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FACTORIZATION OF OPERATORS AND WEIGHTED NORM INEQUALITIES

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The purpose of these lectures is to show how the theory of factorization of operators developed by B. Maurey in the 1970's can be applied to obtain very interesting results about weighted norm inequalities. The idea to carry out this program is due to José Luis Rubio de Francia. He constructed a beautiful theory, which culminates in the extrapolation theorem. This theory is presented in chapter VI of our book [8] in the context of L^p spaces. Here we have chosen to work in a more general class of Banach function spaces, an approach that José Luis Rubio also adopted in some later works [20], [21], [22]. There are two reasons to do this. First of all, the presentation of the main results becomes much clearer, and besides there are very nice applications to Banach lattices to be discussed in Section 5. There are several approaches to extrapolation, giving rise to different results. We have chosen the original approach of José Luis Rubio de Francia, but we have completed the theory so that all the known results become part of it.

§ 1. Banach function spaces

Let $(\Sigma , d\sigma)$ be a complete σ-finite measure space. We shall denote by $\mathcal{M}$ the collection of all extended real-valued measurable functions on Σ and by $\mathcal{M}^+$ the subcollection of $\mathcal{M}$ consisting of those functions whose values lie in $[0,\infty]$.

Definition 1.1. A mapping $\rho : \mathcal{M}^+ \to [0,\infty]$ is called *a function norm* if, for all f, g, f_n $(n = 1,2,3,\dots)$ in $\mathcal{M}^+$, for all constants $a \geq 0$ and for all measurable subsets E of Σ, the following properties hold:

1) $\rho(f) = 0 \iff f = 0$ a.e.; $\rho(af) = a\rho(f)$ and $\rho(f + g) \leq \rho(f) + \rho(g)$

2) $0 \leq g \leq f$ a.e. $\Rightarrow \rho(g) \leq \rho(f)$

3) $0 \leq f_n \uparrow f$ a.e. $\Rightarrow \rho(f_n) \uparrow \rho(f)$

4) $|E| < \infty \Rightarrow \rho(\chi_E) < \infty$

5) $$|E| < \infty \implies \int_E f\, d\sigma \leq C_E \rho(t)$$

for some constant C_E, $0 < C_E < \infty$, depending on E and ρ but independent of f.

If ρ is a function norm, the collection $X = X(\rho)$ of all functions f in $\mathcal{M}$ for which $\rho(|f|) < \infty$ is called *a BANACH FUNCTION SPACE.* For each $f \in X$, we define

$$\|f\|_X = \rho(|f|) .$$

The following result is easy to establish (see [1]):

<u>Theorem 1.2.</u> *Let* ρ *be a function norm, and consider* $X = X(\rho)$. *Then* $(X, \|\ \|_X)$ *is a Banach space and the following properties hold for all* f, g, f_n $(n = 1, 2,\ldots)$ *in* $\mathcal{M}$ *and all measurable subsets* E *in* Σ

a) *(lattice property) If* $|g| \leq |f|$ *a.e. and* $f \in X$, *then* $g \in X$ *and* $\|g\|_X \leq \|f\|_X$

b) *(Fatou property) Suppose* $f_n \in X$, $f_n \geq 0$ $(n = 1,2,\ldots)$ *and* $f_n \uparrow f$ *a.e. If* $f \in X$, *then* $\|f_n\|_X \uparrow \|f\|_X$ *whereas if* $f \notin X$, *then* $\|f_n\|_X \uparrow \infty$.

c) *(Fatou's lemma) If* $f_n \in X$ $(n = 1,2,\ldots)$, $f_n \to f$ *a.e. and* $\liminf_{n\to\infty} \|f_n\|_X < \infty$, *then* $f \in X$ *and* $\|f\|_X \leq \liminf_{n\to\infty} \|f_n\|_X$.

d) *Every simple function belongs to* X.

e) *To each set* E *of finite measure there corresponds a constant* C_E, $0 < C_E < \infty$, *such that*
$$\int_E |f|\, d\sigma \leq C_E \|f\|_X \quad \textit{for all} \quad f \in X .$$

f) *If* $f_n \to f$ *in* X, *then* $f_n \to f$ *in measure on every set of finite measure; in particular, some subsequence of* f_n *converges to* f *a.e.*

In view of theorem 1.2, we shall use the names BANACH FUNCTION SPACE or BANACH LATTICE interchangeably. Here are some examples of Banach lattices:

1) The Lebesgue spaces $L^p = L^p(d\sigma)$, $1 \leq p \leq \infty$, and the weighted Lebesgue spaces $L^p(v) = L^p(v d\sigma)$, $1 \leq p \leq \infty$, given for a weight function $v \geq 0$ by:
$$L^p(v) = \left\{ f \in L^0 : \|f\|_{L^p(v)} = \left(\int_\Sigma |f(\sigma)|^p\, v(\sigma)\, d\sigma \right)^{1/p} < \infty \right\}$$
where $L^0 = \{ f \in \mathcal{M} : |f(\sigma)| < \infty \text{ a.e.} \}$.

2) The Lorentz spaces $L(p,q)$, $1 \leq p,q \leq \infty$, with the **exception** of $L(1,\infty)$ which is not even a normed space. These are defined by:

$$L(p,q) = \{f \in L^0 : \|f\|_{L(p,q)} = \left((q/p) \int_0^\infty \left(t^{1/p} f^*(t)\right)^q dt/t\right)^{1/q} < \infty\}$$

where $f^*(t) = \inf \{s > 0 : |\{|f| > s\}| \leq t\}$ is the non-increasing rearrangement of f .

3) The Orlicz spaces $\Phi(L)$ where Φ is convex, strictly increasing in $[0,\infty)$ and $\Phi(0) = 0$,

$$\Phi(L) = \{f \in L^0 : \int_\Sigma \Phi(|f(\sigma)|/a)\, d\sigma < \infty \text{ for some } a > 0\}$$

with $\|f\|_{\Phi(L)} = \inf \{a > 0 : \int_\Sigma \Phi(|f(\sigma)|/a)\, d\sigma \leq 1\}$.

4) The mixed-norm spaces $L^{p_1,p_2}(\Sigma,d\sigma)$ if $\Sigma = \Sigma_1 \times \Sigma_2$ and $d\sigma = d\sigma_1 \otimes d\sigma_2$ $1 \leq p_1,p_2 \leq \infty$, defined by the condition:

$$\|f\|_{L^{p_1,p_2}} = \left(\int_{\Sigma_2} \left(\int_{\Sigma_1} |f(\sigma_1,\sigma_2)|^{p_1} d\sigma_1\right)^{p_2/p_1} d\sigma_2\right)^{1/p_2} < \infty .$$

Given a Banach lattice $X = X(\rho)$ of functions on $(\Sigma,d\sigma)$, its dual space X^* can not always be identified with a Banach lattice of functions on $(\Sigma,d\sigma)$. This leads us to consider the associate space, which we now define. First of all we consider the associate norm ρ' defined by

$$\rho'(g) = \sup \{ \int_\Sigma f(\sigma)\, g(\sigma)\, d\sigma : f \in \mathcal{M}^+, \ \rho(f) \leq 1\} .$$

It is easy to see that ρ' is also a function norm, and it makes sense to give the following

<u>Definition 1.3.</u> Given a Banach lattice $X = X(\rho)$, the Banach lattice $X(\rho')$ determined by ρ' will be called *the associate space* of X and it will be denoted by X' .

The main properties of the correspondence $X \to X'$ are collected in the following result, whose proof can be seen in [1]:

<u>Theorem 1.4.</u> a) $X' = \{g \in L^0 : fg \in L^1$ *for all* $f \in X\}$ *and*

$$\|g\|_{X'} = \sup \{\left|\int_\Sigma f\, g\, d\sigma\right| : f \in X, \ \|f\|_X \leq 1\} .$$

b) *Every Banach lattice* X *coincides with its second associate space* X'' . *In other words, a function* f *belongs to* X *if and only if it belongs to* X'' *, and in that case* $\|f\|_X = \|f\|_{X''}$.

c) X' *is (canonically isometrically isomorphic to) a closed norming subspace of* X^* . *Norming means that*

$$\|f\|_X = \sup\left\{\left|\int_\Sigma f\, g\, d\sigma\right| : g \in X',\ \|g\|_{X'} \leq 1\right\}$$

for all $f \in X$.

Also in [1] we find this nice characterization of the Banach lattices X for which $X' = X^*$.

Theorem 1.5. *The Banach space dual* X^* *is (canonically isometrically isomorphic to) the associate space* X' *if and only if every* $f \in X$ *satisfies the following property:*

(1.6) *for every* $\varepsilon > 0$ *there is* $\delta > 0$ *such that* $|E| < \delta$ *implies* $\|f\chi_E\|_X < \varepsilon$.

Property (1.6) is referred to by saying that the function f has absolutely continuous norm. When this happens for every $f \in X$, we say that X has absolutely continuous norm. Theorem 1.5 can be rephrased by saying: $X' = X^* \Leftrightarrow X$ has absolutely continuous norm. For example, all the L^p spaces have absolutely continuous norm except L^∞ , and we have: $(L^\infty)' = L^1 \subsetneq (L^\infty)^*$.

The absolute continuity of the norm gives us a version of the dominated convergence theorem.

Proposition 1.7. $f \in X$ *has absolutely continuous norm if and only if the following holds: whenever* f_n $(n = 1,2,\ldots)$ *and* g *are measurable functions satisfying* $|f_n| \leq |f|$ *for all* n *and* $f_n \to g$ *a.e., then* $\|f_n - g\|_X \to 0$.

As a consequence of theorem 1.5, we have the following nice characterization of the reflexive Banach lattices:

Theorem 1.8. *A Banach lattice* X *is reflexive if and only if both* X *and its associate space* X' *have absolutely continuous norm.*

Proof. If X and X' have absolutely continuous norm, then successive applications of theorem 1.5 give: $X^{**} = (X^*)^* = (X')^* = (X')' = X'' = X$. Since all the identifications are the canonical ones, we conclude that X is reflexive. Suppose, conversely that X is reflexive . Recall (theorem 1.4 c)) that X' is a closed norming subspace of X^* . If X' were a proper subspace of X^* , by the Hahn-Banach theorem, there would exist a nonzero functional $\Lambda \in X^{**}$ such that $\Lambda(X') = 0$. The reflexivity of X allows us to represent Λ as

$$\Lambda(f) = \int_\Sigma f\, g\, d\sigma$$

for some $g \in X$ and all $f \in X'$.

But $\Lambda(f) = 0$ for all $f \in X'$. Since X' is norming, this implies $g = 0$ a.e. But then $\Lambda = 0$, which is a contradiction. Thus $X' = X^*$ and, according to theorem 1.5, X has absolutely continuous norm. From this, and the fact that X is reflexive, we get $(X')^* = (X^*)^* = X = X'' = (X')'$. Applying once more theorem 1.5, we get that X' also has absolutely continuous norm. $\square$

For X a Banach lattice we shall use the notation

$$X_+ = \{x \in X : x(\sigma) > 0 \text{ a.e.}\} .$$

<u>Definition 1.9.</u> For X a Banach lattice of measurable functions on $(\Sigma, d\sigma)$ and $a > 0$ we shall consider $X^a = \{y \in L^0 : |y| = x^a \text{ for some } x \in X\}$ and for $y \in X^a$ we shall define $\|y\|_{X^a} = \| |y|^{1/a} \|_X^a$.

<u>Proposition 1.10.</u> *For* $0 < a < 1$, $X^a \cdot X^{1-a} = X$ *and* $\|x \cdot y\|_X \le \|x\|_{X^a} \|y\|_{X^{1-a}}$.

<u>Proof.</u> We may assume $\|x\|_{X^a} = 1 = \|y\|_{X^{1-a}}$. Then

$$|x \cdot y| = \left(|x|^{1/a}\right)^a \left(|y|^{1/(1-a)}\right)^{1-a} \le a|x|^{1/a} + (1-a)|y|^{1/(1-a)}$$

and, consequently,

$$\|x \cdot y\|_X \le a\| |x|^{1/a} \|_X + (1-a)\| |y|^{1/(1-a)} \|_X = 1$$

$\square$

<u>Proposition 1.11.</u> *If* $0 < a \le 1$, $\| \ \|_{X^a}$ *is a norm and* X^a *is a Banach lattice. If* $a > 1$, $\| \ \|_{X^a}$ *is, in general, only a* $(1/a)$*-norm.*

<u>Proof.</u> If $a > 1$, we have

$$|x + y|^{1/a} \le |x|^{1/a} + |y|^{1/a} \quad \text{and this implies}$$

$$\|x + y\|_{X^a}^{1/a} = \| |x + y|^{1/a} \|_X \le \| |x|^{1/a} \|_X + \| |y|^{1/a} \|_X = \|x\|_{X^a}^{1/a} + \|y\|_{X^a}^{1/a} .$$

If $0 < a < 1$, we have

$$\| |x + y|^{1/a} \|_X = \| |x + y| \ |x + y|^{(1/a)-1} \|_X \le$$

$$\le \| |x| \ |x + y|^{(1-a)/a} \|_X + \| |y| \ |x + y|^{(1-a)/a} \|_X .$$

Now we use proposition 1.10 to conclude that

$$\| |x + y|^{1/a} \|_X \le \left(\|x\|_{X^a} + \|y\|_{X^a}\right) \| |x + y|^{1/a} \|_X^{1-a} ,$$

i.e.

$$\|x + y\|_{X^a} \le \|x\|_{X^a} + \|y\|_{X^a} .$$

$\square$

For some Banach lattices X, X^a is still a Banach lattice for some $a > 1$. For example if $X = L^p$, $p > 1$, then $X^a = L^{p/a}$ and this is a Banach lattice for $a \leq p$ and not only for $a \leq 1$. This fact characterizes the p-convex Banach lattices to be defined below.

