TOPOLOGICAL METHODS OF INVESTIGATION OF OPERATOR EQUATIONS AND NONLINEAR BOUNDARY VALUE PROBLEMS

I. V. Skrypnik
Doneck, USSR

Introduction

One of the fundamental methods of qualitative study of nonlinear elliptic and parabolic boundary value problems, which facilitates the study of solvability, branching, bifurcation of solutions, problems of eigenfunctions, is the topological method, based on the theory of degree of nonlinear mappings in Banach spaces. The origin of this method goes back to the remarkable paper by Leray-Schauder [1], where the authors, using Brouwer's theory of degree of finite-dimensional mappings, introduced the degree of a map $I-F : X \to X$, where I is the identity, F a totally continuous mapping of a Banach space X (or its part) into itself. In the same paper the authors gave a method of reducing the quasilinear Dirichlet problem to the operator equation $u-Fu = 0$ with a totally continuous operator F. Applying the theory of degree of a map the authors found that in order to prove an existence theorem it is sufficient to establish an apriori estimate of solutions of a certain parametric family of boundary value problems.

On the other hand, the limited possibilities of application of the Leray-Schauder degree to more general boundary value problems were soon recognized. The application of these methods to the Dirichlet problem for the general nonlinear equation is cumbersome and requires additional restrictions [2]. When studying the second fundamental problem for quasilinear elliptic equations - the Neumann problem - it was found [3] that by applying the Leray-Schauder scheme the differential problem reduces to the equation $u-\Phi u = 0$ with a continuous but not compact operator Φ. Consequently, applying the topological methods in this case we have either to change the way in which we reduce the problem, or to reduce the problem to other classes of operators.

In this way there appeared a demand for topological methods of investigation of more general classes of operator equations in Banach spaces, namely, of such classes to which it is possible to reduce differential boundary value problems for general nonlinear differential elliptic and parabolic equations with general nonlinear boundary value conditions.

The necessity of developing a theory of degree of more extensive classes of mappings also arose from the problem of existence of generalized solutions of boundary value problems for divergent equations. An essential feature of the study of generalized solvability is the sufficiency of considerably weaker apriori estimates of solutions - estimates involving the energy norm. At the beginning of the sixties, Browder, Minty and others (see survey papers [4] - [7]) discovered new classes of operators - the monotone operators - whose essential property is that they preserve the weak convergence under Galerkin's approximations. Problems of finding generalized solutions for divergent elliptic and parabolic equations naturally lead to equations with similar operators. The application of the methods of monotone operators led to a considerable progress in the theory of nonlinear boundary value problems.

The results of various authors concerning solvability of equations with monotone operators (in the coercive case) (see [4] - [7]) were entailed by the establishment of the theory of degree of various classes of monotone operators and their generalizations, given simultaneously and independently by the author [8] and Browder, Petryshyn [9]. A survey of these and other related results is found in [10]. Browder and Petryshyn introduced a multivalued degree of A - proper mappings. The non-uniqueness of the degree (the degree of a mapping is a subset of the set $Z \cup \{-\infty\} \cup \{+\infty\}$) as well as the fact that this degree fails to possess all the properties of the degree of finite-dimensional mappings, make its application to differential problems difficult.

The author introduced a single-valued degree of mappings satisfying a certain condition (α_0). The degree introduced exhibits all the properties of the degree of finite-dimensional mappings. Even the analogue of Hopf's theorem asserting that the degree is the unique homotopic invariant of the class of mappings considered, is valid.

The above mentioned degree of a mapping satisfying the condition (α_0) was in the beginning employed in the study of boundary value problems for divergent quasilinear equations [11]. Further, the author demonstrated [10, 12, 13] that this degree can serve as the basis for developing topological methods of investigation of boundary value problems for general essentially nonlinear elliptic and parabolic equations with general nonlinear boundary value conditions. Explicit constructive methods of reducing boundary value problems to the corresponding classes of operator equations in Sobolev spaces were given.

When verifying the condition (α_0) for the resulting operators, apriori L_p-estimates of elliptic and parabolic linear problems are essentially involved.

In the case of the general nonlinear Dirichlet problem it is possible to give simpler methods of reducing the original problems to operator equations with operators satisfying the condition (α_0). These simplifications are based on coercive estimates for pairs of linear elliptic operators, established in [14, 15] under weak assumptions.

