \in L^1(X)$. Then*

$$b\widehat{\mathcal{H}}^{5R}_1(\{M^R_1 g > b\}) \le C \int_X g(x)\, dx.$$

We shall need also the following modification.

Lemma 7.4. *Suppose that $g \in L^1(X)$. Then*

$$\lim_{r \to 0+} M^r_1 g(x) = 0 \quad \widehat{\mathcal{H}}_m\text{-}a.e.$$

P r o o f. Choose $\varepsilon > 0$ and $R > 0$. Let

$$E_\varepsilon = \{x \in X : \limsup_{r\to 0} M_1^r g(x) > \varepsilon\}.$$

It is clear that $M^R g(x) = +\infty$ at every $x \in E_\varepsilon$. By the maximal theorem, cf. [HaK, Thm. 14.13], $\boldsymbol{m}(E_\varepsilon) = 0$. Let $G \subset X$ be any open set containing E_ε. For every $x \in G$ we can find $r_x > 0$ such that $5r_x < R$, $B(x, r_x) \subset G$ and

$$\int_{B(x,r_x)} g(x')\,dx' > \varepsilon r_x^{-1}\boldsymbol{m}(B(x, r_x)).$$

Using the Vitali-type covering theorem, we observe that

$$\widehat{\mathcal{H}}_1^R(E_\varepsilon) \le C \int_G g(x')\,dx'.$$

Since G was arbitrary, we obtain

$$\widehat{\mathcal{H}}_1^R(E_\varepsilon) = 0, \quad R > 0,\ \varepsilon > 0,$$

which is enough to prove the conclusion. □

8. Upper gradients

There are many non-equivalent ways how to define "Sobolev spaces" on metric spaces, depending on what is considered to serve as a substitute to gradients. We shall use upper gradients introduced by J. Heinonen and P. Koskela in [HeK]. See [Sh], [Ch], [HaK] for various definitions of such spaces. Spaces based on an upper gradient are well fitting not only to abstract domains, but also to abstract targets, see [HKST]. The main reason for our choice is that once assumed Poincaré inequalities, they are stable under truncation, an observation due to S. Semmes [Se].

Definitions. Let $\Omega \subset X$ be an open set. Given points $x, x' \in \Omega$, we denote by $\Gamma_{x,x',\Omega}$ the set of all 1-Lipschitz paths $\gamma : [a, b] \to \Omega$ with $\gamma(a) = x$ and $\gamma(b) = y$. The interval $[a, b]$ depends on γ. Let g be a non-negative Borel measurable function on Ω. We define the *weighted geodetic distance* of points $x, x' \in \Omega$ by

$$d_g(x, x'; \Omega) = \inf_{\gamma\in\Gamma_{x,y,\Omega}} \int_a^b g(\gamma(t))\,dt.$$

The function d_g has all properties of distance except that $d_g(x, x'; \Omega)$ can be 0 or ∞ for distinct points x, x'. Let (Y, d_Y) be a metric space and $u : \Omega \to Y$

be an $\boldsymbol{m}$-measurable function. We say that $g : \Omega \to [0, +\infty]$ is an *upper gradient* to u if

$$d_Y(u(x), u(x')) \le d_g(x, x'; \Omega), \quad x, x' \in \Omega.$$

Throughout this section we suppose that X supports pre-Poincaré inequalities

$$\fint_{B(z,r)} \fint_{B(z,r)} d_g(x, x')\, dx\, dx' \le Pr \fint_{B(z,\tau r)} g(x)\, dx \tag{8.1}$$

for every $z \in X$, $r > 0$, and $g \in L^1(B(z, \tau t))^+$, here the constant P and the scaling parameter $\tau \ge 1$ are fixed parameters of the space X.

Let $u : \Omega \to Y$ be an $\boldsymbol{m}$-measurable function and g be its upper gradient. Then (8.1) easily implies the *Poincaré inequalities*

$$\fint_{B(z,r)} \fint_{B(z,r)} d_Y(u(x), u(x'))\, dx\, dx' \le Pr \fint_{B(z,\tau r)} g(x)\, dx \tag{8.2}$$

for all balls $B(z, r)$ with $B(z, \tau r) \subset \Omega$.

Remarks. (a) If $X = \mathbb{R}^n$, then X supports (8.1) and if u is a Sobolev function, $|\nabla u|$ is a limit of upper gradients to u. More precisely, if $\nabla u = \sum_j g_j$, where the sum converges in L^1, then for every k, $\sum_{j \le k} g_j + \sum_{j>k} |g_j|$ is an upper gradient to u.

(b) There are many examples of spaces supporting Poincaré inequalities in the literature, see eg. [HaK]. A usual assumption is that inequalities equivalent to (8.2) are satisfied for all *scalar* functions $u : X \to \mathbb{R}$ and their upper gradients. Our assumption is probably a bit stronger but realistic, because standard proofs of Poincaré inequalities implicitly show (8.1).

In fact, (8.1) is equivalent to validity of (8.2) for all spaces Y with the constant independent of Y, as the following example shows: Let Y_g be the quotient space $X/\sim$ where

$$x \sim x' \text{ means } d_g(x, x') = 0,$$

and $u : X \to Y_g$ be the quotient mapping. We define the distance d_k on Y_g by

$$d_k(u(x), u(x')) = \min\{d_g(x, x'), k\}, \quad k > 0.$$

Assuming (8.2) for (Y_g, d_k), we obtain

$$\fint_{B(z,r)} \fint_{B(z,r)} d_k(u(x), u(x'))\, dx\, dx' \le Pr \int_{B(z,\tau r)} g(x)\, dx$$

which, letting $k \to \infty$, yields (8.1).

The following lemma is essentially due to Semmes [Se], see also [Sh, Lemma 4.3].

Lemma 8.1. *Suppose that $u : \Omega \to \mathbb{R}$ is an $\boldsymbol{m}$-measurable function and g is an upper gradient to u. If $G \supset \{u \neq 0\}$ is an open set, then $g\chi_G$ is also an upper gradient to u.*

P r o o f. Let $x, x' \in \Omega$ and $\gamma : [a, b] \to \Omega$ is a path, $\gamma \in \Gamma_{x,x'}$. We want to show

$$|u(x) - u(x')| \leq \int_a^b g(\gamma(t))\chi_G(\gamma(t))\,dt.$$

This is evident if $\gamma([a, b]) \subset G$. Otherwise there is $s \in [a, b]$ such that $\gamma(s) \notin G$ and thus $u(\gamma(s)) = 0$. Set

$$\begin{aligned} a' &= \inf\bigl\{t \in [a, s] : \gamma(t) \notin G\bigr\}, \\ b' &= \sup\bigl\{t \in [s, b] : \gamma(t) \notin G\bigr\}. \end{aligned}$$

Then $a \leq a' \leq s \leq b' \leq b$ and $u(\gamma(a')) = u(\gamma(b')) = 0$. If $a' > a$, then $\gamma([a, a']) \subset G$ and thus

$$|u(x)| = |u(\gamma(a')) - u(\gamma(a))| \leq \int_a^{a'} g(\gamma(t))\,dt = \int_a^{a'} g(\gamma(t))\chi_G(\gamma(t))\,dt.$$