Note that for $X = L^1$, proposition 1.10 is simply Hölder's inequality.

Definition 1.12. Let $1 \leq p,q \leq \infty$. The Banach lattice X is said to be

a) p-*convex* if

$$\Big\| \Big(\sum_{j=1}^{n} |x_j|^p \Big)^{1/p} \Big\|_X \leq M \Big(\sum_{j=1}^{n} \|x_j\|_X^p \Big)^{1/p}$$

(or $\| \sup_{1 \leq j \leq n} |x_j| \|_X \leq M \sup_{1 \leq j \leq n} \|x_j\|_X$ if $p = \infty$)

b) q-*concave* if

$$\Big(\sum_{j=1}^{n} \|x_j\|_X^q \Big)^{1/q} \leq M \Big\| \Big(\sum_{j=1}^{n} |x_j|^q \Big)^{1/q} \Big\|_X$$

(or $\sup_{1 \leq j \leq n} \|x_j\|_X \leq M \| \sup_{1 \leq j \leq n} |x_j| \|_X$ if $q = \infty$).

The main properties we shall need concerning these notions are collected in the following statement whose proof can be seen in [19].

Proposition 1.13.

a) *Every Banach lattice is* 1-*convex and* ∞-*concave.*

b) *If* X *is* p_0-*convex and* q_0-*concave, then it is also* p-*convex for every* $1 \leq p \leq p_0$ *and* q-*concave for every* $q_0 \leq q \leq \infty$.

c) *If* X *is* p-*convex and* q-*concave, an equivalent norm can be defined so that inequalities* a) *and* b) *in definition* 1.12 *hold with* $M = 1$.

d) X *is* p-*convex if and only if* X^p *is a Banach lattice, with* X *renormed according to* c).

e) X *is* p-*convex (resp.* q-*concave) if and only if* X' *is* p'-*concave (resp.* q'-*convex) where* p' *is the exponent conjugate to* p *given by* $\frac{1}{p} + \frac{1}{p'} = 1$.

Definition 1.14. a) We say that an operator $T : E \to Y$ from the vector space E to the Banach lattice Y, is *sublinear* if it satisfies the following two conditions:

1) $|T(af)| = |a|\,|Tf|$, a.e., $a \in \mathbb{R}$, $f \in E$

2) $|T(f+g)| \leq |Tf| + |Tg|$ a.e., $f, g \in E$.

b) We say that $T : E \to Y$ is *linearizable* if there exists $S : E \to Y(B)$ linear such that $Tf(\sigma) = \|Sf(\sigma)\|_B$ for a.e. $\sigma \in \Sigma$, where B is a certain

Banach space and

$$Y(B) = \{y : \Sigma \to B \ \text{ s.t. } \ \|y(\cdot)\|_B \in Y\} .$$

c) If $T : X \to Y$ is sublinear and X and Y are both Banach lattices, we say that T is *positive* if $|f| \leq g$ a.e. implies $|Tf| \leq Tg$ a.e.

<u>Observation 1.15.</u> Note that a linearizable operator is sublinear. But it is also non-negative in the sense that $Tf \geq 0$ for every f. Every sublinear operator satisfies the condition

$$\Big|\,|Tf| - |Tg|\,\Big| \leq |T(f - g)| \quad \text{a.e.}$$

However, if T is non-negative sublinear, this condition becomes

$$(1.16) \qquad |Tf - Tg| \leq |T(f - g)| \quad \text{a.e.}$$

This condition also holds for linear operators. All the sublinear operators to be considered will be either linear or non-negative sublinear. This justifies the convention which we shall adopt, of calling T sublinear if and only if (1.16) holds for every f, g. Accordingly when we have $T : E \to B$ and B is simply a Banach space, we shall say that T is sublinear if and only if

$$(1.17) \qquad \|Tf - Tg\| \leq \|T(f - g)\| .$$

Observe that, with this restricted meaning, a sublinear operator $T : A \to B$ between two normed spaces is continuous if and only if it is continuous at 0, and this happens if and only if T is bounded, in the sense that $\|Ta\| \leq C\|a\|$ for every $a \in A$. When for a given sublinear operator T we say that T is bounded from X to Y and also from Z to W, we shall implicitly assume that $X \cap Z$ is dense in both X and Z. This implies the uniqueness of its extension, so that it is reasonable to consider it as the same operator.

<u>§ 2. Factorization of operators</u>

<u>Definition 2.1.</u> Let X be a Banach lattice of measurable functions on $(\Sigma, d\sigma)$. Let $T : X \to B$ be an operator sublinear and continuous into the Banach space B. We say that T *factors through* $L^p = L^p(\Sigma, d\sigma)$ if there exist a continuous operator $T_0 : L^p \to B$ and a function $g(\sigma) > 0$ such that the following diagram is commutative:

$$\begin{tikzcd}
X \arrow[rr, "T"] \arrow[dr, "M_g"'] & & B \\
& L^p \arrow[ur, "T_0"'] &
\end{tikzcd}$$

where M_g is the multiplication operator defined by $M_g(x)(\sigma) = x(\sigma) \cdot g(\sigma)$.

In order for M_g to map X into L^p we must have $|x\cdot g|^p \in L^1$ for every $x \in X$ or equivalently $|g|^p \in (X^p)'$. Thus g must belong to $((X^p)')^{1/p}$. If this is the case, we have $\|M_g(x)\|_{L^p} \le C\|x\|_X$ where $C = \|g\|_{((X^p)')^{1/p}}$.

Also since $T_0(fg) = T(f)$, the continuity of T_0 means

$$\|Tf\|_B^p \le C^p \int_\Sigma |f(\sigma)|^p g(\sigma)^p \, d\sigma$$

that is: $T : L^p(v) \to B$ with $v = g^p \in (X^p)'$. This is a Banach lattice if X is p-convex.

<u>Definition 2.2.</u> Given a p-convex Banach lattice X, $1 \le p < \infty$, we shall write $X_p = (X^p)'$.

We shall prove that if X is p-convex and has absolutely continuous norm, the factorization of $T : X \to B$ through L^p is equivalent to the fact that the vector extension $\tilde{T}$ defined by sending each sequence (x_j) of vectors in X to the sequence (Tx_j), is bounded from $X(\ell^p)$ to ℓ^p_B. This means that we have an inequality:

$$\Big(\sum_j \|Tx_j\|_B^p\Big)^{1/p} \le C\Big\|\Big(\sum_j |x_j|^p\Big)^{1/p}\Big\|_X .$$

Actually we shall formulate a slightly more general theorem valid for a family of perators.

<u>Theorem 2.3.</u> *Let* $\mathcal{T}$ *be a family of sublinear operators* $T : X \to B$ *where* X *is a* p*-convex Banach lattice,* $1 \le p < \infty$, *and* B *a Banach space. Then the sufficient condition for the inequality*

$$(2.4) \qquad \Big(\sum_j \|T_jx_j\|_B^p\Big)^{1/p} \le C\Big\|\Big(\sum_j |x_j|^p\Big)^{1/p}\Big\|_X ; \quad T_j \in \mathcal{T}, \quad x_j \in X ,$$

to hold is that there exists $v \in \tilde{X}_p$, $v > 0$, *with* $\|v\|_{\tilde{X}_p} \le 1$, *such that:*

$$\|Tx\|_B^p \le C^p \int_\Sigma |x(\sigma)|^p v(\sigma)\, d\sigma , \quad T \in \mathcal{T}, \quad x \in X .$$

If X *has absolutely continuous norm, the condition is also necessary.*

The <u>proof</u> of theorem 2.3 will depend on the following version of the mini-max lemma.

<u>Lemma 2.5.</u> *Let* A *and* K *be convex subsets of some real vector spaces, and suppose that* K *is endowed with a topology that makes* K *compact. Let* $\Phi : A \times K \to \mathbb{R} \cup \{+\infty\}$ *be a mapping such that:*

(i) $\Phi(\cdot,b)$ *is concave for each fixed* $b \in K$

(ii) $\Phi(a,\cdot)$ *is convex for each fixed* $a \in A$

(iii) $\Phi(a,\cdot)$ *is lower semicontinuous for each* $a \in A$.

Then $\min_{b\in K} \sup_{a\in A} \Phi(a,b) = \sup_{a\in A} \min_{b\in K} \Phi(a,b)$.

For the proof of this lemma we refer to [8].

Proof of Theorem 2.3. Suppose that v exists. Let us prove (2.4):

$$\sum_j \|T_j x_j\|_B^p \le C^p \int_\Sigma \sum_j |x_j(\sigma)|^p \, v(\sigma)\, d\sigma \le$$

$$\le C^p \Big\| \sum_j |x_j|^p \Big\|_{X^p} \|v\|_{(X^p)'} \le C^p \Big\| \Big(\sum_j |x_j|^p \Big)^{1/p} \Big\|_X^p .$$

Conversely, suppose that X has absolutely continuous norm and (2.4) holds. Let

$$A = \{ \sum_{j=1}^n |x_j|^p : x_j \in X \text{ and } \sum_{j=1}^n \|T_j x_j\|_B^p \le 1 \text{ for some } T_j \in \mathcal{T} \}$$

A is clearly a convex subset of X^p .

Let $K = \{z \in \tilde{X}_p : z \ge 0 , \|z\|_{\tilde{X}_p} \le 1\}$. Since X^p has absolutely continuous norm, $\tilde{X}_p = (X^p)' = (X^p)^*$, K is a convex subset of $\tilde{X}_p$ and if we consider the weak-* topology (the one given by X^p), K is also compact.
Consider now $\Phi : A \times K \to \mathbb{R}$ given by

$$\Phi(y,z) = - \int_\Sigma y(\sigma)\, z(\sigma)\, d\sigma .$$

Φ is actually bilinear and is continuous in z . Therefore, lemma 2.5 applies.
Note that

$$\min_{z\in K} \Phi(y,z) = - \sup_{z\in K} \int_\Sigma y(\sigma)\, z(\sigma)\, d\sigma = - \|y\|_{X^p} \le - \frac{1}{C^p} \quad \text{because of (2.4).}$$

Then $\min_{z\in K} \sup_{y\in A} \Phi(y,z) = \sup_{y\in A} \min_{z\in K} \Phi(y,z) \le - \frac{1}{C^p}$. This means that there exists $z \in K$ such that for every $y \in A$, $\Phi(y,z) \le - \frac{1}{C^p}$. If we take $y = |x|^p / \|Tx\|_B^p$, we get

$$- \int_\Sigma |x(\sigma)|^p \, z(\sigma)\, d\sigma \le - \frac{1}{C^p} \|Tx\|_B^p$$

which is what we wanted with $v = z$. $\square$

Now we want to consider the dual situation of factoring an operator $T : B \to Y$ where B is a Banach space and Y is a Banach lattice of measurable functions on $(\Omega, d\omega)$.

<u>Definition 2.6.</u> $T : B \to Y$ sublinear and continuous factors through $L^p = L^p(\Omega, d\omega)$ if there exist a continuous operator $T_0 : B \to L^p$ and a function $g(\omega) > 0$ such that the following diagram is commutative:

$$\begin{tikzcd} B \arrow[rr, "T"] \arrow[dr, "T_0"'] & & Y \\ & L^p \arrow[ur, "M_g"'] & \end{tikzcd}$$

<u>Proposition 2.7.</u> *Assume* Y *is* p-*concave. Then* M_g *takes* L^p *into* Y *if and only if* $g \in (((Y')^{p'})')^{1/p'}$ *, which is a Banach lattice. In that case* M_g *is a continuous operator whose norm coincides with the norm of* g *in the above mentioned lattice.*

<u>Proof.</u> $fg \in Y \quad \forall f \in L^p \Longleftrightarrow hfg \in L^1 \quad \forall f \in L^p \ , \ \forall h \in Y' \Longleftrightarrow hg \in L^{p'}$ $\forall h \in Y' \Longleftrightarrow k|g|^{p'} \in L^1 \quad \forall k \in (Y')^{p'} \Longleftrightarrow |g| \in (((Y')^{p'})')^{1/p'}$. This is a lattice because Y' is p'-convex, which is equivalent to the fact that Y is p-concave. Also

$$\|M_g(f)\|_Y = \int_\Omega f\ g\ h\ d\omega \quad \text{for some} \quad h \quad \text{with} \quad \|h\|_{Y'} \leq 1 \ .$$

Thus $\|M_g(f)\|_Y \leq \|f\|_{L^p} \, \| |gh|^{p'} \|_{L^1}^{1/p'} \leq \|g\| \, \|f\|_{L^p}$. □

Note that, since $T_0 f = (Tf)/g$, the continuity of T_0 means

$$\int_\Omega |Tf(\omega)|^p \, g(\omega)^{-p} \, d\omega \leq C^p \|f\|_B^p$$

that is $T : B \to g\cdot L^p$ is continuous where $g \in (((Y')^{p'})')^{1/p}$. If $p < \infty$, $g\cdot L^p = L^p(u^{-1})$ where $u = g^p \in (((Y')^{p'})')^{p/p'}$.