The present text of the lectures is mainly based on the author's works. In Chap. 1 we give the definition of the degree of a mapping satisfying the condition (α_0) or one of its analogues, and we establish various properties of the degree. In Chap. 2 we show the methods of reducing the general nonlinear elliptic boundary value problems to the operator equations discussed in Chap. 1. Coercive estimates for pairs of linear differential operators are given in Chap. 3, together with a simpler method of reducing the general nonlinear Dirichlet problem to an operator equation. In Chap. 4 a number of examples of application of the topological methods developed above are given: the Dirichlet problem for the Monge-Ampere equation and a model nonlinear Neumann problem are discussed and an existence theorem for the general Dirichlet problem in a thin layer is established. The ways how to introduce topological characteristics for general nonlinear parabolic equations are shown in Chap. 5.

Let us point out that the methods developed enable us to investigate the behaviour of solutions of families of problems, in particular, branching, bifurcation, eigenvalue problems (see [10], [11]). However, these problems are not dealt with in the present paper.

1. The degree of generalized monotone mappings

1. Throughout this section X is a real separable reflexive Banach space, X^* its adjoint. We denote the strong and the weak convergence by $\rightarrow$ and $\rightharpoonup$, respectively; it will be clear from the context in which space the convergence is considered. Given elements $u \in X$ and $h \in X^*$ we denote by $\langle h,u \rangle$ the value of the functional h at the element u.

We will consider operators A, defined on a set $D_A \subset X$, with values in X^*. An operator A will be called demicontinuous if it maps strongly convergent sequences into weakly convergent ones.

DEFINITION 1. We shall say that an operator A satisfies the condition (α_0) if for any sequence $u_n \in D_A$, the relations $u_n \rightharpoonup u_0$, $Au_n \rightharpoonup 0$ and

$$(1) \qquad \varlimsup_{n\to\infty} \langle Au_n, u_n - u_0\rangle \leqq 0$$

imply the strong convergence of u_n to u_0.

The set of operators satisfying the condition (α_0) is a certain extension of the set of operators satisfying the condition $(S)_+$ [16]. We say that an *operator* A *satisfies the condition* $(S)_+$ if for every sequence $u_n \in D_A$ that satisfies (1), the weak convergence $u_n \rightharpoonup u_0$ implies $u_n \to u_0$.

In this chapter we shall define the degree of a mapping A satisfying the condition (α_0).

2. Operators satisfying the condition $(S)_+$, and hence also (α_0), appear in problems of finding generalized solutions of divergent elliptic equations, under some rather weak assumptions (see [10]). We restrict ourselves to a single simple example of the Neumann problem for the quasilinear elliptic second order equation. This case illustrates in this section the general operator scheme. It is interesting that the application of the Leray-Schauder degree theory meets with considerable difficulties in this case [3].

Let Ω be a bounded domain in an n-dimensional Euclidean space R^n, with a boundary $\partial\Omega$ of a class $C^{2,\lambda}$, $0 < \lambda < 1$. Consider the boundary value problem

$$(2) \qquad L(u) = \sum_{i=1}^{n} \frac{d}{dx_i} a_i\left(x,u,\frac{\partial u}{\partial x}\right) - a_0\left(x,u,\frac{\partial u}{\partial x}\right) = 0 , \quad x \in \Omega ,$$

$$(3) \qquad B(u) = \sum_{i=1}^{n} a_i\left(x,u,\frac{\partial u}{\partial x}\right) \cos(\nu,x_i) + b(x,u) = 0 , \quad x \in \partial\Omega ,$$

where ν is the vector of the outer normal to $\partial\Omega$ at the point x.