Similarly,

$$|u(x')| \leq \int_{b'}^b g(\gamma(t))\chi_G(\gamma(t))\,dt.$$

It follows that

$$|u(x) - u(x')| \leq |u(x)| + |u(x')| \leq \int_a^b g(\gamma(t))\chi_G(\gamma(t))\,dt$$

if $\gamma([a, b]) \not\subset G$. In any case the assertion is valid. □

Lemma 8.2. *Let $\kappa > 0$. Suppose that $u : \Omega \to \mathbb{R}$ is an $\boldsymbol{m}$-measurable function and g is an upper gradient to u. Suppose that $B(z, \tau r) \subset \Omega$. If $a < b$ are real levels such that*

$$\begin{aligned} \boldsymbol{m}\bigl(B(z, r) \cap \{u \geq b\}\bigr) &\geq \kappa\,\boldsymbol{m}(B(z, r)), \\ \boldsymbol{m}\bigl(B(z, r) \cap \{u \leq a\}\bigr) &\geq \kappa\,\boldsymbol{m}(B(z, r)), \end{aligned}$$

then

$$(b - a)\,\boldsymbol{m}(B(z, \tau r)) \leq Cr \int_{B(z,\tau r)\cap\{a<u<b\}} g(x)\,dx,$$

where $C = C(\kappa, \tau, P)$.

P r o o f. Let

$$w = \begin{cases} a & \text{on } \{u \le a\} \\ b & \text{on } \{u \ge b\} \\ u & \text{elsewhere.} \end{cases}$$

Then g is an upper gradient to w. Let G_a be an open set containing $\{u > a\}$ and G_b be an open set containing $\{u < b\}$. Then by Lemma 8.1, $g\chi_{G_a}$ is an upper gradient to $w - a$. Hence it is also a weak upper gradient to $w-b$. Using Lemma 8.1 once more, we obtain that $g\chi_{G_a \cap G_b}$ is a weak upper gradient to $w - b$ and hence also to w. By (8.2),

$$\begin{aligned}(b-a)\,\boldsymbol{m}(B(z,r)) &\le C\,\boldsymbol{m}(B(z,r)) \fint_{B(z,r)} \fint_{B(z,r)} |w(x) - w(x')|\,dx\,dx' \\ &\le Cr \int_{B(z,\tau r)\cap G_a \cap G_b} g(x)\,dx.\end{aligned}$$

Passing to infimum over all open sets $G_a \supset \{u > a\}$ and $G_b \supset \{u < b\}$, we obtain the assertion. □

9. Consequences of the Poincaré inequality

In this chapter we recall some standard estimates modified to the form that we need.

Lemma 9.1. *Suppose that Y is a metric space, $u : \Omega \to Y$ is an $\boldsymbol{m}$-measurable function and $g \in L^1(\Omega)$ is an upper gradient to u. Suppose that $B(z, 2\tau r) \subset \Omega$. Let $0 < s \le t \le r$. Then*

$$\fint_{B(z,s)} \fint_{B(z,t)} d_Y(u(x), u(x'))\,dx\,dx' \le C I_1^{2\tau t} g(z)$$

with $C = C(D, P, \tau)$.

P r o o f. Suppose that $0 < s \le t \le r$. Let k be such that $2^{-k-1}t < s \le 2^{-k}t$.

Then by (8.2),

$$
\begin{aligned}
&\fint_{B(z,s)}\fint_{B(z,t)} d_Y(u(x),u(x'))\,dx\,dx' \\
&\qquad\le \fint_{B(z,s)}\fint_{B(z,2^{-k}t)} d_Y(u(x),u(x'))\,dx\,dx' \\
&\qquad\quad + \sum_{j=0}^{k-1}\fint_{B(z,2^{-j-1}t)}\fint_{B(z,2^{-j}t)} d_Y(u(x),u(x'))\,dx\,dx' \\
&\qquad\le C\sum_{j=0}^{k}\fint_{B(z,2^{-j}t)}\fint_{B(z,2^{-j}t)} d_Y(u(x),u(x'))\,dx\,dx' \\
&\qquad\le C\sum_{j=0}^{k} 2^{-j}t\fint_{B(z,\tau 2^{-j}t)} g(x)\,dx \\
&\qquad\le C\int_0^{2\tau t}\left(\fint_{B(z,s)} g(x)\,dx\right) ds \\
&\qquad= C I_1^{2\tau t} g(z)
\end{aligned}
$$

as required. □

Lemma 9.2. *Suppose that Y is a metric space, $u:\Omega\to Y$ is an $\boldsymbol{m}$-measurable function and $g\in L^1(\Omega)$ is an upper gradient to u. Suppose that*

$$\lim_{r\to 0+} M_1^r g(z) = 0.$$

Then there exists $\{y_r : r>0\}\subset Y$ such that

$$\lim_{r\to 0}\fint_{B(z,r)} d_Y(u(x),y_r)\,dx = 0.$$

P r o o f. We find $r_k \searrow 0$ such that

$$Pr\fint_{B(z,\tau r)} g(x)\,dx < 2^{-k-1}, \quad 0<r<r_k.$$

Set $x_r = z$ for $r\ge r_1$. Let $r_{k+1}\le r<r_k$. By (8.2), we obtain

$$\fint_{B(z,r)}\fint_{B(z,r)} d_Y(u(x),u(x'))\,dx\,dx' \le 2^{-k-1}.$$

It follows that

$$\begin{aligned} 2^{-k}\boldsymbol{m}\Big(\Big\{x'\in B(z,r): \fint_{B(z,r)} d_Y(u(x),u(x'))\,dx > 2^{-k}\Big\}\Big) &\le \int_{B(z,r)}\fint_{B(z,r)} d_Y(u(x),u(x'))\,dx\,dx' \\ &\le 2^{-k-1}\boldsymbol{m}(B(z,r)) \end{aligned}$$

and thus there exists a point $x_r \in B(z,r)$ such that

$$\fint_{B(z,r)} d_Y(u(x_r),u(x))\,dx \le 2^{-k}.$$

Now we can set $y_r = u(x_r)$. $\square$

10. Hausdorff content of level sets continued

In this section we estimate level sets of functions in terms of integrability of their upper gradient. For $m > 1$ this is a consequence of previous results on Riesz potentials. If $m = 1$, then this needs to use the truncation structure of upper gradients. In the Euclidean case the equivalence of $(n-1)$-dimensional Hausdorff content and $W^{1,1}$-capacity has been obtained by W. H. Fleming in [Fl]. Here we follow the proof in [MSZ1].