<u>Definition 2.8.</u> Given Y a p-concave Banach lattice, $1 < p < \infty$, we shall write $\hat{Y}_p = (((Y')^{p'})')^{p/p'}$.

Under certain conditions, we shall prove that the factorization of $T : B \to Y$ through L^p is equivalent to the boundedness of the vector extension $\tilde{T}$ from ℓ^p_B to $Y(\ell^p)$. This means that we have the inequality

$$\left\| \left(\sum_j |Tf_j|^p \right)^{1/p} \right\|_Y \leq C \left(\sum_j \|f_j\|_B^p \right)^{1/p} \ .$$

As in the previous case, we shall formulate a general theorem valid for a family of operators.

<u>Theorem 2.9.</u> *Let* $\mathcal{T}$ *be a family of sublinear operators* $T : B \to Y$ *where* B *is a Banach space and* Y *is a* p-*concave Banach lattice*, $1 < p < \infty$. *Then a*

sufficient condition for the inequality

$$(2.10)\qquad \Big\| \Big(\sum_j |T_j f_j|^p \Big)^{1/p} \Big\|_Y \leq C \Big(\sum_j \|f_j\|_B^p \Big)^{1/p} \quad ; \quad T_j \in \mathcal{T}, \quad f_j \in B,$$

to hold is that there exists $u \in \hat{Y}_p$, $u > 0$ *with* $\|u\|_{\hat{Y}_p} \leq 1$, *such that*

$$\int_\Omega |Tf(\omega)|^p (u(\omega))^{-1} \, d\omega \leq C^p \|f\|_B^p , \quad T \in \mathcal{T}, \quad f \in B .$$

If $(Y')^{p'}$ *is reflexive, the condition is also necessary.*

<u>Proof.</u> The key observation is that

$$(2.11)\qquad \|y\|_Y = \min \Big\{ \Big(\int_\Omega |y(\omega)|^p \, z(\omega)^{-p/p'} d\omega \Big)^{1/p} : z \in ((Y')^{p'})', \quad \|z\| \leq 1 \Big\}.$$

Indeed, if $y \in Y$, we have, for some $y' \in Y'$ with $\|y'\| = 1$:

$$\|y\|_Y = \int_\Omega yy' = \int_\Omega yz^{-1/p'} y' z^{1/p'} \leq \Big(\int_\Omega |y|^p \, z^{-p/p'} \Big)^{1/p} \Big(\int_\Omega |y'|^{p'} z \Big)^{1/p'} \leq$$

$$\leq \Big(\int_\Omega |y|^p \, z^{-p/p'} \Big)^{1/p} \quad \text{for every } z \in ((Y')^{p'})'_+ \text{ with } \|z\| \leq 1$$

and equality is achieved for some z .
Assuming u exists, let us prove (2.10):

$$\Big\| \Big(\sum_j |T_j f_j|^p \Big)^{1/p} \Big\|_Y^p \leq \int_\Omega \sum_j |T_j f_j(\omega)|^p \, u(\omega)^{-1} \, d\omega \leq C^p \sum_j \|f_j\|_B^p .$$

Conversely, assume that (2.10) holds and also that $(Y')^{p'}$ is reflexive. We shall apply again the mini-max lemma 2.5. In order to do that, we define

$$A = \Big\{ \sum_{j=1}^n |T_j f_j|^p : \sum_{j=1}^n \|f_j\|_B^p \leq 1 \Big\} .$$

This is a convex set of Y^p . Let

$$K = \{ z \in ((Y')^{p'})' : z \geq 0 , \quad \|z\| \leq 1 \} .$$

This is a convex set and, since $((Y')^{p'})' = ((Y')^{p'})^*$ (theorem 1.8) it is also weak-* compact.

Define $\Phi : A \times K \to \mathbb{R} \cup \{+\infty\}$ by

$$\Phi(x,z) = \int_\Omega x(\omega) \, z(\omega)^{-p/p'} \, d\omega .$$

Φ is linear in x , and therefore concave. The convexity of Φ in z follows from the fact that the mapping $t \mapsto t^{-a}$, $a > 0$, is convex in $[0,\infty)$. Finally, in order to see that Φ is lower-semicontinuous in z , we need to see that, for every x and every a, the set $E = \{z \in K : \Phi(x,z) \leq a\}$ is closed in the weak-*

topology. But this set is convex. Also, since $((Y')^{p'})^{**} = (Y')^{p'}$, the weak-* topology coincides with the weak topology. These two facts imply that we just need to see that E is closed in the norm topology of $((Y')^{p'})'$ (see, for example [23], theorem 3.12).
Now if $z_j \to z$ in the norm, there is a subsequence converging a.e.. Fatou's lemma (theorem 1.2 c)) can be applied to show that $z_j \in E$ implies $z \in E$. We are in a position to apply lemma 2.5 to Φ :

$$\min_{z \in K} \Phi(x,z) = \| |x|^{1/p} \|_Y^p \le C^p .$$

Thus

$$\min_{z \in K} \sup_{x \in A} \Phi(x,z) = \sup_{x \in A} \min_{z \in K} \Phi(x,z) \le C^p .$$

In other words: there exists $z \in K$ such that for every $x \in A$ $\Phi(x,z) \le C^p$.
In particular, if we take $x = |Tf|^p / \|f\|_B^p$ for some $f \in B$ and some $T \in \mathcal{T}$, we get

$$\int_\Omega |Tf(\omega)|^p \, z(\omega)^{-p/p'} \, d\omega \le C^p \|f\|_B^p .$$

This is what we wanted with $u = z^{p/p'} \in \hat{Y}_p$ which satisfies $\|u\|_{\hat{Y}_p} \le 1$. □

There is a version of theorem 2.9 for $p = \infty$. In that case, note that $g \in Y$ and T maps B into $g \cdot L^\infty = \{f \in L^0 : \|f/g\|_\infty < \infty\}$.

Theorem 2.12. *Let* $\mathcal{T}$ *be a family of sublinear operators* $T : B \to Y$. *Then the inequality*

$$(2.13) \qquad \Big\| \sup_j |T_j f_j| \Big\|_Y \le C \sup_j \|f_j\|_B , \quad T_j \in \mathcal{T} , \quad f_j \in B ,$$

holds if and only if there exists $u \in Y_+$ *with* $\|u\|_Y \le 1$ *such that*

$$\|Tf\|_{u \cdot L^\infty} = \Big\| \frac{Tf}{u} \Big\|_\infty \le C \|f\|_B , \quad T \in \mathcal{T} , \quad f \in B .$$

Proof. If u exists,

$$\Big\| \sup_j |T_j f_j| \Big\|_Y \le \Big\| \sup_j \Big| \frac{T_j f_j}{u} \Big| \Big\|_\infty \|u\|_Y \le C \sup_j \|f_j\|_B$$

and we get (2.13). Conversely if (2.13) holds, let

$$A = \{ \sup_{1 \le j \le n} |T_j f_j| : \sup_{1 \le j \le n} \|f_j\|_B \le 1 \} .$$

This is a directed subset of the closed ball of center 0 and radius C in Y . Then A has a least upper bound $G \in Y$ such that $\|G\|_Y \le C$ (see [24]). If we set $u = G/C$, we have

$$\sup_j |T_j f_j| \le C \sup_j \|f_j\|_B \cdot u$$

and, consequently,

$$\left\|\frac{Tf}{u}\right\|_\infty \leq C\|f\|_B . \qquad \square$$

§ 3. Equivalence between weighted inequalities and vector-valued inequalities

Theorem 3.1. *Let* X *and* Y *be* p*-convex Banach lattices of functions on* $(\Sigma, d\sigma)$ *and* $(\Omega, d\omega)$ *respectively,* $1 \leq p < \infty$ *. Let* $\mathcal{T}$ *be a family of sublinear operators* $T : X \longrightarrow Y$ *. Then a sufficient condition for the vector-valued inequality*

$$\text{(3.2)} \qquad \left\|\left(\sum_j |T_j x_j|^p\right)^{1/p}\right\|_Y \leq C\left\|\left(\sum_j |x_j|^p\right)^{1/p}\right\|_X \;; \quad T_j \in \mathcal{T}, \quad x_j \in X,$$

to hold is that for every positive $u \in \tilde{Y}_p$ *there exists a positive* $v \in X_p$ *such that* $\|v\|_{\tilde{X}_p} \leq \|u\|_{\tilde{Y}_p}$ *and*

$$\int_\Omega |Tx(\omega)|^p \, u(\omega) \, d\omega \leq C^p \int_\Sigma |x(\sigma)|^p \, v(\sigma) \, d\sigma , \quad T \in \mathcal{T}, \quad x \in X .$$

If X *has absolutely continuous norm, the condition is also necessary.*

Proof. If the condition holds, let us prove (3.2)

$$\left\|\left(\sum_j |T_j x_j|^p\right)^{1/p}\right\|_Y^p = \left\|\sum_j |T_j x_j|^p\right\|_{Y^p} \leq \int_\Omega \sum_j |T_j x_j|^p \, u \, d\omega$$

for some $u \in \tilde{Y}_p$, $u > 0$, and $\|u\|_{\tilde{Y}_p} \leq 1$. We get the corresponding v and continue

$$\leq C^p \int_\Sigma \sum_j |x_j|^p \, v \, d\sigma \leq C^p \left\|\sum |x_j|^p\right\|_{X^p} \|v\|_{\tilde{X}_p} \leq C^p \left\|\left(\sum_j |x_j|^p\right)^{1/p}\right\|_X .$$

Conversely, let us assume that (3.2) holds and that X has absolutely continuous norm. Let $u > 0$, $u \in \tilde{Y}_p$. We may assume $\|u\|_{\tilde{Y}_p} = 1$. We see that $Y \hookrightarrow L^p(u)$. Indeed, if $y \in Y$,

$$\int_\Omega |y(\omega)|^p \, u(\omega) \, d\omega \leq \left\| |y|^p \right\|_{Y^p} \|u\|_{\tilde{Y}_p} = \|y\|_Y^p .$$

Thus we have a family of $T : X \longrightarrow L^p(u) \equiv B$ such that

$$\left(\sum_j \|T_j x_j\|_B^p\right)^{1/p} = \left\|\left(\sum_j |T_j x_j|^p\right)^{1/p}\right\|_B \leq C\left\|\left(\sum_j |x_j|^p\right)^{1/p}\right\|_X .$$

Theorem 2.3 applies, and we get $v \in \tilde{X}_p$, $v > 0$ with $\|v\|_{\tilde{X}_p} \leq 1$ such that

$$\int_\Omega |Tx(\omega)|^p\, u(\omega)\, d\omega = \|Tx\|_B^p \leq C^p \int_\Sigma |x(\sigma)|^p\, v(\sigma)\, d\sigma\ , \quad T \in \mathcal{T}\ , \quad x \in X\ .$$

Thus, the condition is necessary. □

Note that in theorem 3.1, we actually get that T is bounded from $L^p(v)$ to $L^p(u)$ because X is dense in $L^p(v)$. Indeed, if $h \in (L^p(v))^* = L^{p'}(v^{-p'/p})$ is such that $\int_\Sigma x(\sigma)\, h(\sigma)\, d\sigma = 0$ for every $x \in X$, we have $h \in L^{p'}(v^{-p'/p}) \hookrightarrow X'$ and, consequently $h(\sigma) = 0$ a.e.

<u>Theorem 3.3.</u> *Let* X *and* Y *be* p*-concave Banach lattices of functions on* $(\Sigma, d\sigma)$ *and* $(\Omega, d\omega)$ *respectively,* $1 < p < \infty$. *Let* $\mathcal{T}$ *be a family of sublinear operators* $T : X \to Y$. *Then a sufficient condition for the vector-valued inequality* (3.2) *to hold is that for every positive* $v \in \hat{X}_p$ *there exists a positive* $u \in \hat{Y}_p$ *such that* $\|u\|_{\hat{Y}_p} \leq \|v\|_{\hat{X}_p}$ *and*

$$\int_\Omega |Tx(\omega)|^p (u(\omega))^{-1}\, d\omega \leq C^p \int_\Sigma |x(\sigma)|^p (v(\sigma))^{-1}\, d\sigma\ , \quad T \in \mathcal{T}.$$

If $(Y')^{p'}$ *is reflexive, the condition is also necessary.*

<u>Proof.</u> If the condition holds, let us prove (3.2). By (2.11) applied to the lattice X,

$$\left\| \left(\sum_j |x_j|^p \right)^{1/p} \right\|_X^p = \int_\Sigma \sum_j |x_j(\sigma)|^p (v(\sigma))^{-1}\, d\sigma$$

for some $v \in \hat{X}_p$, $v > 0$ with $\|v\|_{\hat{X}_p} \leq 1$. By considering the associated $u \in \hat{Y}_p$, we can continue writing:

$$\geq \frac{1}{C^p} \int_\Omega \sum_j |T_j x_j(\omega)|^p (u(\omega))^{-1}\, d\omega \geq \frac{1}{C^p} \left\| \left(\sum_j |T_j x_j|^p \right)^{1/p} \right\|$$

again by (2.11).