We assume that the following conditions are fulfilled with some positive constants C_1, C_2 and $p > 1$:

(i) the functions $a_i(x,u,\xi)$, $a_0(x,u,\xi)$, $b(y,u)$ are defined, continuous for $x \in \overline{\Omega}$, $y \in \partial\Omega$, $u \in R^1$, $\xi = (\xi_1,\dots,\xi_n) \in R^n$ and satisfy the inequalities

$$(4) \qquad |a_i(x,u,\xi)| \leqq C_1(1+|u|+|\xi|)^{p-1} , \quad i = 0,1,\dots,n ,$$

(5) $$|b(x,u)| \leq C_1(1+|u|)^{p-1} ;$$

(ii) for arbitrary $x \in \overline{\Omega}$, $u \in R^1$, $\xi, \eta \in R^n$ the following inequalities hold:

$$\sum_{i=1}^{n} a_i(x,u,\xi)\xi_i \geq C_2|\xi|^p - C_1|u|^p - C_1 ,$$

$$\sum_{i=1}^{n} [a_i(x,u,\xi) - a_i(x,u,\eta)](\xi_i-\eta_i) > 0 \quad \text{for} \quad \xi \neq \eta .$$

REMARK. The conditions (4), (5) can be further weakened (see [10]).

If the conditions (i), (ii) are fulfilled, it is possible to introduce the generalized solution of the problem (2), (3) belonging to $W_p^1(\Omega)$, which will be simply called a solution. A function $u \in W_p^1(\Omega)$ is called a solution of the problem (2), (3), if an arbitrary function $\phi \in W_p^1(\Omega)$ satisfies the integral identity

(6) $$\int_{\Omega}\Big[\sum_{i=1}^{n} a_i\big(x,u, \tfrac{\partial u}{\partial x}\big) \frac{\partial \phi}{\partial x} + a_0\big(x,u, \tfrac{\partial u}{\partial x}\big)\phi\Big]dx + \int_{\partial\Omega} b(x,u)\phi dS = 0 .$$

We associate the problem (2), (3) with an operator $A : W_p^1(\Omega) \rightarrow$ $\rightarrow (W_p^1(\Omega))^*$, which is defined by the identity

(7) $$\langle Au,\phi\rangle = \int_{\Omega}\Big[\sum_{i=1}^{n} a_i\big(x,u,\tfrac{\partial u}{\partial x}\big)\frac{\partial \phi}{\partial x_i} + a_0\big(x,u,\tfrac{\partial u}{\partial x}\big)\phi\Big]dx + \int_{\partial\Omega} b(x,u)\phi dS .$$

It immediately follows from the given definitions that the solutions of the problem (2), (3) coincide with those of the operator equation $Au = 0$.

LEMMA 1. The operator $A : W_p^1(\Omega) \rightarrow (W_p^1(\Omega))^*$ *defined by* (7) *satisfies the condition* $(S)_+$ *provided* (i), (ii) *are fulfilled.*

P r o o f . Let a sequence $u_n \in W_p^1(\Omega)$ satisfy $u_n \rightharpoonup u_0$, $\lim_{n\to\infty} \langle Au_n,u_n-u_0\rangle \leq 0$. By virtue of compactness of the imbedding of $W_p^1(\Omega)$ in $L_p(\Omega)$ and $L_p(\partial\Omega)$ we obtain

$$\lim_{n\to\infty} \Big[\int_{\Omega} a_0\big(x,u_n,\tfrac{\partial u_n}{\partial x}\big)(u_n-u_0)dx + \int_{\partial\Omega} b(x,u_n)(u_n-u_0)dS\Big] = 0 ,$$

$$\lim_{n\to\infty} \int_{\Omega}\sum_{i=1}^{n} a_i\big(x,u_n,\tfrac{\partial u_0}{\partial x}\big)\frac{\partial(u_n-u_0)}{\partial x_i} dx = 0 .$$

Consequently, the sequence u_n satisfies the inequality

$$\lim_{n\to\infty} \int_{\Omega} \sum_{i=1}^{n} \left[a_i\left(x,u_n,\frac{\partial u_n}{\partial x}\right) - a_i\left(x,u_n,\frac{\partial u_0}{\partial x}\right)\right]\frac{\partial}{\partial x_i}(u_n-u_0)dx \leqq 0 .$$

The condition (ii) implies that $\frac{\partial u_n}{\partial x_i}$ converges to $\frac{\partial u_0}{\partial x_i}$ in measure, and yields the estimate

$$\int_E \left|\frac{\partial u_n}{\partial x}\right|^p dx \leqq \alpha_n + C_0\int_E \left(\left|\frac{\partial u_0}{\partial x}\right|^p + |u_n|^p + 1\right)dx$$

with a constant C_0 independent of n and with $\alpha_n \to 0$, which is valid for any measurable set $E \subset \Omega$. This yields the strong convergence of u_n to u_0 in $W_p^1(\Omega)$.