Theorem 10.1. *Suppose that $m \ge 1$. Let $a < b$ be real numbers and let $R > 0$ be a fixed radius. Let $v : \Omega \to \mathbb{R}$ be an $\boldsymbol{m}$-measurable function with an upper gradient $g \in L^1(\Omega)$. Let $E \subset \Omega$. Suppose that for every $z \in E$ there exist radii r_z, R_z such that $0 < r_z < R_z \le R$, $B(z, 2\tau R_z) \subset \Omega$,*

$$\begin{aligned} \boldsymbol{m}\big(B(z,R_z)\cap\{v>a\}\big) &\le \frac{1}{2}\,\boldsymbol{m}(B(z,R_z)), \\ \boldsymbol{m}\big(B(z,r_z)\cap\{v>b\}\big) &> \frac{1}{2}\,\boldsymbol{m}(B(z,r_z)). \end{aligned} \tag{10.1}$$

Then

$$(b-a)^m \widehat{\mathcal{H}}_m^{20\tau R}(E) \le C\|g\|^m_{L_{m,1}(\Omega)},$$

where $C = C(D, P, \tau, m)$.

Proof. First, suppose that $m > 1$. By Lemma 9.1,

$$b-a \le C\fint_{B(z,r)}\fint_{B(z,R)} d_Y(u(x),u(x'))\,dx\,dx' \le C I_1^{2\tau R} g(z), \quad z\in E,$$

and the same certainly holds with g replaced by $g\chi_\Omega$. Using Theorem 7.1, we obtain

$$(b-a)^m \widehat{\mathcal{H}}_m^{20\tau R}(E) \le C\|g\|^m_{L_{m,1}(\Omega)}.$$

Now, suppose that $m = 1$. Let us consider the monotone real function

$$\psi(s) = \int_{\{a<v<s\}} g(x)\,dx.$$

Then ψ is a.e. differentiable and

$$\int_a^b \psi'(s)\,ds \le \psi(b) - \psi(a) = \psi(b).$$

Hence there exists $s_0 \in (a,b)$ such that $(b-a)\psi'(s_0) < 2\psi(b)$. We find $\delta > 0$ such that

$$\frac{\psi(s)-\psi(s_0)}{s-s_0} \le \frac{2\psi(b)}{b-a} \quad \text{for every } s \in (s_0, s_0+\delta). \tag{10.2}$$

Choose $z \in E$. By (10.1), the set

$$S_z = \left\{ t > 0 : \frac{\boldsymbol{m}(B(z,t) \cap \{v > s_0\})}{\boldsymbol{m}(B(z,t))} > \frac{1}{2} \right\}$$

contains r_z but does not contain R_z. Then, with every $z \in E$ we can associate $t_z \in (0, R_z]$ by

$$t'_z = \sup\,(0, R_z] \cap S_z, \quad t_z = \min\{R_z, 2t'_z\}.$$

Obviously, we have

$$\begin{aligned}
\boldsymbol{m}(B(z,t'_x) \cap \{v > s_0\}) &\ge \frac{1}{2}\,\boldsymbol{m}(B(z,t'_x)),\\
\boldsymbol{m}(B(z,t_x) \cap \{v > s_0\}) &\le \frac{1}{2}\,\boldsymbol{m}(B(z,t_x)),
\end{aligned}$$

and thus

$$\begin{aligned}
\boldsymbol{m}(B(z,t_x) \cap \{v > s_0\}) &\ge \frac{1}{2D}\,\boldsymbol{m}(B(z,t_x)),\\
\boldsymbol{m}(B(z,t_x) \cap \{v \le s_0\}) &\ge \frac{1}{2}\,\boldsymbol{m}(B(z,t_x)).
\end{aligned} \tag{10.3}$$

We use the Vitali-type covering theorem to extract a (finite or infinite) sequence $\{B_j\}_{j\in I}$ of pairwise disjoint balls $B_j = B(z_j, t_j)$ from $\{B(z, \tau t_x)$: $x \in E\}$ such that

$$E \subset \bigcup_{j\in I} B(z_j, 5t_j).$$

Here $I = \mathbb{N}$ or $I = \{1, 2, \ldots, i_{\max}\}$. Fix $i \in I$. Using (10.3), we find a level $s_i \in (s_0, s_0 + \delta)$ such that

$$\boldsymbol{m}\big(B(z_j, t_j/\tau) \cap \{v > s_i\}\big) \geq \frac{1}{4D}\,\boldsymbol{m}\big(B(z_j, t_j/\tau)\big), \quad j = 1, \ldots, i. \tag{10.4}$$

From Lemma 8.2, (10.3) and (10.4) we infer that

$$t_j^{-1}(s_i - s_0)\,\boldsymbol{m}(B(z_j, t_j)) \leq C \int_{B(z_j,t_j)\cap\{s_0<g<s_i\}} g(x)\,dx.$$

Summing over $j = 1, \ldots, i$ and using (10.2), we obtain

$$\begin{aligned}(s_i - s_0)\sum_{j=1}^{i} t_j^{-1}\boldsymbol{m}(B(z_j, t_j)) &\leq \sum_{j=1}^{i}\int_{B(z_j,t_j)\cap\{s_0<g<s_i\}} g(x)\,dx\\ &\leq \int_{\{s_0<g<s_i\}} g(x)\,dx\\ &\leq \psi(s_i) - \psi(s_0)\\ &\leq (s_i - s_0)\,\frac{2\psi(b)}{b-a}.\end{aligned}$$

Passing i to $i_{\max}$ or ∞, we obtain

$$\begin{aligned}\widehat{\mathcal{H}}_1^{5\tau R}(E) &\leq \sum_j (5t_j)^{-1}\boldsymbol{m}(B(z_j, 5t_j)) \leq C\sum_j t_j^{-1}\boldsymbol{m}(B(z_j, t_j))\\ &\leq C\frac{\psi(b)}{b-a} = \frac{2C}{b-a}\int_\Omega g(x)\,dx,\end{aligned}$$

as required. □

Remark. We cannot prove the $m = 1$ part of Theorem 10.1 similarly to the $m > 1$ part because of the lack of $\widehat{\mathcal{H}}_1$-estimates of level sets of Riesz potentials, see Example 7.2. Also the estimate of Lemma 7.3 is not enough. Indeed, the inequality

$$v(x) \leq CM_1 g(x), \quad x \in E,$$

does not hold under the assumptions of Theorem 10.1; consider the functions

$$v_\varepsilon(x) = \varepsilon \log(1/|x|)^+, \quad \varepsilon \to 0+,$$

as an example in $\mathbb{R}^n$, $n > 1$.

11. Lebesgue points

H. Federer and W. P. Ziemer [FZ] showed that Sobolev functions have Lebesgue points q.e. in the sense of the Sobolev capacity. The case Sobolev-Orlicz and Sobolev-Lorentz spaces was treated by P. Malý, D. Swanson and W. P. Ziemer in [MSZ2]. The results concerning Sobolev-Orlicz spaces also follow from Aissaoui's work [A] on Lebesgue points of potentials.

In metric spaces, a $W^{1,p}$ result for $p > 1$ has been obtained by J. Kinnunen and V. Latvala [KL].