Conversely, suppose that (3.2) holds and $(Y')^p$ is reflexive. Given $v \in \hat{X}_p$, $v > 0$ with $\|v\|_{\hat{X}_p} = 1$ we have $B \equiv L^p(v^{-1}) \hookrightarrow X$ by (2.11). We may view $\mathcal{T}$ as a family of operators $T : B \to Y$ such that

$$\left\| \left(\sum_j |T_j f_j|^p \right)^{1/p} \right\|_Y \leq C \left\| \left(\sum_j |f_j|^p \right)^{1/p} \right\|_B = \left(\sum_j \|f_j\|_B^p \right)^{1/p} .$$

We can apply theorem 2.9 to conclude that there exists $u \in \hat{Y}_p$, $u > 0$, with $\|u\|_{\hat{Y}_p} \leq 1$ such that

$$\int_\Omega |Tx(\omega)|^p (u(\omega))^{-1}\, d\omega \leq C^p \|x\|_B^p = C^p \int_\Sigma |x(\sigma)|^p (v(\sigma))^{-1}\, d\sigma\ .$$

□

The case $p = \infty$ is much simpler.

Theorem 3.4. *Let $\mathcal{T}$ be a family of sublinear operators* $T : X \to Y$ *, where* X *and* Y *are Banach lattices of functions on* $(\Sigma, d\sigma)$ *and* $(\Omega, d\omega)$*, respectively. Then the inequality*

$$\left\| \sup_j |T_j x_j| \right\|_Y \leq C \left\| \sup_j |x_j| \right\|_X \tag{3.5}$$

holds if and only if for every $v \in X_+$ *, there exists* $u \in Y_+$ *such that* $\|u\|_Y \leq \|v\|_X$ *and the operators* $T \in \mathcal{T}$ *are uniformly bounded from* $v \cdot L^\infty$ *to* $u \cdot L^\infty$ *with* $\|T\| \leq C$ *, that is:*

$$\left\| \frac{Tx}{u} \right\|_\infty \leq C \left\| \frac{x}{v} \right\|_\infty .$$

Proof. To prove the sufficiency, let $v = \sup_j |x_j|$. Then we have the corresponding u , and since $\|x_j/v\|_\infty \leq 1$, we get:

$$\left\| \sup_j |T_j x_j| \right\|_Y \leq \left\| \sup_j \left| \frac{T_j x_j}{u} \right| \right\|_\infty \|u\|_Y \leq C \|v\|_X$$

which is (3.5). Conversely, if (3.5) holds and we are given $v \in X_+$ with $\|v\|_X = 1$, we have $v \cdot L^\infty \hookrightarrow X$ since $\|x\|_X \leq \|x/v\|_\infty \|v\|_X = \|x/v\|_\infty$. Let $B = v \cdot L^\infty$. Then (3.5) gives

$$\left\| \sup_j |T_j x_j| \right\|_Y \leq C \cdot \left\| \sup_j \left| \frac{x_j}{v} \right| \right\|_\infty \leq C \sup_j \|x_j\|_B .$$

Theorem 2.12 can be applied and we get $u \in Y_+$ with $\|u\|_Y \leq \|v\|_X$ and

$$\left\| \frac{Tx}{u} \right\|_\infty \leq C \|x\|_B = C \left\| \frac{x}{v} \right\|_\infty . \qquad \square$$

When we are dealing with a single Banach lattice X of functions on $(\Sigma, d\sigma)$, theorem 3.1 can be improved obtaining an inequality with the same weight in both sides. This unification of the weight is achieved by the Rubio de Francia algorithm ([19]) which we describe below.

Theorem 3.6. *Let* X *be a* p*-convex Banach lattice of functions on* $(\Sigma, d\sigma)$, $1 \leq p < \infty$ *, and let* $\mathcal{T}$ *be a family of sublinear operators* $T : X \to X$ *. Then a sufficient condition for the vector-valued inequality:*

$$\left\| \left(\sum_j |T_j x_j|^p \right)^{1/p} \right\|_X \leq C_1 \left\| \left(\sum_j |x_j|^p \right)^{1/p} \right\|_X ; \quad T_j \in \mathcal{T} , \quad x_j \in X, \tag{3.7}$$

to hold is that for every positive $u \in \tilde{X}_p$ *there exists a positive* $v \in X_p$ *such that* $u \leq v$, $\|v\|_{\tilde{X}_p} \leq 2\|u\|_{\tilde{X}_p}$ *and*

$$\int_\Sigma |Tx(\sigma)|^p \, v(\sigma) \, d\sigma \leq C_2^p \int_\Sigma |x(\sigma)|^p \, v(\sigma) \, d\sigma ; \quad T \in \mathcal{T}, \quad x \in X .$$

If X *has absolutely continuous norm, the condition is also necessary. Moreover,*

$$2^{-1/p} \le C_1/C_2 \le 2^{1/p}.$$

Proof. The sufficiency is proved exactly as in theorem 3.1, yielding $C_1 \le 2^{1/p} C_2$.

Conversely, if X has absolutely continuous norm, let us prove the necessity. We assume that (3.7) holds, and that we have $u > 0$, $u \in \tilde{X}_p$. Theorem 3.1 gives us $U \in \tilde{X}_p$, $U > 0$ with $\|U\|_{\tilde{X}_p} \le \|u\|_{\tilde{X}_p}$ and such that

$$\int_\Sigma |Tx(\sigma)|^p u(\sigma)\, d\sigma \le C_1^p \int_\Sigma |x(\sigma)|^p U(\sigma)\, d\sigma , \quad T \in \mathcal{T}.$$

Let us call $u_0 = u$ and $u_1 = U$, and use theorem 3.1 again to obtain $U_1 \in \tilde{X}_p$, $U_1 > 0$ with

$$\|U_1\|_{\tilde{X}_p} \le \|u_1\|_{\tilde{X}_p} \le \|u\|_{\tilde{X}_p}$$

and such that

$$\int_\Sigma |Tx(\sigma)|^p u_1(\sigma)\, d\sigma \le C_1^p \int_\Sigma |x(\sigma)|^p U_1(\sigma)\, d\sigma , \quad T \in \mathcal{T} .$$

Now call $u_2 = U_1$ and continue. By induction we get $u_j \in \tilde{X}_p$, $u_j > 0$ with $\|u_j\|_{\tilde{X}_p} \le \|u\|_{\tilde{X}_p}$ and

$$\int_\Sigma |Tx(\sigma)|^p u_j(\sigma)\, d\sigma \le C_1^p \int_\Sigma |x(\sigma)|^p u_{j+1}(\sigma)\, d\sigma , \quad T \in \mathcal{T}, \quad j = 0,1,2,\dots .$$

Let $v = \sum_{j=0}^{\infty} 2^{-j} u_j$. Then $v \ge u_0 = u$, $\|v\|_{\tilde{X}_p} \le 2\|u\|_{\tilde{X}_p}$ and

$$\int_\Sigma |Tx(\sigma)|^p v(\sigma)\, d\sigma \le C_1^p \int_\Sigma |x(\sigma)|^p \sum_{j=0}^{\infty} 2^{-j} u_{j+1}(\sigma)\, d\sigma \le$$

$$\le 2C_1^p \int_\Sigma |x(\sigma)|^p v(\sigma)\, d\sigma$$

which is our condition with $C_2 \le 2^{1/p} C_1$. □

There does not seem to be a direct way to unify the weight in theorem 3.3 or 3.4 when $X = Y$. However, if we are dealing with linear operators, we can achieve the unification by using duality.

Theorem 3.8. *Let* $\mathcal{T}$ *be a family of linear operators* $T : X \to X$ *where* X *is a* p-*concave Banach lattice,* $1 < p < \infty$. *Then, a sufficient condition for* (3.7) *to hold is that for every positive* $u \in \hat{X}_p$ *there exists a positive* $v \in \hat{X}_p$ *such that* $u \le v$, $\|v\|_{\hat{X}_p} \le 2^{p/p'}\|u\|_{\hat{X}_p}$ *and*

$$\int_\Sigma |Tx(\sigma)|^p (v(\sigma))^{-1}\, d\sigma \le C_2^p \int_\Sigma |x(\sigma)|^p (v(\sigma))^{-1}\, d\sigma \ , \quad T \in \mathcal{T} \ .$$

If X *is reflexive, the condition is also necessary. Moreover,* $2^{-1/p'} \le C_1/C_2 \le 2^{1/p}$.

Proof. The sufficiency is proved as in theorem 3.3, giving $C_1 \le 2^{1/p} C_2$. Conversely, let us assume that X is reflexive and that (3.7) holds. The reflexivity implies (theorem 1.8), that $X' = X^*$. Thus, the family $\mathcal{T}^*$ consisting of the adjoint operators T^* of the operators $T \in \mathcal{T}$, is well defined as a class of operators on X' . Besides (3.7) implies

$$\left\| \left(\sum_j |T_j^* y_j|^{p'} \right)^{1/p'} \right\|_{X'} \le C_1 \left\| \left(\sum_j |y_j|^{p'} \right)^{1/p'} \right\|_{X'} \ ; \quad T_j^* \in \mathcal{T}.$$

Now X' is p'-convex and has absolutely continuous norm because X is reflexive. Thus, we can apply theorem 3.6. Observe that $(X')^{\sim}_{p'} = \left((X')^{p'}\right)' = \hat{X}_p^{p'/p}$. Now, given a positive $u \in \hat{X}_p$, let $U = u^{p'/p} \in (X')^{\sim}_{p'}$. We know that there is $V \in (X')^{\sim}_{p'}$ such that $U \le V$, $\|V\|_{(X')^{\sim}_{p'}} \le 2\|U\|_{(X')^{\sim}_{p'}}$ and

$$\int_\Sigma |T^* y(\sigma)|^{p'}\, V(\sigma)\, d\sigma \le 2C_1^{p'} \int_\Sigma |y(\sigma)|^{p'}\, V(\sigma)\, d\sigma \ , \quad T^* \in \mathcal{T}^* \ .$$

But this is equivalent to

$$\int_\Sigma |Tx(\sigma)|^p\, V(\sigma)^{-p/p'} d\sigma \le 2^{p/p'} C_1^p \int_\Sigma |x(\sigma)|^p\, V(\sigma)^{-p/p'} d\sigma \ .$$

If we write $V^{p/p'} = v \in \hat{X}_p$ we have:

$$u = U^{p/p'} \le V^{p/p'} = v \ ,$$

$$\|v\|_{\hat{X}_p} = \|V\|^{p/p'}_{(X')^{\sim}_{p'}} \le 2^{p/p'} \|U\|^{p/p'}_{(X')^{\sim}_{p'}} = 2^{p/p'} \|u\|_{\hat{X}_p}$$

and

$$\int_\Sigma |Tx(\sigma)|^p (v(\sigma))^{-1}\, d\sigma \le 2^{p/p'} C_1^p \int_\Sigma |x(\sigma)|^p (v(\sigma))^{-1}\, d\sigma$$

which is what we wanted with $C_2 \le 2^{1/p'} C_1$.

□

We can also use duality for the case $p = \infty$, improving theorem 3.4 when $X = Y$ and the operators are linear.

Theorem 3.9. *Let* $\mathcal{T}$ *be a family of linear operators* $T : X \to X$, *where* X *is a Banach lattice. Then a sufficient condition for the inequality*

(3.10) $$\left\| \sup_j |T_j x_j| \right\|_X \le C_1 \left\| \sup_j |x_j| \right\|_X \ , \quad T_j \in \mathcal{T},$$

to hold is that for every $u \in X_+$ *there exists* $v \in X_+$ *such that* $u \leq v$, $\|v\|_X \leq 2\|u\|_X$ *and*

$$\left\|\frac{Tx}{v}\right\|_\infty \leq C_2\left\|\frac{x}{v}\right\|_\infty \ ; \quad T \in \mathcal{T} \ .$$

If X *is reflexive, the condition is also necessary. Moreover, we have:* $1/2 \leq C_1/C_2 \leq 2$.

<u>Proof.</u> The sufficiency is proved as in theorem 3.4 and gives $C_1 \leq 2C_2$. If X is reflexive, we have $X' = X^*$, and we may consider the class $\mathcal{T}^*$ formed by the adjoint operators $T^* : X' \to X'$. If (3.10) holds, we claim that

$$\text{(3.11)} \qquad \Big\| \sum_j |T_j^* y_j| \Big\|_{X'} \leq C_1 \Big\| \sum_j |y_j| \Big\|_{X'} \ .$$

Indeed, the left hand side equals

$$\int_\Sigma \sum_j |T_j^* y_j| \ x = \int_\Sigma \sum_j (T_j^* y_j) \ a_j(\omega) \ x(\omega) \ d\omega$$

for some $x \in X$ with $\|x\| \leq 1$ and $\sup_j |a_j(\omega)| \leq 1$. But the last integral can be written as

$$\int_\Sigma \sum_j y_j T_j(a_j x) \leq \int_\Sigma \sum_j |y_j| \cdot \sup_j T_j(a_j x) \ d\sigma \leq$$

$$\leq \Big\| \sum_j |y_j| \Big\|_{X'} \Big\| \sup_j T_j(a_j x) \Big\|_X \leq C_1 \Big\| \sup_j |a_j| \ |x| \Big\|_X \Big\| \sum_j |y_j| \Big\|_{X'} \leq$$

$$\leq C_1 \Big\| \sum_j |y_j| \Big\|_{X'} \ .$$

Once we know (3.11), we can apply theorem 3.6. Note that X' has absolutely continuous norm because X is reflexive, and also $(X')^\sim_1 = X'' = X$. Thus, given $u \in X_+$, we have $v \in X_+$ such that $u \leq v$, $\|v\|_X \leq 2\|u\|_X$ and

$$\int_\Sigma |T^* y(\sigma)| \ v(\sigma) \ d\sigma \leq 2C_1 \int_\Sigma |y(\sigma)| \ v(\sigma) \ d\sigma \ .$$

But, since $(L^1(v))^* = v \cdot L^\infty$ this is equivalent to

$$\left\|\frac{Tx}{v}\right\|_\infty \leq 2C_1\left\|\frac{x}{v}\right\|_\infty$$

which is what we wanted to prove with $C_2 \leq 2C_1$. □

Duality can also be used to obtain a variant of theorem 3.3 for a family of linear operators. For the necessity we require that X has absolutely continuous norm and also that Y is reflexive. These conditions are slightly different from those in theorem 3.3.