3. Let us go back to the case of a general abstract operator $A : D_A \longrightarrow X^*$, which was introduced in Sec. 1.

Let $\{v_i\}$, $i = 1,2,\ldots$ be an arbitrary complete system of the space X and assume that the elements $v_1,\ldots,v_N$ are linearly independent for every N. Denote by F_n the linear hull of the elements $v_1,\ldots,v_n$.

Let us assume that the interior of the set D_A is nonempty and let D be an arbitrary bounded open subset of the space X, such that $\overline{D} \subset D_A$. We will define the degree $\mathrm{Deg}(A,D,0)$ of the mapping A of the set $\overline{D}$ with respect to the point 0 of the space X^*.

Let us notice that in the author's papers [10, 11] the term of degree of a mapping is replaced by the term "rotation of a vector field on the boundary of a set". These two notions are equivalent.

For every $n = 1,2,\ldots$ we shall introduce finite-dimensional approximations A_n of the map A, $A_n : D_n = \overline{D} \cap F_n \longrightarrow F_n$, by

$$A_n(u) = \sum_{i=1}^{n} \langle Au,v_i\rangle v_i . \tag{8}$$

LEMMA 2. Let $A : D_A \longrightarrow X^*$ *be a demicontinuous bounded operator satisfying the condition* (α_0), $\overline{D} \subset D_A$ *and* $Au \neq 0$ *for* $u \in \partial\Omega$. *Then there is a number* N_1 *such that Brouwer's degree* $\deg(A_n,D_n,0)$ *is defined provided* $n \geqq N_1$.

P r o o f . It is sufficient to verify that for large n,

$A_n u \neq 0$ for $u \in \partial D_n$. This will be proved by contradiction. Let there exist a sequence $u_k \in \partial D_{n_k}$ such that $n_k \longrightarrow \infty$, $A_{n_k} u_k = 0$. Evidently $u_k \in \partial D$. We may assume $u_k \rightharpoonup u_0$ and show that then $u_k \longrightarrow u_0$. It is easily seen that $Au_k \rightharpoonup 0$. We choose a sequence $W_k \in F_{n_k}$ such that $W_k \longrightarrow u_0$. Then

$$\langle Au_k, u_k - u_0 \rangle = \langle Au_k, W_k - u_0 \rangle$$

and the right hand side converges to zero with $k \to \infty$. In virtue of the condition (α_0) this implies the strong convergence of u_k to u_0 and we obtain $Au_0 = 0$, $u_0 \in \partial D$, which contradicts the assumptions of the lemma.

Further, we shall establish the stabilization of Brouwer's degree of the mappings A_n .

<u>LEMMA 3.</u> *The limit* $\lim\limits_{n\to\infty} \deg(A_n, D_n, 0)$ *exists.*

P r o o f . We have to prove the existence of such a number N_2 that

$$\deg(A_n, D_n, 0) = \deg(A_m, D_m, 0) \quad \text{for} \quad n, m \geqq N_2 . \tag{9}$$

Let us consider an auxiliary mapping

$$\tilde{A}_n(u) = \sum_{i=1}^{n-1} \langle Au, v_i \rangle v_i + \langle h_n, u \rangle v_n ,$$

where h_n is an arbitrary fixed element of X^* which satisfies

$$\langle h_n, v_i \rangle = \delta_{in} , \quad i \leqq n .$$

Here δ_{in} are the Kronecker symbols. According to the Leray-Schauder Lemma [1] we obtain

$$\deg(A_{n-1}, D_{n-1}, 0) = \deg(\tilde{A}_n, D_n, 0) \tag{10}$$

for $n \geqq N_1 + 1$. To prove (9) we only have to verify that for large n , the mappings $\tilde{A}_n$ and A_n are homotopic on D_n . Let us consider a family of mappings $A_n^{(t)} : D_n \longrightarrow F_n$, $t \in [0,1]$ depending on t :

$$A_n^{(t)}(u) = \sum_{i=1}^{n-1} \langle Au, v_i \rangle v_i + \{t \langle h_n, u \rangle + (1-t) \langle Au, v_n \rangle\} v_n .$$

Evidently $A_n^{(0)} = A_n$, $A_n^{(1)} = \tilde{A}_n$ and in virtue of the properties of Brouwer's degree, in order to establish the identity

$$(11) \qquad \deg(A_n, D_n, 0) = \deg(\tilde{A}_n, D_n, 0)$$

for large n it suffices to verify that $A_n^{(t)}(u) \neq 0$ for $t \in [0,1]$, $u \in \partial D_n$ and n sufficiently large.