Definitions. Let $\Omega \subset X$ be an open set, $z \in \Omega$ and $u : X \to Y$ be an $\boldsymbol{m}$-measurable mapping. A point $y \in Y$ is said to be a Lebesgue limit of u at z and denoted by $\text{L-lim}_{x\to z} u(x)$ if

$$\lim_{r\to 0} \fint_{B(z,r)} d_Y(u(x), y)\, dx = 0.$$

We say that an $\boldsymbol{m}$-measurable function $u : \Omega \to Y$ is *Lebesgue precise* if

$$\underset{x\to z}{\text{L-lim}}\, u(x) = u(z)$$

whenever the Lebesgue limit of u at z exists.

Lemma 11.1. *Suppose that Y is a complete metric space, $u : \Omega \to Y$ is an $\boldsymbol{m}$-measurable function and $g \in L^1(\Omega)$ is an upper gradient to u. Suppose that $B(z, 2\tau R) \subset \Omega$ and*

$$I_1^{2\tau R} g(z) < \infty. \tag{11.1}$$

Then u has a Lebesgue limit y at z and

$$\fint_{B(z,r)} d_Y(u(x), y)\, dx \le C I_1^{2\tau r} g(z) \tag{11.2}$$

for every $r \in (0, R]$, with $C = C(D, P, \tau)$.

Proof. By (11.1), we have

$$\lim_{r\to 0+} I_1^r g(z) = 0, \tag{11.3}$$

and thus also

$$\lim_{r\to 0+} M_1^r g(z) \le C \lim_{r\to 0+} \int_r^{2r} \left(\fint_{B(z,t)} g(x)\, dx \right) dt = 0.$$

By Lemma 9.2 and (11.3), there exist $r_k \searrow 0$ and $y_k \in Y$ such that

$$\fint_{B(z,r_k)} d_Y(u(x), y_k)\, dx < 2^{-k}, \quad I_1^{2\tau r_k}(z) < 2^{-k}. \tag{11.4}$$

For $k < j \in \mathbb{N}$ we estimate

$$\begin{gathered} d_Y(y_k, y_j) \le d_Y(y_k, u(x)) + d_Y(u(x), u(x')) + d_Y(u(x'), y_j), \\ x \in B(z, r_k),\ x' \in B(z, r_j). \end{gathered}$$

Integrating with respect to x and x' and using (11.4) and Lemma 9.1, we obtain

$$\begin{aligned} d_Y(y_k, y_j) \le & \fint_{B(z,r_k)} d_Y(y_k, u(x))\, dx \\ & + \fint_{B(z,r_k)} \fint_{B(z,r_j)} d_Y(u(x), u(x'))\, dx'\, dx \\ & + \fint_{B(z,r_j)} d_Y(y_j, u(x'))\, dx' \\ \le & \, 2^{-k+1} + C I_1^{2\tau r_k} g(z) \le C 2^{-k}. \end{aligned} \tag{11.5}$$

Hence $\{y_k\}_k$ is a Cauchy sequence and, since Y is complete, there exists $y \in Y$ such that

$$y = \lim_k y_k.$$

Now, for $0 < r_k < r \le R$, by Lemma 9.1, (11.4) and (11.5) we have

$$\begin{aligned} \fint_{B(z,r)} d_Y(u(x), y)\, dx \le & \fint_{B(z,r)} \fint_{B(z,r_k)} d_Y(u(x), u(x'))\, dx'\, dx \\ & + \fint_{B(z,r_k)} d_Y(u(x'), y_k)\, dx' + d_Y(y_k, y) \\ \le & \, C\big(I_1^{2\tau r} g(z) + 2^{-k}\big). \end{aligned}$$

Letting $k \to \infty$, we obtain (11.2). □

Theorem 11.2. *Let $m > 1$. Suppose that Y is a complete metric space, $u : \Omega \to Y$ is an $\boldsymbol{m}$-measurable function and $g \in L_{m,1}(\Omega)$ is an upper gradient to u. Then for $\widehat{\mathcal{H}}_m$-a.e. $x \in \Omega$ there exists a Lebesgue limit of u at x. In particular, if u is Lebesgue precise, then $\widehat{\mathcal{H}}_m$-a.e. points of Ω are Lebesgue points for u.*

P r o o f. This is a combination of Lemma 11.1 and Theorem 7.1. □

Theorem 11.3. *Suppose that Y is boundedly compact. Let $\Omega \subset X$ be an open set and $u : \Omega \to Y$ be an $\boldsymbol{m}$-measurable function with an upper gradient $g \in L^1(\Omega)$. Then for $\widehat{\mathcal{H}}_1$-a.e. $x \in \Omega$ there exists a Lebesgue limit of u at x. In particular, if u is Lebesgue precise, then $\widehat{\mathcal{H}}_1$-a.e. points of Ω are Lebesgue points for u.*

P r o o f. *The scalar case.*
Set

$$Z = \bigl\{z \in \Omega : \limsup_{r \to 0+} M_1^r g(z) > 0\bigr\}.$$

By Lemma 7.4,

$$\widehat{\mathcal{H}}_1(Z) = 0.$$

Hence we may consider only $z \in \Omega \setminus Z$. Then by Lemma 9.2, there exists a sequence $\{y_k\}$, $y_k = y_k(z) \in \mathbb{R}$, such that

$$\lim_{k \to \infty} \fint_{B(z,r_k)} |u(x) - y_k| \, dx = 0, \quad r_k = 2^{-k}.$$

Using the doubling property of $\boldsymbol{m}$ and the special choice of r_k, we easily observe that the values

$$\varphi(z) = \limsup_{k \to \infty} y_k(z), \quad \psi(z) = \liminf_{k \to \infty} y_k(z)$$

do not depend on the choice of y_k, and that $\text{L-lim}_{x \to z} u(x) \in \mathbb{R}$ exists if and only if $\varphi(z) = \psi(z) \in \mathbb{R}$. Therefore, it remains to prove that

$$\varphi(z) = \psi(z) \in \mathbb{R} \quad \text{for } \widehat{\mathcal{H}}_1\text{-a.e. } x \in \Omega \setminus Z. \tag{11.6}$$

Suppose that $\varphi(z) = \infty$. Then we consider a ball $B(z_0, R)$ containing z and such that $B(z_0, 3\tau R) \subset \Omega$. There exists a number $a \in \mathbb{R}$ satisfying

$$\boldsymbol{m}\bigl(B(z_0, 2R) \cap \{u > a\}\bigr) \le \frac{1}{2D^2} \, \boldsymbol{m}(B(z_0, 2R)).$$

Then, since $B(z, R) \subset B(z_0, 2R) \subset B(z, 4R)$,

$$\begin{aligned}
\boldsymbol{m}\bigl(B(z, R) \cap \{u > a\}\bigr) &\le \boldsymbol{m}\bigl(B(z_0, R) \cap \{u > a\}\bigr) \\
&\le \frac{1}{2D^2} \, \boldsymbol{m}(B(z_0, 2R)) \\
&\le \frac{1}{2} \, \boldsymbol{m}(B(z, 2R)).
\end{aligned}$$