We shall end this section by writing versions of the previous theorems for Lebesgue spaces.

Lemma 3.12. a) *For* $q > p$, $(L^q)^{\sim}_p = L^\alpha$ *where* $\frac{1}{\alpha} = 1 - \frac{p}{q}$.

b) *For* $q < p$, $(L^q)^{\wedge}_p = L^\beta$ *where* $\frac{1}{\beta} = \frac{p}{q} - 1$.

Proof. a) L^q is q-convex and, consequently, p-convex. Now $(L^q)^{\sim}_p = (L^{q/p})' = L^{(q/p)'}$. But $\frac{1}{(q/p)'} = 1 - \frac{p}{q} = \frac{1}{\alpha}$.

b) L^q is q-concave and, consequently, p-concave $(L^q)^{\wedge}_p = L^{(q'/p')'(p'/p)}$. But $\frac{1}{(q'/p')'(p'/p)} = \left(1 - \frac{p'}{q'}\right)\frac{p}{p'} = \left(\frac{1}{p'} - \frac{1}{q'}\right)p = \left(\frac{1}{q} - \frac{1}{p}\right)p = \frac{p}{q} - 1 = \frac{1}{\beta}$. □

It is interesting that in both cases the reciprocal of the exponent turns out to be $|1 - \frac{p}{q}|$. This allows us to combine in a single statement the versions of theorems 3.1 and 3.3 for Lebesgue spaces.

We have a family $\mathcal{T}$ of sublinear operators $T : L^q(\Sigma,d\sigma) \longrightarrow L^r(\Omega,d\omega)$. We are interested in knowing when the following vector-valued inequality holds:

$$\left\|\left(\sum_j |T_j f_j|^p\right)^{1/p}\right\|_r \le C\left\|\left(\sum_j |f_j|^p\right)^{1/p}\right\|_q \ ; \quad T_j \in \mathcal{T} , \quad f_j \in L^q . \tag{3.13}$$

The answer is as follows:

Theorem 3.14. *Let* $1 \le p,q,r < \infty$, *and define* α *and* β *by* $\frac{1}{\alpha} = |1 - \frac{p}{r}|$, $\frac{1}{\beta} = |1 - \frac{p}{q}|$. *Then*

a) *If* $p < q,r$, (3.13) *holds if and only if for every* $u \in L^\alpha_+(d\omega)$, *there exists* $v \in L^\beta_+(d\sigma)$ *such that* $\|v\|_\beta \le \|u\|_\alpha$ *and*

$$\int_\Omega |Tf(\omega)|^p u(\omega)\, d\omega \le C^p \int_\Sigma |f(\sigma)|^p v(\sigma)\, d\sigma , \quad T \in \mathcal{T} .$$

b) *If* $p > q,r$, $r > 1$, (3.13) *holds if and only if for every* $v \in L^\beta_+(d\sigma)$, *there exists* $u \in L^\alpha_+(d\omega)$ *such that* $\|u\|_\alpha \le \|v\|_\beta$ *and*

$$\int_\Omega |Tf(\omega)|^p (u(\omega))^{-1}\, d\omega \le C^p \int_\Sigma |f(\sigma)|^p (v(\sigma))^{-1}\, d\sigma , \quad T \in \mathcal{T} .$$

Actually, theorem 3.14 is true even for $0 < p,q,r < \infty$. The same proofs work with some minor changes.

The version of theorem 3.6 for $X = L^q$ will be this:

Theorem 3.15. *Let* $1 \le p < q < \infty$ *and let* $\mathcal{T}$ *be a family of sublinear operators*

$T : L^q(\Sigma,d\sigma) \to L^q(\Sigma,d\sigma)$. *Define* α *by* $\frac{1}{\alpha} = 1 - \frac{p}{q}$. *Then the following conditions are equivalent:*

a) $$\Big\| \Big(\sum_j |T_j f_j|^p \Big)^{1/p} \Big\|_q \le C_1 \Big\| \Big(\sum_j |f_j|^p \Big)^{1/p} \Big\|_q \ ; \quad T_j \in \mathcal{T} , \quad f_j \in L^q .$$

b) *For every* $u \in L^\alpha_+(d\sigma)$, *there exists* $v \in L^\alpha_+(d\sigma)$ *such that* $u \le v$, $\|v\|_\alpha \le 2\|u\|_\alpha$ *and*

$$\int_\Sigma |Tf(\sigma)|^p \, v(\sigma) \, d\sigma \le C_2^p \int_\Sigma |f(\sigma)|^p \, v(\sigma) \, d\sigma , \quad T \in \mathcal{T}.$$

Moreover, $2^{-1/p} \le C_1/C_2 \le 2^{1/p}$.

And here is the version of theorem 3.8:

<u>Theorem 3.16.</u> *Let* $1 < q < p < \infty$ *and let* $\mathcal{T}$ *be a family of linear operators* $T : L^q(\Sigma,d\sigma) \to L^q(\Sigma,d\sigma)$. *Define* α *by* $\frac{1}{\alpha} = \frac{p}{q} - 1$. *Then the following conditions are equivalent:*

a) $$\Big\| \Big(\sum_j |T_j f_j|^p \Big)^{1/p} \Big\|_q \le C_1 \Big\| \Big(\sum_j |f_j|^p \Big)^{1/p} \Big\|_q , \quad T_j \in \mathcal{T} , \quad f_j \in L^q .$$

b) *For every* $u \in L^\alpha_+(d\sigma)$, *there exists* $v \in L^\alpha_+(d\sigma)$ *such that* $u \le v$, $\|v\|_\alpha \le 2^{p/p'}\|u\|_\alpha$ *and*

$$\int_\Sigma |Tf(\sigma)|^p (v(\sigma))^{-1} \, d\sigma \le C_2^p \int_\Sigma |f(\sigma)|^p (v(\sigma))^{-1} \, d\sigma , \quad T \in \mathcal{T} .$$

Moreover, $2^{-1/p'} \le C_1/C_2 \le 2^{1/p}$.

<u>§ 4. Extrapolation theorems</u>

We are going to reformulate the theorems in section 3 for linear operators, replacing the conditions given there by seemingly weaker ones which do not assume any size relation between the weights. These conditions will suffice because we can apply the general principles of Linear Analysis.

We start by recalling the relation between the lattices and the weighted Lebesgue spaces.

<u>Lemma 4.1.</u> *Let* X *be a Banach lattice.*

a) *If* X *is* p*-convex, then* $X = \bigcap L^p(u)$, $u \in (\tilde{X}_p)_+$ *and*

$$\|x\|_X = \max \Big\{ \Big(\int_\Sigma |x(\sigma)|^p \, u(\sigma) \, d\sigma \Big)^{1/p} , \quad \|u\|_{\tilde{X}_p} \le 1 , \quad u > 0 \Big\} .$$

b) *If* X *is* p-*concave, then* $X = \bigcup L^p(v^{-1})$, $v \in (\hat{X}_p)_+$ *and* .

$$\|x\|_X = \min \left\{ \left(\int_\Sigma |x(\sigma)|^p \big(v(\sigma)\big)^{-1} \, d\sigma \right)^{1/p} , \ \|v\|_{\hat{X}_p} \le 1 \ , \ v > 0 \right\} .$$

Proof. a) $|x|^p u \in L^1 \quad \forall u \in (\tilde{X}_p)_+ \Leftrightarrow |x|^p \in (X^p)'' = X^p \Leftrightarrow x \in X$. The expression for the norm follows from $\|x\|_X = \| |x|^p \|_{X^p}^{1/p}$.

b) The identity for the norm has been proved already. It is a reformulation of (2.11). The expression of X as a union of L^p spaces is an immediate consequence. □

Here is the new formulation of theorem 3.1.

Theorem 4.2. *Let* $\mathcal{T}$ *be a family of linear operators* $T : X \to Y$, *where* X *and* Y *are* p-*convex Banach lattices. Then a sufficient condition for the inequality* (3.2) *to hold is that for every positive* $u \in \tilde{Y}_p$ *there exists a positive* $v \in \tilde{X}_p$ *such that all* $T \in \mathcal{T}$ *are uniformly bounded from* $L^p(v)$ *to* $L^p(u)$.

Proof. It follows from lemma 4.1 that $X(\ell^p) = \bigcap L^p(v)(\ell^p)$, $v \in (\tilde{X}_p)_+$ and $Y(\ell^p) = \bigcap L^p(u)(\ell^p)$, $u \in \tilde{Y}_p)_+$. What we have to prove is that all the operators $\tilde{T} : (x_j) \mapsto (T_j x_j)$ obtained by choosing $T_j \in \mathcal{T}$ are (uniformly) bounded from $X(\ell^p)$ to $Y(\ell^p)$. What we assume implies that for every $u \in (\tilde{Y}_p)_+$, there exists $v \in (\tilde{X}_p)_+$ such that $\tilde{T}$ maps $L^p(v)(\ell^p)$ to $L^p(u)(\ell^p)$. Thus, $X(\ell^p)$ is carried into all the $L^p(u)(\ell^p)$'s and, consequently, into $Y(\ell^p)$. Once we know that $\tilde{T}$ carries $X(\ell^p)$ to $Y(\ell^p)$, the fact that it is continuous follows from the closed graph theorem. Indeed, the graph of T is closed in $X(\ell^p) \times Y(\ell^p)$ because it is closed in $L^p(v)(\ell^p) \times L^p(u)(\ell^p)$ and $X(\ell^p) \times Y(\ell^p) \hookrightarrow L^p(v)(\ell^p) \times L^p(u)(\ell^p)$. □

Observation 4.3. Theorem 4.2 continues to hold for a family $\mathcal{T}$ of linearizable operators.

There are two ways to prove this:

1) If $T_j : X \to Y$ is given by

$$T_j x(\omega) = \|S_j x(\omega)\|_{B_j} \quad \text{with} \quad S_j : X \to Y(B_j)$$

linear, we consider the operator $\tilde{S} : (x_j) \to (S_j x_j)$ and we have to prove that $\tilde{S}$ is continuous from

$$X(\ell^p) \quad \text{to} \quad Y\big(\bigoplus_{\ell^p} B_j\big) .$$

What we are assuming implies that for every $u \in (\tilde{Y}_p)_+$, there exists $v \in (\tilde{X}_p)_+$ such that $\tilde{S}$ maps

$$L^p(v)(\ell^p) \quad \text{to} \quad L^p(u)\Big(\bigoplus_{\ell^p} B_j\Big) .$$

Now we just have to use the fact that

$$Y\Big(\bigoplus_{\ell^p} B_j\Big) = \bigcap L^p(u)\Big(\bigoplus_{\ell^p} B_j\Big) , \quad u \in (\tilde{Y}_p)_+$$

to conclude that $\tilde{S}$ takes

$$X(\ell^p) \quad \text{to} \quad Y\Big(\bigoplus_{\ell^p} B_j\Big) .$$

Since $\tilde{S}$ is linear, the theorem follows.

2) The other approach consists in associating to the operator T given by $Tx(\omega) = \|Sx(\omega)\|_B$, the family of linear operators $\{T_h\}$ where h ranges over all functions in $L^\infty(B^*)$ having $\|h\| \leq 1$, and $T_h x(\omega) = \langle Sx(\omega), h(\omega)\rangle$. Now we consider the family $\mathcal{T}'$ which is the union of all the families $\{T_h\}$ when $T \in \mathcal{T}$. Since $|T_h x(\omega)| \leq \|Sx(\omega)\|_B = |Tx(\omega)|$, the new family $\mathcal{T}'$, whose elements are linear operators, satisfies the same condition that $\mathcal{T}$ in terms of weights. Thus, the family $\mathcal{T}'$ satisfies (3.2) and, consequently, the family $\mathcal{T}$ also satisfies (3.2).

There is also a version for p-concave lattices :

Theorem 4.4. *Let* $\mathcal{T}$ *be a family of linear operators, uniformly bounded from* X *to* Y *, both* p*-concave lattices with absolutely continuous norm. Then, a sufficient condition for* (3.2) *to hold is that for every positive* $v \in \hat{X}_p$ *, there exists a positive* $u \in \hat{Y}_p$ *such that all* T *are uniformly bounded from* $L^p(v^{-1})$ *to* $L^p(u^{-1})$.

Proof. It is clear that $\tilde{T}$ sends $X(\ell^p)$ to $Y(\ell^p)$. To prove that it is continuous, we can apply theorem 4.2 to the family of adjoint operators. □

Observation 4.5. Theorem 4.4 is also valid for a family $\mathcal{T}$ of linearizable operators. The second approach adopted in observation 4.3 works equally well in the p-concave case.