On the contrary, let us assume that there are sequences u_k, t_k such that

$$(12) \qquad A_{n_k}^{(t_k)}(u_k) = 0\ , \quad u_k \in \partial D_{n_k}\ , \quad t_k \in [0,1]\ , \quad n_k \to \infty\ .$$

We may assume $u_k \rightharpoonup u_0$, $t_k \to t_0$. Then (12) yields

$$(13) \qquad \begin{aligned} &\langle Au_k, v_i\rangle = 0\ , \quad 1 \leq i \leq n_k - 1\ , \\ &t_k \langle h_{n_k}, u_k\rangle + (1-t_k)\langle Au_k, v_{n_k}\rangle = 0\ . \end{aligned}$$

Hence $Au_k \rightharpoonup 0$. According to Lemma 2, $t_k \neq 0, 1$ for $n_k \geq N_1 + 1$. We choose a sequence $W_k \in F_{n_k - 1}$ so that $W_k \to u_0$. Then

$$\begin{aligned} \langle Au_k, u_k - u_0\rangle &= \langle Au_k, W_k - u_0\rangle + \langle h_{n_k}, u_k\rangle\langle Au_k, u_k\rangle = \\ &= \langle Au_k, W_k - u_0\rangle - \frac{t_k}{1-t_k}\langle h_{n_k}, u_k\rangle^2\ , \end{aligned}$$

which in virtue of the condition (α_0) implies the strong convergence of the sequence u_k to u_0. Hence $Au_0 = 0$, $u_0 \in \partial D$, which contradicts the assumptions. The identities (10), (11) imply the assertion of the lemma.

Denote

$$(14) \qquad D\{v_i\} = \lim_{n\to\infty} \deg(A_n, D_n, 0)\ .$$

We shall further prove that the limit $D\{v_i\}$ is independent of the choice of the system of elements $\{v_i\}$. Let $\{w_i\}$ be another system possessing the same properties as the system of elements $\{v_i\}$. Denote by E_n the linear hull of the elements $w_1, \ldots, w_n$ and define the mapping

$$A_n' : D_n' = \overline{D} \cap E_n \to E_n\ , \quad A_n'(u) = \sum_{i=1}^{n} \langle Au, w_i\rangle w_i\ .$$

<u>LEMMA 4.</u> *Let the assumptions of Lemma 2 be fulfilled. Then* $D\{v_i\} = D\{w_i\} = D$, *where* $D\{v_i\}$ *is defined by* (14) *and* $D\{w_i\}$ *is defined analogously.*

P r o o f . It suffices to prove that there is N_3 such that

$$\deg(A_n, D_n, 0) = \deg(A'_n, D'_n, 0) \tag{15}$$

provided $n \geqq N_3$.

Moreover, we may assume that for any n the systems $v_1, \dots, v_n$, $w_1, \dots, w_n$ are linearly independent. If this is not the case, we in addition construct intermediate system $\{g_i\}$, $i = 1, 2, \dots$. Denote by L_{2n} the linear space spanned by the elements $v_1, \dots, v_n$, $w_1, \dots$ $\dots, w_n$. Let $D''_{2n} = D \cap L_{2n}$ and

$$A''_{2n}(u) = \sum_{i=1}^{n} \{<Au, v_i> v_i + <Au, w_i> w_i\} , \quad u \in D''_{2n} .$$

In order to prove (15) we only have verify that

$$\deg(A_n, D_n, 0) = \deg(A''_{2n}, D''_{2n}, 0) \tag{16}$$

for large n , in virtue of the full symmetry of the left- and right- -hand sides of (15). For every n define elements $f_i^{(n)}$, $i = 1, \dots$ $\dots, n$, belonging to X^* and such that

$$<f_i^{(n)}, v_j> = 0 , \quad <f_i^{(n)}, w_j> = \delta_{ij} \tag{17}$$

holds for $i, j = 1, \dots, n$.