Suppose that $b > a$ and $y_k > b + 1$. Then

$$\frac{\boldsymbol{m}(B(z, r_k) \cap \{u \le b\})}{\boldsymbol{m}(B(z, r_k))} \le \fint_{B(z,r_k)} |u - y_k|\, dx \to 0,$$

which implies that there exists $r_z \in (0, R)$ such that

$$\boldsymbol{m}(B(z, r_z) \cap \{u > b\}) > \frac{1}{2}\, \boldsymbol{m}(B(z, r_z)).$$

By Theorem 10.1, we have

$$(b - a)\, \widehat{\mathcal{H}}_1^{20\tau R}(B(z_0, R) \cap \{\varphi(z) = \infty\}) \le C \int_\Omega g(x)\, dx.$$

On letting $b \to \infty$, we obtain that

$$\widehat{\mathcal{H}}_1(\{z : \varphi(z) = \infty\}) = 0, \tag{11.7}$$

and similarly

$$\widehat{\mathcal{H}}_1(\{z : \psi(z) = -\infty\}) = 0. \tag{11.8}$$

Now, given rational numbers $a < b$, consider the set

$$E := \{z : \psi(z) < a < b < \varphi(z)\}.$$

Since $\boldsymbol{m}$-a.e. point $z \in \Omega$ is a Lebesgue point for u, it follows that $\boldsymbol{m}(E) = 0$. Let $G \subset \Omega$ be an arbitrary open set containing E and $a' \in (\psi(z), a)$. Then for big k, $y_k < a'$ and thus

$$(a - a')\, \frac{\boldsymbol{m}(B(z, r_k) \cap \{u > a\})}{\boldsymbol{m}(B(z, r_k))} \le \fint_{B(z,r_k)} |u - y_k|\, dx \to 0.$$

Hence there exists $R_z > 0$ such that $B(z, 2\tau R_z) \subset G$ and

$$\boldsymbol{m}(B(z, R_z) \cap \{u > a\}) \le \frac{1}{2}\, \boldsymbol{m}(B(z, R_z)).$$

Similarly, there exists $r_z \in (0, R_z)$ such that

$$\boldsymbol{m}(B(z, r_z) \cap \{u > b\}) > \frac{1}{2}\, \boldsymbol{m}(B(z, r_z)).$$

By Theorem 10.1,

$$\widehat{\mathcal{H}}_1^{20\tau R}(E) \le C \int_G g(x)\,dx.$$

Since G containing E was arbitrary, we have

$$\widehat{\mathcal{H}}_1^{20\tau R}(E) = 0.$$

Letting $R \to 0$ and passing to union over all rational couples (a, b), we obtain that

$$\widehat{\mathcal{H}}_1(\{z : \psi(z) < \varphi(z)\}) = 0. \tag{11.9}$$

Getting together (11.8), (11.9) and (11.10) we verify (11.6), which concludes the proof.

The metric space valued case.
Set

$$A = \bigcup_{y\in Y} \big\{z \in \Omega : \operatorname*{L-lim}_{x\to z} d_Y(u(x), y) \text{ does not exist}\big\}.$$

Let us consider a dense countable set $S \subset Y$. If $y \in Y$ and $z \in \Omega$, then there exists a sequence $\{y_k\}$ of points of S such that $y_k \to y$. Assuming that

$$\operatorname*{L-lim}_{x\to z} d_Y(u(x), y_k) = L_k \in \mathbb{R},$$

we observe that

$$|L_k - L_j| \le d_Y(y_k, y_j), \quad k, j \in \mathbb{N},$$

so that there exists

$$L = \lim_k L_k \in \mathbb{R}.$$

Then

$$\begin{aligned}&\limsup_{r\to 0+} \left| \fint_{B(z,r)} d_Y(u(x), y)\,dx - L \right| \\ &\qquad \le \limsup_{r\to 0+} \left| \fint_{B(z,r)} d_Y(u(x), y_k)\,dx - L_k \right| + d_Y(y_k, y) + |L_k - L|,\end{aligned}$$

and thus

$$L = \operatorname*{L-lim}_{x\to z} d_Y(u(x), y).$$

This shows that

$$A = \bigcup_{y\in S} \big\{z \in \Omega : \operatorname*{L-lim}_{x\to z} d_Y(u(x), y) \text{ does not exist}\big\},$$

and since S is countable, by the scalar part of the proof applied to the functions
$$y \mapsto d_Y(u(x), y), \quad y \in S,$$
we infer that
$$\widehat{\mathcal{H}}_1(A) = 0. \tag{11.10}$$
Fix $z \in \Omega \setminus A$ and set
$$f(y) = \operatorname*{L-lim}_{x \to z} d_Y(u(x), y), \quad y \in Y.$$
Then clearly
$$|f(y) - f(y')| \le d_Y(y, y') \le f(y) + f(y'), \quad y, y' \in Y.$$
Hence f is a 1-Lipschitz function and
$$\lim_{d_Y(y,y') \to \infty} f(y) = \infty, \quad y' \in Y.$$
Since Y is boundedly compact, these observations imply that f attains its minimum at some point $y_0 \in Y$. Set
$$L_0 = f(y_0) = \min_{y \in Y} f(y).$$
Assume that $L_0 > 0$. The set
$$K = \{y \in Y : |d_Y(y, y_0) - L_0| \le L_0/2\}$$
is compact and thus there exists a finite set
$$\{y_k : k = 1, \dots, q\} \subset K$$
such that the balls $B(y_k, L_0/4)$ cover K. For every $\varepsilon > 0$ we find $\delta > 0$ such that
$$\left| \fint_{B(z,r)} d_Y(u(x), y_k) - f(y_k) \right| < \varepsilon, \quad 0 < r < \delta.$$
Fix $r \in (0, \delta)$. Then
$$B(z, r) = \bigcup_{k=0}^{q} E_k,$$

where

$$\begin{aligned} E_0 &= \big\{x \in B(z,r) : |d_Y(u(x),y_0) - L_0| > L_0/2\big\}, \\ E_k &= \big\{x \in B(z,r) : |d_Y(u(x),y_k) - f(y_k)| < L_0/4\big\}, \quad k = 1,\dots,q. \end{aligned}$$

If $x \in E_0$, then

$$d_Y(u(x),y_0) \le |d_Y(u(x),y_0) - L_0| + L_0 \le L_0/2 + 2|d_Y(u(x),y_0) - L_0|.$$

If $x \in E_k$, $k \in \{1,\dots,q\}$, then by the minimum property of L_0,

$$\begin{aligned} d_Y(u(x),y_0) &\le 2L_0 \le 2f(y_k) \le 2|d_Y(u(x),y_k) - f(y_k)| + 2d_Y(u(x),y_k) \\ &\le 2|d_Y(u(x),y_k) - f(y_k)| + L_0/2. \end{aligned}$$