When we have one single lattice X , we get results with one single weight, which correspond to theorems 3.6 and 3.8. We write them together as follows:

Theorem 4.6. *Let* X *be a* p*-convex (resp.* p*-concave lattice),* $1 < p < \infty$ *, with absolutely continuous norm (resp. reflexive) and let* $\mathcal{T}$ *be a family of linearizable operators. Then* (3.7) *holds if and only if for every* $u \in \tilde{X}_p$ *(resp.* $\hat{X}_p$ *),*

$u > 0$, *there exists* $v \in \tilde{X}_p$ *(resp.* $\hat{X}_p$ *), such that* $u \leq v$ *and all* $T \in \mathcal{T}$ *are uniformly bounded in* $L^p(v)$ *(resp.* $L^p(v^{-1})$ *).*

Proof. That (3.7) implies our condition follows from theorems 3.6 and 3.8. Note that even though theorem 3.8 is stated for linear operators, it can be applied to linearizable ones just as in observation 4.3 2). Conversely, let us see that our condition implies 3.7. For X p-convex, we just need to apply theorem 4.2, extended to linearizable operators. Indeed, $u \leq v$ implies $L^p(v) \hookrightarrow L^p(u)$ so that all $T \in \mathcal{T}$ are uniformly bounded from $L^p(v)$ to $L^p(u)$, and this is precisely the condition needed in theorem 4.2. For X p-concave, we apply theorem 4.4, extended to linearizable operators. This requires that for each $u \in (\hat{X}_p)_+$, we find $v \in (\hat{X}_p)_+$ such that all $T \in \mathcal{T}$ are uniformly bounded from $L^p(u^{-1})$ to $L^p(v^{-1})$. What we have now is $v \geq u$ and T uniformly bounded in $L^p(v^{-1})$. But $L^p(u^{-1}) \hookrightarrow L^p(v^{-1})$ and we actually have what we wanted. □

The case $p = \infty$ requires a special formulation, but it is proved with the same method:

Theorem 4.7. *Let* $\mathcal{T}$ *be a family of linearizable operators uniformly bounded in the reflexive Banach lattice* X . *Then* (3.10) *holds if and only if for every* $u \in X_+$ *there exists* $v \in X_+$ *such that* $u \leq v$ *and all* $T \in \mathcal{T}$ *are uniformly bounded in* $v \cdot L^\infty$.

The key to the extrapolation theorems is going to be a boundedness criterion for linear operators obtained from theorems 4.2, 4.4 and 4.6 by means of the following fundamental result due to Grothendieck and Krivine (see [13] and [14]).

Theorem 4.8. *Let* $T : X \to Y$ *be a linear operator, bounded from* X *to* Y , *both Banach lattices. Then*

$$\left\| \left(\sum_j |Tx_j|^2 \right)^{1/2} \right\|_Y \leq K_G \cdot \|T\| \cdot \left\| \left(\sum_j |x_j|^2 \right)^{1/2} \right\|_X , \quad x_j \in X ,$$

where K_G *is Grothendieck's universal constant whose value is still unknown although* $1 < K_G < 2$.

The corresponding theorem for Lebesgue spaces is due to Marcinkiewicz and Zygmund and it is much simpler (see [8], chapter V).

Here are the boundedness criteria we get, where for simplicity we write $\tilde{X}$ instead of $\tilde{X}_2$ and $\hat{X}$ instead of $\hat{X}_2$.

Theorem 4.9. *Let* X *and* Y *be reflexive Banach lattices of measurable functions* *on* $(\Sigma, d\sigma)$ *and* $(\Omega, d\omega)$, *respectively and suppose we have a linear operator* T *sending measurable functions on* $(\Sigma, d\sigma)$ *to measurable functions on* $(\Omega, d\omega)$.

Then:

a) *If* X *and* Y *are 2-convex,* T *is bounded from* X *to* Y *if and only if for every* $u \in \tilde{Y}_+$ *there exists* $v \in \tilde{X}_+$ *such that* T *is bounded from* $L^2(v)$ *to* $L^2(u)$.

b) *If* X *and* Y *are 2-concave,* T *is bounded from* X *to* Y *if and only if for every* $v \in \hat{X}_+$, *there exists* $u \in \hat{Y}_+$ *such that* T *is bounded from* $L^2(v^{-1})$ *to* $L^2(u^{-1})$.

<u>Theorem 4.10.</u> *Let* X *be a reflexive, 2-convex (resp. 2-concave) Banach lattice of measurable functions on* $(\Sigma, d\sigma)$ *and suppose* T *is a linear operator sending measurable functions on* $(\Sigma, d\sigma)$ *to measurable functions on* $(\Sigma, d\sigma)$. *Then* T *is bounded in* X *if and only if for every* $u \in \tilde{X}_+$ *(resp.* $\hat{X}_+$ *) there exists* $v \in \tilde{X}_+$ *(resp.* $\hat{X}_+$ *) such that* $u \leq v$ *and* T *is bounded in* $L^2(v)$ *(resp.* $L^2(v^{-1})$*).*

In theorems 4.9 and 4.10 the exponent 2 was crucial because of the Grothendieck-Krivine inequality which is false in general for $p \neq 2$. However, if T is linear and positive, the boundedness $T : X \to Y$ implies the boundedness $\tilde{T} : X(\ell^p) \to Y(\ell^p)$ for any $1 \leq p \leq \infty$, as we see next.

<u>Proposition 4.11.</u> *Let* $T : X \to Y$ *be bounded, linear and positive, where* X *and* Y *are Banach lattices. Then*

$$\left\| \left(\sum_j |Tx_j|^p \right)^{1/p} \right\|_Y \leq \|T\| \left\| \left(\sum_j |x_j|^p \right)^{1/p} \right\|_X , \quad 1 \leq p < \infty,$$

and

$$\left\| \sup_j |Tx_j| \right\|_Y \leq \|T\| \left\| \sup_j |x_j| \right\|_X .$$

<u>Proof.</u> Assume $1 \leq p < \infty$. The proof for $p = \infty$ is analogous, only the notation is different.

$$\left(\sum_{j=1}^n |x_j|^p \right)^{1/p} \geq \left| \sum_{j=1}^n a_j x_j \right| \quad \text{whenever} \quad \sum_{j=1}^n |a_j|^{p'} \leq 1 .$$

Since T is positive,

$$T\left(\left(\sum_{j=1}^n |x_j|^p \right)^{1/p} \right) \geq \left| \sum_{j=1}^n a_j Tx_j \right| .$$

Hence

$$\left(\sum_{j=1}^{n}|Tx_j|^p\right)^{1/p} = \sup\left\{\left|\sum_{j=1}^{n}a_jTx_j\right| : \sum_{j=1}^{n}|a_j|^{p'} \le 1\right\} \le T\left(\left(\sum_{j=1}^{n}|x_j|^p\right)^{1/p}\right)$$

and, consequently,

$$\left\|\left(\sum_{j=1}^{n}|Tx_j|^p\right)^{1/p}\right\|_Y \le \left\|T\left(\left(\sum_{j=1}^{n}|x_j|^p\right)^{1/p}\right)\right\|_Y \le \|T\| \left\|\left(\sum_{j=1}^{n}|x_j|^p\right)^{1/p}\right\|_X . \quad \square$$

We can make the following

Observation 4.12. If T is linear and positive, theorems 4.9 and 4.10 continue to hold for $p \neq 2$, $1 \le p < \infty$. Naturally we have to replace $\tilde{X}$ by $\tilde{X}_p$ and $\hat{X}$ by $\hat{X}_p$. Even $p = \infty$ is admissible in the concave case. We just have to use X instead of $\hat{X}_p$ and $v\cdot L^\infty$ instead of $L^p(v^{-1})$.

When the lattices are Lebesgue spaces, proposition 4.11 has the following version valid for linearizable operators.

Proposition 4.13. *Let* $T : X \to Y$ *be bounded linearizable and positive where* $X = L^q(Ud\sigma)$ *and* $Y = L^q(Vd\omega)$, $U > 0$, $V > 0$. *Then, for every* $q \le p < \infty$, *we have* $\tilde{T}$ *bounded from* $X(\ell^p)$ *to* $Y(\ell^p)$ *with* $\|\tilde{T}\| = \|T\|$.

Proof. When $p = q$ the result is obvious, so that we have $\tilde{T} : X(\ell^q) \to Y(\ell^q)$ bounded. By the positivity we also have $\tilde{T} : X(\ell^\infty) \to Y(\ell^\infty)$ bounded since $\sup|Tx_j| \le T(\sup|x_j|)$. Now the result follows by interpolation, which is perfectly legitimate since we are really interpolating a linear operator $\tilde{T}_0 : X(\ell^q) \to Y(\ell^q_B)$, $\tilde{T}_0 : X(\ell^\infty) \to Y(\ell^\infty_B)$ associated to T_0 such that $|Tx| = \|T_0x\|_B$. $\square$

We are finally ready to obtain the extrapolation theorems. We shall use the following notations:
For an operator T sending measurable functions on $(\Sigma, d\sigma)$ to measurable functions on $(\Omega, d\omega)$ and $1 \le p < \infty$,

$$V_p(T) = \{(u,v) : u > 0 \text{ a.e. on } \Omega,\ v > 0 \text{ a.e. on } \Sigma \text{ and } T \text{ is bounded from } L^p(v) \text{ to } L^p(u)\}.$$

In particular for $p = 2$ we shall simply write $V(T)$ for $V_2(T)$. Also

$$V_\infty(T) = \{(u,v) : u > 0 \text{ a.e. on } \Omega,\ v > 0 \text{ a.e. on } \Sigma \text{ and } T \text{ is bounded from } v\cdot L^\infty \text{ to } u\cdot L^\infty\}.$$

Observation 4.14. If T is sublinear and positive, $(u,v) \in V_\infty(T)$ if and only if $|Tv| \le Cu$ a.e. for some constant C.

Indeed, $T : v\cdot L^{\infty} \to u\cdot L^{\infty}$ bounded implies $|Tv/u| \leq \|T\| \, \|v/v\|_{\infty} = \|T\|$. Conversely, if $|Tv| \leq Cu$, we have

$$|f| = \left|\frac{f}{v}\cdot v\right| \leq \left\|\frac{f}{v}\right\|_{\infty} v .$$

Since T is positive and sublinear,

$$|Tf| \leq \left\|\frac{f}{v}\right\|_{\infty} |Tv| \leq C\left\|\frac{f}{v}\right\|_{\infty} u .$$

Thus $T : v\cdot L^{\infty} \to u\cdot L^{\infty}$ is bounded with $\|T\| \leq C$.

When $\Sigma = \Omega$, we shall consider

$$W_p(T) = \{w > 0 \text{ a.e. on } \Sigma = \Omega : T \text{ is bounded in } L^p(w)\},$$

$1 \leq p < \infty$. Also in this case when $p = 2$ we shall write $W(T)$ for $W_2(T)$. Finally,

$$W_{\infty}(T) = \{w > 0 \text{ a.e. on } \Sigma = \Omega : T \text{ is bounded in } w\cdot L^{\infty}\} .$$

When T is positive sublinear we have, as before: $w \in W_{\infty}(T) \Leftrightarrow |Tw| \leq Cw$ a.e.

Here is, first of all, the extrapolation theorem from L^2 .

<u>Theorem 4.15.</u> *Let* S *and* T *be operators sending functions on* $(\Sigma,d\sigma)$ *to functions on* $(\Omega,d\omega)$ *, such that:*

a) S *is linear*

b) T *is linearizable*

c) $V(S) \subset V(T)$.

If X *and* Y *are reflexive Banach lattices of functions on* $(\Sigma,d\sigma)$ *and* $(\Omega,d\omega)$*, respectively, both 2-convex or both 2-concave and if* S *is bounded from* X *to* Y *, then* T *is also bounded from* X *to* Y .

In case $X = Y$, c) *can be replaced by the weaker assumption*

c') $W(S) \subset W(T)$.

<u>Proof.</u> Suppose that X and Y are 2-convex. Let $u \in \tilde{Y}_+$. Since $S : X \to Y$ is bounded, theorem 4.9 implies that there exists $v \in \tilde{X}_+$ such that $(u,v) \in V(S) \subset V(T)$. Now we can proceed as in theorem 4.2 and observation 4.3, obtaining in particular that T is bounded from X to Y . The argument is similar in the case of 2-concave lattices or in the case $X = Y$. □

If S is linear and positive, extrapolation works from any $1 < p < \infty$.

<u>Theorem 4.16.</u> *Let* $1 < p < \infty$ *and suppose that* S *and* T *are operators sending functions on* $(\Sigma,d\sigma)$ *to functions on* $(\Omega,d\omega)$ *, such that:*

a) S *is linear and positive*

b) T *is linearizable*

c) $V_p(S) \subset V_p(T)$.

If X *and* Y *are reflexive Banach lattices of functions on* $(\Sigma, d\sigma)$ *and* $(\Omega, d\omega)$*, respectively, both* p-*convex or both* p-*concave, and if* S *is bounded from* X *to* Y *, then* T *is also bounded from* X *to* Y .

In case $X = Y$ *condition* c) *can be replaced by the weaker assumption*

c') $W_p(S) \subset W_p(T)$.