For $t \in [0,1]$ we define a parametric family of mappings $A_{2n,t} : D''_{2n} \to L_{2n}$ by

$$A_{2n,t}(u) = \sum_{i=1}^{n} <Au, v_i> v_i + \sum_{i=1}^{n} \{t<Au, w_i> + (1-t)<f_i^{(n)}, u>\} w_i .$$

Evidently $A_{2n,1} = A''_{2n}$ and for $n \geqq N_1$,

$$\deg(A_{2n,0}, D''_{2n}, 0) = \deg(A_n, D_n, 0) \tag{18}$$

holds in virtue of the Leray-Schauder Lemma. To establish (16) it suffices to verify that for large n , $A_{2n,t}(u) \neq 0$ for $t \in [0,1]$, $u \in \partial D''_{2n}$. On the contrary, assume that there are sequences $u_k \in$ $\in \partial D''_{2n_k}$, $t_k \in [0,1]$, such that $A_{2n_k, t_k}(u_k) = 0$, $n_k \to \infty$. Then

$$\begin{aligned} &<Au_k, v_i> = 0 , \quad i = 1, \dots, n_k , \\ &t_k <Au_k, w_i> + (1-t_k)<f_i^{(n_k)}, u_k> = 0 , \quad i = 1, \dots, n_k . \end{aligned} \tag{19}$$

We may assume that $u_k \rightharpoonup u_0$, $t_k \to t_0$. It follows from (19) that $Au_k \rightharpoonup 0$. Choose a sequence $\tilde{w}_k \in F_{n_k}$ such that $\tilde{w}_k \to u_0$.

Then

$$<Au_k,u_k-u_0> = <Au_k,\tilde{w}_k-u_0> + \sum_{i=1}^{n_k} <f_i^{(n_k)},u_k><Au_k,w_i> =$$

$$= <Au_k,\tilde{w}_k-u_0> - \sum_{i=1}^{n_k} <f_i^{(n_k)},u_k>^2 \frac{1-t_k}{t_k} ,$$

for $n_k \geqq N_1$. This together with the condition (α_0) yields the strong convergence of the sequence u_k to u_0 , $Au_0 = 0$, $u_0 \in \partial D$, a contradiction.

Lemmas 2 - 4 enable us to introduce the following definition.

DEFINITION 2. The number D from Lemma 4 is called the degree of the mapping A of the set $\bar{D}$ with respect to the point $0 \in X^*$ and is denoted by $\mathrm{Deg}(A,\bar{D},0)$.

Hence we have by definition

$$\mathrm{Deg}(A,\bar{D},0) = D = D\{v_i\} = \lim_{n\to\infty} \deg(A_n,D_n,0) .$$

4. The degree $\mathrm{Deg}(A,\bar{D},0)$ of a mapping introduced in Sec. 3 possesses all the properties of the degree of finite-dimensional mappings (see [10,11]). We mention only the most important of them, omitting the proofs.

DEFINITION 3. Let D_0 be an arbitrary set in the space X and $A_t : D_0 \to X$, $t \in [0,1]$, a parametric family of mappings. The family A_t is said to satisfy the condition $(\alpha_0^{(t)})$, if for any sequences $u_n \in D_0$, $t_n \in [0,1]$, the relations $u_n \rightharpoonup u_0$, $t_n \to t_0$, $A_{t_n}(u_n) \rightharpoonup 0$ and

$$\lim_{n\to\infty} <A_{t_n}(u_n),u_n-u_0> \leqq 0$$

imply the strong convergence of u_n to u_0 .

DEFINITION 4. Let A' $A'' : D_0 \to X$ be bounded demicontinuous operators satisfying the condition (α_0), D a bounded open set such that $\bar{D} \subset D_0$, and let $A'u \neq 0$, $A''u \neq 0$ for $u \in \partial D$. The mappings A' , A'' are called homotopic on $\bar{D}$, if there is a parametric family of mappings $A_t : \bar{D} \to X^*$ satisfying the condition $(\alpha_0^{(t)})$ and the following conditions:

(a) $A_t(u) \neq 0$ for $u \in \partial D$, $t \in [0,1]$; $A_0 = A'$, $A_1 = A''$;

(b) for each $t \in [0,1]$, the operator A_t is continuous;

(c) there is a function $\omega : [0,1] \to R^1$, $\omega(\rho) \to 0$ with $\rho \to 0$, such that $\sup_{u \in \overline{D}} ||A_t u - A_s u|| \leq \omega(|t-s|)$.