Integrating these inequalities, we obtain

$$\begin{aligned} \fint_{B(z,r)} d_Y(u(x),y_0)\,dx &\le \frac{L_0}{2} + \fint_{B(z,r)} \Big(d_Y(u(x),y_0) - \frac{L_0}{2}\Big)^+ dx \\ &\le \frac{L_0}{2} + \frac{1}{\boldsymbol{m}(B(z,R))} \sum_{k=0}^{q} \int_{E_k} \Big(d_Y(u(x),y_0) - \frac{L_0}{2}\Big)^+ dx \\ &\le \frac{L_0}{2} + 2\sum_{k=0}^{q} \fint_{B(z,r)} \big|d_Y(u(x),y_k) - f(y_k)\big|\,dx \\ &\le \frac{L_0}{2} + 2(q+1)\varepsilon. \end{aligned}$$

For sufficiently small ε this implies that

$$\fint_{B(z,r)} d_Y(u(x),y_0)\,dx < \frac{3}{4}\,L_0, \quad 0 < r < \delta,$$

and thus $f(y_0) < L_0$. This contradiction yields

$$f(y_0) = L_0 = 0,$$

so that y_0 is the Lebesgue limit of f at z. We have shown that the Lebesgue limit at z exists for every $z \in \Omega \setminus A$. By (11.11), this concludes the proof. $\square$

12. Coarea property

Now we are ready to establish metric space versions of results already discussed in Section 2.

Definition. We say that a function $u : \Omega \to Y$ satisfies the *m-coarea property* in Ω if for every Lebesgue null set $E \subset \Omega$ and $\mathcal{H}_m$-almost every $y \in Y$ we have $\widehat{\mathcal{H}}_m(E \cap u^{-1}(y)) = 0$.

Notation. Consider a mapping $u : X \to Y$. We denote by $\mathcal{L}_u$ the set of all Lebesgue points of u. For any $z \in X$ and $r > 0$, we denote

$$G_u(z,r) = \big\{x \in B(z,r) : u(x) \in B(u(z),r)\big\}.$$

Lemma 12.1. *Suppose that $m \geq 1$. Let $u : \Omega \to Y$ be a mapping with an upper gradient $g \in L^1(X)$. Let $z \in \Omega$, $y \in Y$ and let $R > 0$ be a radius such that $B(z, 3\tau R) \subset \Omega$. Then*

$$R^m \widehat{\mathcal{H}}_m^{20\tau R}(G_u(z,R) \cap \mathcal{L}_u) \leq C\|(1+g)\chi_{G_u(z,3\tau R)}\|_{L_{m,1}}^m, \tag{12.1}$$

with $C = C(D, P, \tau, m)$.

P r o o f. Set

$$v(x) = \big(2R - d_Y(u(x), u(z))\big)^+$$

and consider an open set G containing $\{v > 0\}$. Then g is an upper gradient to v and, by Lemma 8.1, $g\chi_G$ is an upper gradient to v.

Suppose first that

$$\boldsymbol{m}\big(B(x,R) \cap G_u(z,2R)\big) \leq \frac{1}{2}\,\boldsymbol{m}(B(x,R)) \quad \text{for every } x \in B(z,R). \tag{12.2}$$

Then, taking into account the definition of Lebesgue points, there exists $r_x \in (0,R)$ such that

$$\boldsymbol{m}\big(B(x,r_x) \cap \{|u - u(x)| < R\}\big) > \frac{1}{2}\,\boldsymbol{m}(B(x,r_x)).$$

Hence

$$\boldsymbol{m}\big(B(x,R) \cap \{v > 0\}\big) \leq \frac{1}{2}\,\boldsymbol{m}(B(z,R)),$$
$$\boldsymbol{m}\big(B(x,r_x) \cap \{v > R\}\big) > \frac{1}{2}\,\boldsymbol{m}(B(z,r_x)).$$

By Theorem 10.1,

$$R^m \widehat{\mathcal{H}}_m^{20\tau R}(G_u(z,R) \cap \mathcal{L}_u) \le C \|g\chi_{G_u(z,3\tau R)}\|_{L_{m,1}}^m.$$

Suppose now that (12.2) is violated. Then there exists $x \in B(z,R)$ such that

$$\boldsymbol{m}(B(x,R) \cap G_u(z,2R)) > \frac{1}{2}\,\boldsymbol{m}(B(x,R)).$$

We estimate the Hausdorff measure via the trivial covering of the set by the ball $B(z,R)$. Thus

$$\begin{aligned} R^m \widehat{\mathcal{H}}_m^R((G_u(z,R) \cap \mathcal{L}_u)) &\le \boldsymbol{m}(B(z,R)) \\ &\le D\,\boldsymbol{m}(B(x,R)) \\ &\le 2D\,\boldsymbol{m}(G_u(z,2R)) \\ &\le 2D\|\chi_{G_u(z,2R)}\|_{L_{m,1}}^m. \end{aligned}$$

In any case we have (12.1). □

Definition 12.2. For integration with respect to the Hausdorff measure, it is useful to consider the functionals

$$\Lambda_m^\delta : f \mapsto \inf\bigl\{\sum_j \gamma_j (\operatorname{diam} A_j)^m : \gamma_j \ge 0,\ \operatorname{diam} A_j \le \delta,\ f \le \sum_j \gamma_j \chi_{A_j}\bigr\},\ \delta > 0,$$

defined on non-negative functions f on Y. By [F, 2.10.24],

$$\int_Y^* f\,d\mathcal{H}_m = \lim_{\delta \to 0} \Lambda_m^\delta(f)$$

for any such an f provided that Y is boundedly compact. Recall that $\int^*$ stands for the upper integral.

The following theorem is our final statement on the m-coarea property.

Theorem 12.3. *Suppose that $m \ge 1$ and that Y is boundedly compact. Let $\Omega \subset X$ be an open set and $u : \Omega \to Y$ be a Lebesgue precise $\boldsymbol{m}$-measurable mapping with an upper gradient $g \in L_{m,1}(\Omega)$. Then u satisfies the m-coarea property in Ω.*

P r o o f. Let $E \subset \Omega$ be a set of $\boldsymbol{m}$-measure 0. By Theorems 11.2, 11.3, we may assume that E consists only of Lebesgue points for u. In the Cartesian product $X \times Y$ we shall use "balls" $B([x,y],r) = B(x,r) \times B(y,r)$. Given $\varepsilon > 0$, we find an open set $G \subset \Omega$ such that

$$E \subset G \text{ and } \|(1+g)\chi_G\|^m_{L_{m,1}} < \varepsilon.$$

Choose $\delta > 0$. Let $x \in E$. We decompose

$$E = E' \cup E'',$$

where

$$\begin{aligned} E'' = E''_\varepsilon = \bigl\{x \in E:\ & \text{there exist } t_{x,j} \to 0 \text{ such that} \\ & \widehat{\mathcal{H}}^\varepsilon_m\bigl(E \cap u^{-1}(u(x)) \cap B(x, 5t_{x,j})\bigr) \le \boldsymbol{m}(B(x, t_{x,j}))\bigr\}, \\ E' = E'_\varepsilon = E \setminus E''. & \end{aligned}$$