Proof. Suppose that X and Y are p-convex. Let $u \in (\tilde{Y}_p)_+$. Proposition 4.11 implies that S is bounded from $X(\ell^p)$ to $Y(\ell^p)$. Then theorem 3.1 implies that there exists $v \in (\tilde{X}_p)_+$ such that $(u,v) \in V_p(S) \subset V_p(T)$. Then observation 4.3 gives the boundedness of T .

Similar arguments work for p-concave lattices or for $X = Y$ with c'). $\square$

From theorem 4.7 we can obtain an extrapolation theorem from $p = \infty$. However, when $X \neq Y$ the result is trivial because $V_\infty(S) \subset V_\infty(T) \Rightarrow |Tf| \leq CS(|f|)$. Indeed,

$$(Sv,v) \in V_\infty(S) \subset V_\infty(T) \Rightarrow T : v \cdot L^\infty \to (Sv) \cdot L^\infty \quad \text{i.e.} \quad \left|\frac{Tf}{Sv}\right| \leq C \left|\frac{f}{v}\right| .$$

Putting $v = |f|$ we get $|Tf| \leq CS(|f|)$. However for $X = Y$ we get an interesting result.

Theorem 4.17. *Suppose that* X *is a reflexive Banach lattice and*

a) S *is linear and positive*

b) T *is linearizable*

c) $W_\infty(S) \subset W_\infty(T)$.

Then if S *is bounded in* X , T *is also bounded in* X .

For every fixed p_0 and p , and weights $u, v > 0$, the lattices $X = L^p(v)$ and $Y = L^p(u)$ are either p_0-convex if $p_0 \leq p$ or p_0-concave if $p_0 \geq p$. Besides, they are reflexive provided $1 < p < \infty$. If we apply theorems 4.15, 4.16 and 4.17 to this case, we obtain

Theorem 4.18. *Let* S *be a linear operator and* T *a linearizable operator.*

a) *If* $V(S) \subset V(T)$ *, then* $V_p(S) \subset V_p(T)$ *for every* $1 < p < \infty$.

b) *If we have a single measure space and* $W(S) \subset W(T)$ *, then* $W_p(S) \subset W_p(T)$ *for every* $1 < p < \infty$.

c) *If* S *is positive and* $V_{p_0}(S) \subset V_{p_0}(T)$ *for some* $1 < p_0 \leq \infty$ *, then* $V_p(S) \subset V_p(T)$ *for every* $1 < p < \infty$.

d) *If we have a single measure space,* S *is positive and* $W_{p_0}(S) \subset W_{p_0}(T)$ *for some* $1 < p_0 \leq \infty$ *, then* $W_p(S) \subset W_p(T)$ *for every* $1 < p < \infty$ *.*

If the operator S is not linear, but still linearizable and positive, we can get an extrapolation theorem from above, by making use of proposition 4.13.

<u>Theorem 4.19.</u> *Let* S *and* T *be linearizable operators. Assume also that* S *is positive.*

a) *If* $V_{p_0}(S) \subset V_{p_0}(T)$ *for some* $1 < p_0 \leq \infty$ *, then* $V_p(S) \subset V_0(T)$ *for every* $1 < p \leq p_0$ *.*

b) *If we have a single measure space and* $W_{p_0}(S) \subset W_{p_0}(T)$ *for some* $1 < p_0 \leq \infty$*, then* $W_p(S) \subset W_p(T)$ *for every* $1 < p \leq p_0$ *.*

<u>Proof.</u> Let us see, for example, how to prove b). $w \in W_p(S)$ implies that S is bounded in $X = L^p(w)$. Since $p < p_0$, proposition 4.13 applies to S , which is linearizable and positive. Thus, $\tilde{S}$ is bounded in $X(\ell^{p_0})$. Now theorems 4.6 or 4.7 can be used. Note that X is p_0-concave and reflexive. From the boundedness of $\tilde{S}$, we get that for every $U \in \hat{X}_{p_0}$ if $p_0 < \infty$ (or $U \in X$ if $p_0 = \infty$) with $U > 0$, there exists $V \in \hat{X}_{p_0}$ if $p_0 < \infty$ (or $V \in X$ if $p_0 = \infty$) such that $U \leq V$ a.e. and $V^{-1} \in W_{p_0}(S)$ if $p_0 < \infty$ (or $V \in W_\infty(S)$ if $p_0 = \infty$). But $W_{p_0}(S) \subset W_{p_0}(T)$ and we can apply again theorems 4.6 or 4.7, this time to T and in the opposite direction. We conclude that, in particular, T is bounded in X , that is: $w \in W_p(T)$. □

Sometimes the classes $V_p(S)$ or $W_p(S)$ behave well under duality, and this can be used to extrapolate from a given p_0 to any other p . Here is a result in this direction:

<u>Theorem 4.20.</u> *Let* S *be an operator linearizable and positive such that for every* $1 < p < \infty$ *,*

$$(4.21) \qquad w \in W_p(S) \Leftrightarrow w^{-p'/p} \in W_{p'}(S) \ .$$

Let T *be a linearizable operator such that* $W_{p_0}(S) \subset W_{p_0}(T)$ *for some* $1 < p_0 \leq \infty$ *. Then* $W_p(S) \subset W_p(T)$ *for every* $1 < p < \infty$ *.*

<u>Proof.</u> We just need to consider $p_0 < p < \infty$, since the remaining cases are covered by the previous theorem. Assume T is linear, $w \in W_p(S) \Rightarrow w^{-p'/p} \in W_{p'}(S)$. But $p' < p_0'$ and

$$W_{p_0'}(S) = \left(W_{p_0}(S)\right)^{-p_0'/p_0} \subset \left(W_{p_0}(T)\right)^{-p_0'/p_0} = W_{p_0'}(T^*) .$$

Theorem 4.19 yields $W_{p'}(S) \subset W_{p'}(T^*)$. Thus $w^{-p'/p} \in W_{p'}(T^*)$ and this is equivalent to $w \in W_p(T)$.

For T non-linear, we just need to consider the linearizations $\{T_h\}$ as in observation 4.3 2) and apply theorem 4.19 to the collection of the adjoints $\{T_h^*\}$. □

The last six theorems are variants of an abstract extrapolation theorem. By making specific choices of the operator S , we get concrete extrapolation theorems. Here are some candidates for S .

1) Let M be the Hardy-Littlewood maximal operator, sending a function $f \in L^1_{loc}(\mathbb{R}^n)$ to

$$Mf(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y)| \, dy ,$$

where the supremum is taken over all the cubes Q with sides parallel to the coordinate axes and containing the point x . It was proved by Muckenhoupt [16] (see also [4]) that for $1 < p < \infty$ $W_p(M) = A_p$, the class of weights defined by

$$(4.22) \qquad w \in A_p \Leftrightarrow \sup_Q \left(\frac{1}{|Q|} \int_Q w\right)^{1/p} \left(\frac{1}{|Q|} \int_Q w^{-p'/p}\right)^{1/p'} < \infty ,$$

where the sup is taken over all the cubes with sides parallel to the coordinate axes. Also

$$W_\infty(M) = A_1 = \{w \geq 0 : Mw \leq Cw \quad \text{a.e.}\} .$$

2) Let M^* be the strong maximal operator, sending a function $f \in L^1_{loc}(\mathbb{R}^n)$ to

$$M^*f(x) = \sup_{R \ni x} \frac{1}{|R|} \int_R |f(y)| \, dy ,$$

where the supremum is now taken over all the intervals (i.e.: Cartesian products of intervals) containing x .
It can be seen (for example in [8] IV.6) that for $1 < p < \infty$ $W_p(M^*) = A_p^*$, the class of weights defined by

$$(4.23) \qquad w \in A_p^* \Leftrightarrow \sup_R \left(\frac{1}{|R|} \int_R w\right)^{1/p} \left(\frac{1}{|R|} \int_R w^{-p'/p}\right)^{1/p'} < \infty ,$$

where the sup is taken over all intervals. Also

$$W_\infty(M^*) = A_1^* = \{w \geq 0 : M^*w \leq Cw \quad \text{a.e.}\} .$$

Alternatively, for $1 \leq p < \infty$, $w \in A_p^*$ if and only if it is A_p in each variable, with uniform constant, for a.e. determination of the other variables.

3) The operator M is linearizable and positive but it is very interesting to know that there is a linear operator giving rise to the same classes of weights.

If $n = 1$, we can take the Hilbert transform H given by:

$$Hf(x) = \text{p.v. } \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{f(y)}{x - y} \, dy$$

and we have ([8]): $W_p(H) = A_p$, $1 < p < \infty$.

If $n > 1$, we can take the Riesz transforms $R_1, R_2, \ldots, R_n$ given by:

$$R_j f(x) = \text{p.v. } c_n \int_{\mathbb{R}^n} \frac{x_j - y_j}{|x - y|^{n+1}} f(y) \, dy \; ; \quad c_n = \frac{\Gamma((n+1)/2)}{\pi^{(n+1)/2}}$$

If we consider $R = R_1 + R_2 + \ldots + R_n$, we have (see [4] and [8])

$$W_p(R) = A_p , \quad 1 < p < \infty .$$

4) Also for M^* we can find a linear substitute which turns out to be the multiple Hilbert transform H^* given by:

$$H^* f(x) = \lim_{\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n \to 0} \int_{|x_j - y_j| > \varepsilon_j} \frac{f(y) \, dy}{(x_1 - y_1)(x_2 - y_2) \ldots (x_n - y_n)} .$$

We have (see [8] IV.6.) $W_p(H^*) = A_p^*$, $1 < p < \infty$

For these particular examples we get the following extrapolation theorem:

<u>Theorem 4.24.</u> *Let* T *be a linearizable operator whose domain and range consist of measurable functions on* $\mathbb{R}^n$. *Suppose that either*

(i) *for some* $1 < p_0 < \infty$, T *is bounded in* $L^{p_0}(w)$ *for every* $w \in A_{p_0}$ *(resp.* $A_{p_0}^*$ *) or*

(ii) T *is bounded in* $w \cdot L^\infty$ *for every* $w \in A_1$ *(resp.* A_1^* *).*

Then

a) *for every* $1 < p < \infty$ *and every* $w \in A_p$ *(resp.* A_p^* *),* T *is bounded in* $L^p(w)$.

Not only that, but in general

b) *f* X *is a reflexive Banach lattice 2-convex or 2-concave and* R *(resp.* H^* *) is bounded in* X, *then* T *also is bounded in* X.

<u>Proof.</u> Note that the classes A_p and A_p^* satisfy condition (4.21), so that theorem 4.20 can be applied with $S = M$ or M^*, obtaining a). In particular, a)

holds with $p = 2$ and we can apply theorem 4.15 with $S = R$ or H^* to obtain b)

$\square$

<u>Observation 4.25.</u> Condition (ii) can be replaced by this weaker one:

T maps $w\cdot L^\infty$ into B.M.O.(w) for every $w \in A_1$, where

$$g \in \text{B.M.O.}(w) \Leftrightarrow \sup_Q \frac{1}{w(Q)} \int_Q |g(x) - g_Q| \, dx < \infty$$

with $g_Q = \frac{1}{|Q|} \int_Q g$.

If we define $g^\#(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |g(y) - g_Q| \, dy$,

we observe that for $w \in A_1$,

$$g \in \text{B.M.O.}(w) \Rightarrow g^\#(x) = \sup_{Q \ni x} \frac{w(Q)}{|Q|} \frac{1}{w(Q)} \int_Q |g(y) - g_Q| \, dy \leq$$

$$\leq C.Mw(x) \leq Cw(x) \Rightarrow g^\# \in w\cdot L^\infty$$

If we define $T^\# f = (Tf)^\#$ we see that $T^\#$ satisfies (ii). Then if $w \in A_p$, $1 < p < \infty$, we have

$$\int_{\mathbb{R}^n} |Tf|^p w \leq C \int_{\mathbb{R}^n} |(Tf)^\#|^p w \leq C \int_{\mathbb{R}^n} |f|^p w .$$

For the first inequality see [8], chapter IV, theorem 2.20. The weaker condition is sometimes easy to check whereas (ii) fails for many natural operators satisfying a).

We shall finish this section by recalling briefly the history of the extrapolation theorem. It was first discovered by Rubio de Francia [19] with a non-constructive proof of the type we have given, but only for the A_p classes. Then García-Cuerva [6] gave a constructive proof that used the particular definition of the A_p classes plus the Rubio de Francia algorithm. Jawerth [12] proved a general theorem for L^p spaces by using the Rubio de Francia algorithm plus interpolation. Then it came the general formulation of Rubio de Francia [20], [21] in lattices, which is the one we have presented. The case $p_0 = \infty$ was treated in [10] and [7].

<u>§ 5. Applications</u>

We shall give a characterization of U.M.D. Banach lattices in terms of A_p weights due to José Luis Rubio de Francia [22]. U.M.D. stands for unconditional martingale differences. Here is the original definition of this condition:

Definition 5.1. We shall say that a Banach space B is U.M.D. if and only if the following inequality holds:

$$\Big\| \sum_{k=1}^{n} \varepsilon_k d_k \Big\|_{L^p(B)} \le C_{p,B} \Big\| \sum_{k=1}^{n} d_k \Big\|_{L^p(B)}$$

for all n, all $\varepsilon_k = \pm 1$, and all B-valued martingale differences (that is: $d_k = f_k - f_{k-1}$ for a B-valued martingale $\{f_k\}$), p is some fixed exponent with $1 < p < \infty$.