Let $y \in Y$. Using Vitali's covering theorem, we find a pairwise disjoint system of balls $B(x'_i, t'_i)$ selected from $\{B(x, t_{x,j})\}$ such that $5t'_i < \varepsilon$, $B(x'_i, t'_i) \subset G$ and $B(x'_i, 5t'_i)$ cover $E'' \cap u^{-1}(y)$. Then we obtain

$$\begin{aligned} \widehat{\mathcal{H}}^\varepsilon_m\bigl(E'' \cap u^{-1}(y)\bigr) &\le \sum_i \widehat{\mathcal{H}}^\varepsilon_m\bigl(E'' \cap u^{-1}(y) \cap B(x'_i, 5t'_i)\bigr) \\ &\le \sum_i \boldsymbol{m}(B(x'_i, t'_i)) \le \boldsymbol{m}(G) < \varepsilon. \end{aligned} \tag{12.3}$$

Now, consider $x \in E'$. We find $r_{x,0} > 0$ such that

$$15\tau r_{x,0} < \delta/2, \quad B(x, 3\tau r_{x,0}) \subset G,$$

and denote $r_{x,i} = (15\tau)^{-i} r_{x,0}$. Observe that if $\{a_i\}$ is a bounded sequence of positive real numbers, then there exists i such that $a_{i+1} \le 2a_i$. Applying this trick to

$$a_i = \frac{\boldsymbol{m}(B(x, r_{x,i}))}{\widehat{\mathcal{H}}^\varepsilon_m\bigl(E \cap G_u(x, r_{x,i})\bigr)}$$

and using (3.1), we find r_x among radii $3\tau r_{x,i}$ such that

$$\widehat{\mathcal{H}}^\varepsilon_m\bigl(E \cap G_u(x, 5r_x)\bigr) \le C\widehat{\mathcal{H}}^\varepsilon_m\bigl(E \cap G_u(x, r_x/(3\tau))\bigr). \tag{12.4}$$

The system $\{B(x, r_x) \times B(u(x), r_x)\}$ forms a covering of the graph of u over E and, by a Vitali-type covering theorem, we find a pairwise disjoint sequence

$\{B_j\}$ of "balls" $B_j = B(x_j, r_j) \times B(u(x_j), r_j)$ such that $x_j \in E$, $r_j = r_{x_j}$, and

$$\big\{[x, u(x)] : x \in E'\big\} \subset \bigcup_j B(x_j, 5r_j) \times B\big(u(x_j), 5r_j\big). \tag{12.5}$$

We denote

$$\begin{aligned} A_j &= B(u(x_j), 5r_j), \\ \gamma_j &= \widehat{\mathcal{H}}^{\varepsilon}_m\big(E \cap B(x_j, 5r_j) \cap u^{-1}(A_j)\big) = \widehat{\mathcal{H}}^{\varepsilon}_m\big(E \cap G_u(x_j, 5r_j)\big). \end{aligned}$$

Then for every $y \in Y$, we have by (12.5),

$$\begin{aligned} \widehat{\mathcal{H}}^{\varepsilon}_m\big(E' \cap u^{-1}(y)\big) &\le \sum_j \widehat{\mathcal{H}}^{\varepsilon}_m\big(E \cap B(x_j, 5r_j) \cap u^{-1}(A_j)\big)\chi_{A_j}(y) \\ &= \sum_j \gamma_j \chi_{A_j}(y), \end{aligned}$$

so that for the functional introduced in Definition 12.2 and

$$f_\varepsilon(y) = \widehat{\mathcal{H}}^{\varepsilon}_m\big(E'_\varepsilon \cap u^{-1}(y)\big)$$

we have

$$\begin{aligned} \Lambda^{\delta}_m(f_\varepsilon) &\le \sum_j \gamma_j \operatorname{diam}(A_j)^m \\ &\le C \sum_j \widehat{\mathcal{H}}^{\varepsilon}_m\big(E \cap G_u(x_j, 5r_j)\big) \operatorname{diam}(A_j)^m. \end{aligned} \tag{12.6}$$

By Lemma 12.1 and (12.4),

$$\begin{aligned} \widehat{\mathcal{H}}^{\varepsilon}_m\big(E \cap G_u(x_j, 5r_j)\big) \operatorname{diam}(A_j)^m &\le C r_j^m \widehat{\mathcal{H}}^{\varepsilon}_m\big(E \cap G_u(x_j, r_j/(3\tau))\big) \\ &\le C \|(1+g)\chi_{G_u(x_j, r_j)}\|^m_{L_{m,1}}. \end{aligned} \tag{12.7}$$

Since the sets $G_u(x_j, r_j)$ are pairwise disjoint and contained in G, by Lemma 4.1, (12.7) and (12.6), we obtain

$$\Lambda^{\delta}_m(f_\varepsilon) \le C \|(1+g)\chi_G\|^m_{L_{m,1}} \le C\varepsilon.$$

Letting $\delta \to 0$, we obtain

$$\int_Y^* \widehat{\mathcal{H}}^{\varepsilon}_m(E'_\varepsilon \cap u^{-1}(y))\, d\mathcal{H}_m(y) = 0,$$

so that, by (12.3), for $\mathcal{H}_m$-almost every $y \in Y$ we have

$$\widehat{\mathcal{H}}^{\varepsilon}_m(E \cap u^{-1}(y)) \le \widehat{\mathcal{H}}^{\varepsilon}_m(E'_\varepsilon \cap u^{-1}(y)) + \widehat{\mathcal{H}}^{\varepsilon}_m(E''_\varepsilon \cap u^{-1}(y)) \le \varepsilon.$$

Since $\varepsilon > 0$ was arbitratily small, it concludes the proof. $\square$

13. The Eilenberg inequality

In this section we generalize the part (a) of Theorem 2.5 to metric spaces. A combination with Theorem 12.3 verifies that the Eilenberg inequality in our metric space setting is valid for mappings with upper gradients in the Lorentz space $L_{m,1}$.

Lemma 13.1. *There is a constant $\ell = \ell(D, P, \tau)$ with the following property: Suppose that Y is a metric space and $u : \Omega \to Y$ is an $\boldsymbol{m}$-measurable function with a strictly positive upper gradient $g \in L^1(\Omega)$. Then for every Borel set $E \subset \Omega$ and $\varepsilon > 0$ there is a pairwise disjoint decomposition*

$$E = N \cup \bigcup_j E_j$$

such that $\boldsymbol{m}(N) = 0$ and

$$d_Y(u(z), u(z')) \le (1 + \varepsilon)\ell d_X(z, z') \inf_{x \in E_j} g(x)$$

for every $z, z' \in E_j$.

P r o o f. Let $\varepsilon > 0$ be fixed. First, we decompose X into the sets

$$X_k = \{x : (1 + \varepsilon)^{k-1} \le g(x) < (1 + \varepsilon)^k\}, \quad k \in \mathbb{Z}.$$

We may assume that $E \subset X_k$ for some k. By the Lebesgue differentiation theorem [HaK], almost every point x of E is a Lebesgue point for g. This means that for every $\varepsilon > 0$ there exists a radius R_x such that $B(x, R_x) \subset \Omega$,

$$\fint_{B(x,t)} |g| \, dx' \le (1 + \varepsilon)^k, \quad 0 < t < R_x.$$

We can decompose E into countably many pieces on which R_x is bounded away from zero, so that we may assume that $R_x \ge \delta > 0$ on E. Suppose that $z, z' \in E$ are Lebesgue points for u, $d_X(z, z') = r < \delta/(4\tau)$. Then $B(z, r) \subset B(z', 2r)$ and, by the Poincaré inequality (8.2),

$$\begin{aligned} \fint_{B(z,r)} \fint_{B(z',2r)} d_Y(u(x), u(x')) \, dx' \, dx &\le Cr \fint_{B(z',2\tau r)} g(x') \, dx' \\ &\le Cr(1 + \varepsilon)^k. \end{aligned} \tag{13.1}$$

By Lemma 11.1,

$$\fint_{B(z,r)} d_Y(u(x),u(z))\,dx \le C I_1^{2\tau r} g(z),$$
$$\fint_{B(z',2r)} d_Y(u(x'),u(z'))\,dx' \le C I_1^{4\tau r} g(z').$$

Since

$$I_1^{2\tau r} g(z) = \int_0^{2\tau r} \left(\fint_{B(z,t)} g(x)\,dx \right) dt \le Cr(1+\varepsilon)^k$$

and similarly

$$I_1^{4\tau r} g(z') \le Cr(1+\varepsilon)^k,$$

together with (13.1) we obtain

$$d_Y(u(z),u(z')) \le Cr(1+\varepsilon)^k \le (1+\varepsilon)\,\ell\, d_X(z,z') \inf_{x\in E_j} g(x)$$

for an appropriate constant ℓ. □

R e m a r k. The proof of Lemma 13.1 is fairly general, but it hardly leads to the optimal constant. In some situations, e.g. in Euclidean spaces, one obtains Lemma 13.1 with $\ell = 1$. Indeed, we have $\ell = 1$ if u is smooth. For Sobolev functions, a theorem of Lusin-type (see e.g. [Z]) enables us to approximate u by a smooth function v such that the set where u differs from v is small.

Now, we are ready to establish the Eilenberg-type inequality.

Theorem 13.2. *Suppose that Y is boundedly compact and $m \ge 1$. Let $u : \Omega \to Y$ be an $\boldsymbol{m}$-measurable function satisfying the m-coarea property and let $g \in L^m(\Omega)$ be an upper gradient to u. Let ω be a non-negative $\boldsymbol{m}$-measurable function on Ω. Then*

$$\int_Y^* \left(\int_{u^{-1}(y)} \omega(x)\,d\widehat{\mathcal{H}}_m(x) \right) d\mathcal{H}_m(y) \le (2\ell)^m \boldsymbol{\alpha}_k \int_\Omega \omega\, g^m\,dx, \tag{13.2}$$

where ℓ is the constant from Lemma 13.1.

P r o o f. We may assume that ω is a characteristic function of a measurable set E. Also, we may neglect sets of $\boldsymbol{m}$-measure zero, because for (characteristic functions of) such sets (13.2) holds by the m-coarea property. An easy approximation argument shows that we may assume that g is strictly positive. Given $\varepsilon > 0$, by Lemma 13.1, we may decompose E into Borel sets E_k such that

$$d_Y(u(x), u(x')) \le (1+\varepsilon)\,\ell\, d_X(x,x') \inf_{E_k} g, \quad x, x' \in E_k.$$

Taking away a set of $\boldsymbol{m}$-measure zero, we may assume that every point of E_k is a Lebesgue density point of E_k. Fix $k \in \mathbb{N}$ and choose $\delta > 0$. Let $G_k \subset \Omega$ be an open set such that

$$\int_{G_k} g^m(x)\,dx \le \int_{E_k} (g^m(x) + \varepsilon^m)\,dx.$$

By the fine Vitali covering theorem [He, Thm. 1.6.], we find a sequence of pairwise disjoint balls $B(x_j, r_j) \subset G$ such that $r_j < \varepsilon$,

$$\begin{aligned} \operatorname{diam} u(B(x_j,r_j)) &\le 2(1+\varepsilon)\,\ell r_j \inf_{E_k} g < \delta, \\ \boldsymbol{m}(B(x_j,r_j)) &\le (1+\varepsilon)\,\boldsymbol{m}(B(x_j,r_j)\cap E_k), \end{aligned} \tag{13.3}$$

and

$$\boldsymbol{m}\Big(E_k \setminus \bigcup_j B(x_j,r_j)\Big) = 0.$$

For each j, by (13.3),

$$\begin{aligned} &\boldsymbol{m}(B(x_j,r_j))\big(\operatorname{diam} u(B(x_j,r_j))\big)^m \\ &\qquad \le (1+\varepsilon)\,\boldsymbol{m}(E_k\cap B(x_j,r_j))\big(\operatorname{diam} u(B(x_j,r_j))\big)^m \\ &\qquad \le (1+\varepsilon)^{m+1}\,\ell^m\, r_j^m \int_{B(x_j,r_j)} g^m(x)\,dx. \end{aligned} \tag{13.4}$$

We denote

$$A_j = u(B(x_j,r_j)).$$

Then for every $y \in Y$ we have by (12.5)

$$\widehat{\mathcal{H}}^{\varepsilon}_m\big(E_k \cap u^{-1}(y)\big) \le \sum_j r_j^{-m}\boldsymbol{m}(B(x_j,r_j))\chi_{A_j}(y),$$

so that for the functional introduced at Definition 12.2 and

$$f_{k,\varepsilon}(y) = \widehat{\mathcal{H}}^{\varepsilon}_m\big(E_k \cap u^{-1}(y)\big),$$

appealing to (13.4), we have

$$\begin{aligned}
\Lambda^\delta_m(f_{k,\varepsilon}) &\le \sum_j r_j^{-m}\boldsymbol{m}(B(x_j,r_j))\boldsymbol{\alpha}_k(\operatorname{diam} A_j)^m\\
&\le (2\ell)^m(1+\varepsilon)^{m+1}\boldsymbol{\alpha}_k\sum_j\int_{B(x_j,r_j)} g^m(x)\,dx\\
&\le (2\ell)^m(1+\varepsilon)^{m+1}\boldsymbol{\alpha}_k\int_{G_k} g^m(x)\,dx\\
&\le (2\ell)^m(1+\varepsilon)^{m+1}\boldsymbol{\alpha}_k\int_{E_k} (g^m(x)+\varepsilon^m)\,dx.
\end{aligned}$$

Letting $\delta\to 0$, we obtain

$$\int_Y^* \widehat{\mathcal{H}}^\varepsilon_m\big(E_k\cap u^{-1}(y)\big)\,d\mathcal{H}_m(y)\le (2\ell)^m(1+\varepsilon)^{m+1}\boldsymbol{\alpha}_k\int_{E_k}(g^m+\varepsilon^m)\,dx.$$

Summing over k and then letting $\varepsilon\to 0$, we obtain the required estimate. □

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