Even though this definition seems to depend on p, it does not. Actually Burkholder [3] gave a geometrical characterization called ρ-convexity, which is obviously independent of p.

We shall not use any of these characterizations. Instead, we shall base our discussion upon the following

Theorem 5.2. *The Banach space B is U.M.D. if and only if the B-valued extension H_B of the Hilbert transform H (say, on the torus $\mathbf{T}$) is bounded in $L^p(B)$ for some $1 < p < \infty$. This means, that the operator H_B defined on*

$$L^p \otimes B = \{ \sum_j \phi_j(t) b_j : \phi_j \in L^p, \ b_j \in B \} \ \text{by} \ H_B\Big(\sum_j \phi_j b_j \Big) = \sum_j (H\phi_j) b_j$$

satisfies: $\|H_B(f)\|_{L^p(B)} \le C\|f\|_{L^p(B)}$

so that it extends continuously to $L^p(B)$.

That the boundedness of H_B is necessary was proved by Burkholder [3] and that it is sufficient by Bourgain [2]. When X is a Banach lattice of functions on $(\Sigma, d\sigma)$ we can view $L^p(X) = L^p(\mathbf{T})(X)$ as a lattice of functions of two variables $f(t,\sigma)$, $t \in \mathbf{T}$, $\sigma \in \Sigma$. If we have an operator A bounded in L^p, we can define $\tilde{A}$ at least in $L^p \otimes X$ by

$$\tilde{A}f(t,\sigma) = A\big(f(\cdot,\sigma)\big)(t) .$$

Note that $\tilde{H} = H_X$ on $L^p \otimes X$.

Here is the characterization of U.M.D. Banach lattices in terms of A_p weights:

Theorem 5.3. *Let X be a Banach lattice, reflexive and p_0-convex for some $p_0 > 1$. Fix p such that $1 < p < p_0$. Then X is U.M.D. if and only if for every $u \in L^2(\tilde{X}_p)$, $u \ge 0$, there exists $w \in L^2(\tilde{X}_p)$ such that $u \le w$, $\|w\| \le 2\|u\|$ and $w(\cdot,\sigma) \in A_p$ uniformly in σ for a.e. $\sigma \in \Sigma$.*

Proof. Let X be U.M.D. and take $q = 2p$. Then H_X is bounded in $L^q(X)$.

Not only that. It is easy to see that $X(\ell^p)$ is also U.M.D., so that, H_X has an ℓ^p extension. We can apply theorem 3.6 with $Y = L^q(X)$ in place of the lattice X appearing there. Observe that

$$\tilde{Y}_p = (L^{q/p}(X^p))'' = L^2(\tilde{X}_p) .$$

Thus, given $u \in L^2(\tilde{X}_p)$, $u \geq 0$, we have $w \in L^2(\tilde{X}^p)$ such that $u \leq w$, $\|w\| \leq 2\|u\|$ and

$$(5.4) \qquad \int_\Sigma \int_{\mathbf{T}} |H_X f(t,\sigma)|^p \, w(t,\sigma) \, dt \, d\sigma \leq C \int_\Sigma \int_{\mathbf{T}} |f(t,\sigma)|^p \, w(t,\sigma) \, dt \, d\sigma .$$

Let us apply this inequality to $f(t,\sigma) = \phi(t) \chi_E(\sigma)$ where ϕ is a trigonometric polynomial with rational coefficients and E is a subset of Σ with $|E| < \infty$. Since $H_X f(t,\sigma) = H\phi(t)\cdot\chi_E(\sigma)$, we have

$$\int_E \int_{\mathbf{T}} |H\phi(t)|^p \, w(t,\sigma) \, dt \, d\sigma \leq C \int_E \int_{\mathbf{T}} |\phi(t)|^p \, w(t,\sigma) \, dt \, d\sigma$$

and, consequently,

$$(5.5) \qquad \int_{\mathbf{T}} |H\phi(t)|^p \, w(t,\sigma) \, dt \leq C \int_{\mathbf{T}} |\phi(t)|^p \, w(t,\sigma) \, dt$$

for every $\sigma \notin E_0$, a set independent of ϕ and having $|E_0| = 0$.

This implies that $w(\cdot,\sigma)$ is an A_p-weight with uniform constant for every $\sigma \notin E_0$. The converse is even easier. If given $u \in L^2(\tilde{X}_p)$, $u \geq 0$, we have $w \in L^2(\tilde{X}_p)$ such that $u \leq w$, $\|w\| \leq 2\|u\|$ and (5.5) holds, we can obtain (5.4) by a limiting process. Then, by the easy part of theorem 3.6, H_X is bounded in $L^q(X)$ so that X is U.M.D. □

The condition in theorem 5.3 will be abreviated by saying that " $L^2(\tilde{X}_p)$ has enough A_p weights ".

Incidentally, note that in theorem 5.3 the exponent 2 does not play any role and we can use $L^s(\tilde{X}_p)$ for any $1 < s < \infty$. Theorem 5.3 can be used to prove the following

<u>Theorem 5.6.</u> *Let* X *be a U.M.D. lattice. Then there exists* $\varepsilon > 0$ *such that* X^a *is U.M.D. for every* $0 < a < 1 + \varepsilon$.

<u>Proof.</u> There are two different parts in this result. The deepest one is for $1 < a < 1 + \varepsilon$. It is proved in the following way. If X is U.M.D., it is known that X is superreflexive, and consequently p_0-convex for some $p_0 > 1$. Fix $1 < p < p_0$. By theorem 5.3, $L^2(\tilde{X}_p)$ has enough A_p weights. It is a well known fact in A_p theory that A_p weights satisfy a so called reverse Hölder's inequality (see [8]) and, consequently, every A_p weight is an $A_{p-\varepsilon}$ weight for

some $\varepsilon > 0$ depending only on the A_p constant. Thus $L^2(\tilde{X}_p)$ has enough $A_{p-\varepsilon}$ weights for some ε depending only on p and the U.M.D. constant of X. Let $Y = X^{p/(p-\varepsilon)}$, so that $Y^{p-\varepsilon} = X^p$. Then Y satisfies the condition in theorem 5.3 with $p - \varepsilon$ instead of p. It follows that $Y = X^a$ for some $a > 1$ is a U.M.D. lattice.

The second part of the result, the one for $0 < a < 1$, is based upon the so called "magical identity". This identity works for the conjugate function operator which sends the trigonometric polynomial $\sum a_n e^{int}$ to the trigonometric polynomial $\sum -i(\operatorname{sgn} n) a_n e^{int}$. The difference between this operator and the Hilbert transform H is dominated by the Hardy-Littlewood maximal operator so that its associated weights are again the A_p weights, and it is also good to characterize U.M.D. This justifies that we call H to the conjugate function also. With this notation we have the magic formula

$$(Hf)^2 - f^2 = 2H(fHf)$$

simply because $(f + iHf)^2$ is analytic. Now the magic formula works equally well for $\tilde{H}$, so that we have

$$\tilde{H}f(t,\sigma)^2 = f(t,\sigma)^2 + 2\tilde{H}(f\tilde{H}f)(t,\sigma)$$

and this implies:

$$\begin{aligned}
\|\tilde{H}f\|_{L^4(X^{1/2})} &= \int \|\tilde{H}f\|^4_{X^{1/2}} = \int \| |\tilde{H}f|^2 \|^2_X \leq \\
&\leq \int \left(\|f(t,\sigma)^2\|_X + 2\|\tilde{H}(f\tilde{H}f)\|_X \right)^2 \leq \\
&\leq 2 \int \|f(t,\sigma)^2\|^2_X + 8 \int \|\tilde{H}(f\tilde{H}f)\|^2_X \leq \\
&\leq 2\|f\|^4_{L^4(X^{1/2})} + 8C \int \|f\tilde{H}f\|^2_X \leq \\
&\leq 2\|f\|^4_{L^4(X^{1/2})} + C \int \|f\|^2_{X^{1/2}} \|\tilde{H}f\|^2_{X^{1/2}} \leq \\
&\leq 2\|f\|^4_{L^4(X^{1/2})} + C\|f\|^2_{L^4(X^{1/2})} \|\tilde{H}f\|^2_{L^4(X^{1/2})} \leq \\
&\leq C_\varepsilon \|f\|^4_{L^4(X^{1/2})} + \varepsilon\|\tilde{H}f\|^4_{L^4(X^{1/2})}
\end{aligned}$$

This implies that $X^{1/2}$ is U.M.D. By iteration we get that X^a is U.M.D. if $a = 2^{-k}$, $k = 1,2,\ldots$. We obtain the result for every $0 < a < 1$ by interpolation. □

In order to present the next result, we shall use the notation $[\ ,\ ]_\Theta$, $0 < \Theta < 1$, for Calderon's complex interpolation method, and we shall also recall Calderon's identity for Banach lattices. This holds provided at least one of the lattices X_0 , X_1 is reflexive:

$$[X_0,X_1]_\Theta = X_0^{1-\Theta}X_1^\Theta = \{x : |x| = x_0^{1-\Theta}x_1^\Theta , \quad x_0 \in X_0 , \quad x_1 \in X_1\}$$

Corollary 5.7. *Let* X *be a U.M.D. lattice in* $(\Sigma,d\sigma)$. *Then there exist* Θ *with* $0 < \Theta < 1$ *and another U.M.D. lattice* X_0 *such that* $X = [L^2(d\sigma),X_0]_\Theta$.

Proof. Take $a > 1$ such that X^a and $(X^*)^a$ are U.M.D. Call $Y = ((X^*)^a)^*$, also U.M.D. The key observation is that $X = [L^1,Y]_{1/a}$. In order to verify this, we just need to check that the duals coincide. But

$$([L^1,Y]_{1/a})^* = [L^\infty,(X^*)^a]_{1/a} = (L^\infty)^{1/a'}((X^*)^a)^{1/a} = X^*$$

Then

$$X = X^{1/2}X^{1/2} = X^{1/2}([L^1,Y]_{1/a})^{1/2} = X^{1/2}((L^1)^{1/a'}Y^{1/a})^{1/2} =$$

$$= X^{1/2}(L^1)^{1/2a'}Y^{1/2a} = (L^2)^{1/a'}((X^a)^{1/2}Y^{1/2})^{1/a} =$$

$$= [L^2,[X^a,Y]_{1/2}]_{1/a} = [L^2,X_0]_{1/a}$$

which is what we wanted because $X_0 = [X^a,Y]_{1/2}$ is U.M.D. □

Corollary 5.7 extends a previous result of Pisier [17] in which X_0 was just a Banach lattice, not necessarily U.M.D.

The next theorem will allow us to obtain the boundedness of the vector-valued extension to a U.M.D. Banach lattice of a huge class of operators.

Theorem 5.8. *Let* X *be a U.M.D. Banach lattice and let* T *be a linearizable operator which is bounded in* $L^p(w) = L^p(\mathbb{T},w(t)dt)$ *for every* $w \in A_p$ *and every* $1 < p < \infty$. *Then* $\tilde{T}f(t,\sigma) = T(f(\cdot,\sigma))(t)$ *is bounded in* $L^p(X) = L^p(\mathbb{T})(X)$ *for every* $1 < p < \infty$.

Proof. If $L^p(X)$ is 2-convex or 2-concave, we just need to apply theorem 4.15 to the lattice $L^p(X)$, the linear operator $\tilde{H}$ and the linearizable operator $\tilde{T}$. We need to see that $W(\tilde{H}) \subset W(\tilde{T})$. In order to do that we proceed as in the proof of theorem 5.3. We realize that if a weight $w(t,\sigma)$ belongs to $W(\tilde{H})$, then $w(\cdot,\sigma)$ is an A_2 weight with uniform constant for a.e. $\sigma \in \Sigma$. Now by the hypothesis we have assumed about T , we get an inequality like (5.5) with T in place of H . Integrating on Σ and completing gives $w \in W(\tilde{T})$. In the general case, since X is U.M.D., we know that $L^p(X)$ is p_0-convex for some $1 < p_0 < \infty$. We use the fact that $\tilde{M}$ is bounded in $L^p(X)$ (see [22]) and apply theorem 4.16 exactly as we did before. □

Corollary 5.9. *Let us denote by*

$$S_n f(t) = \sum_{-n}^{n} \hat{f}(k) e^{ikt} ,$$

the n*-th partial sum of the Fourier series of* f . *Then if* X *is a U.M.D. Banach lattice, we have the following convergence a.e. of the* X*-valued Fourier series*

$$\|\tilde{S}_n f(t) - f(t)\|_X \to 0 \quad a.e., \quad f \in L^p(X) , \quad 1 < p < \infty .$$

Proof. Let $Tf(t) = \sup_n |S_n f(t)|$ be Carleson's maximal partial sum operator. Theorem 5.8 can be applied to it (see [11]), and concludes that $\tilde{T}$ is bounded in $L^p(X)$, $1 < p < \infty$.

But then

$$\int_{\mathbb{T}} \sup_n \|S_n f(t)\|_X^p \, dt \leq \int_{\mathbb{T}} \|\sup_n |S_n f(t,\cdot)|\|_X^p \, dt =$$

$$= \int_{\mathbb{T}} \|\tilde{T} f(t,\cdot)\|_X^p \, dt = \|\tilde{T} f\|_{L^p(X)}^p \leq C \|f\|_{L^p(X)}^p$$

which is enough for our purposes. □

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