The following theorems provide a classification of mappings that satisfy the condition (α_0), in terms of the degree of a mapping.

THEOREM 1. Let $A' : \overline{D} \to X^*$, $A'' : \overline{D} \to X^*$ *be two mappings that are homotopic to each other in the sense of Definition 4. Then* $\mathrm{Deg}(A',\overline{D},0) = \mathrm{Deg}(A'',\overline{D},0)$.

THEOREM 2. Let D *be a convex bounded open set in the space* X *and* $A' : \overline{D} \to X^*$, $A'' : \overline{D} \to X^*$ *bounded demicontinuous mappings satisfying the condition* (α_0)*, such that* $A'u \neq 0$, $A''u \neq 0$ *for* $u \in \partial D$ *and* $\mathrm{Deg}(A',\overline{D},0) = \mathrm{Deg}(A'',\overline{D},0)$. *Then the mappings* A' , A'' *are homotopic on* $\overline{D}$.

The application of the theory of degree of a mapping to the solvability of the operator equation

$$Au = 0 \tag{20}$$

is based on the following principle.

Principle of Nonzero Rotation.

Let $A : \overline{D} \to X^*$ be a bounded demicontinuous operator satisfying the condition (α_0). A sufficient condition for the equation (20) to be solvable in D is that $\mathrm{Deg}(A,\overline{D},0) \neq 0$.

Let us present two criteria of non-vanishing of the degree of a mapping.

THEOREM 3. Let $B(0,r) = \{u \in X : ||u|| \leq r\}$, *let* $A : B(0,r) \to X^*$ *be a bounded demicontinuous operator satisfying the condition* (α_0). *Assume that for every* $u \in \partial B(0,r)$ *the inequalities*

$$Au \neq 0 \ , \quad \frac{Au}{\|Au\|_*} \neq \frac{A(-u)}{\|A(-u)\|_*} \tag{21}$$

hold. Then $\mathrm{Deg}(A,B(0,r),0)$ *is an odd number.*

THEOREM 4. Let D *be an arbitrary bounded domain in the space* X $0 \notin \partial D$ *and* $A : \overline{D} \to X^*$ *a bounded demicontinuous operator satisfying the condition* (α_0). *Assume that for* $u \in \partial D$ *the inequalities*

(22) $\qquad Au \neq 0 \ , \quad \langle Au,u\rangle \geq 0$

hold. Then $\operatorname{Deg}(A,\bar{D},0) = 1$ *for* $0 \in D$, $\operatorname{Deg}(A,\bar{D},0) = 0$ *for* $0 \notin \bar{D}$.

Let us formulate an easy consequence of Theorem 1 and the principle of nonzero rotation.

THEOREM 5. Let D *be a bounded domain in* X *and* $A_t : \bar{D} \to X^*$ *a family of operators that realize the homotopy between* A_0 *and* A_1 *in the sense of Definition 4. Assume that* $\operatorname{Deg}(A_0,\bar{D},0) \neq 0$. *Then the equation* $A_1 u = 0$ *has in* D *at least one solution.*

5. In this section we present the application of the results of the previous section to the proof of solvability of the problem (2), (3).

First of all, it is easy to formulate conditions on the functions $a_i(x,u,\xi)$, $a_0(x,u,\xi)$, $\phi(x,u)$ guaranteeing that the operator A defined by the identity (7) satisfies the condition (22), provided that for D we choose a ball $B(0,R)$ of a sufficiently large radius R . In that case the existence theorem immediately follows.

Here we give a less evident result on solvability of the problem (2), (3), proving existence of a classical solution belonging to $C^{2,\delta}(\bar{\Omega})$ with some $\delta > 0$. Naturally, this requires stronger smoothness assumptions to be imposed on the functions $a_i(x,u,\xi)$, $a_0(x,u,\xi)$ than those from Sec. 2.

Let $p > 1$ and for $(x,u,\xi) \in \bar{\Omega} \times R^1 \times R^n$ let the following conditions be fulfilled:

(i) $a_i(x,u,\xi)$, $b(x,u)$ are three times, and $a_0(x,u,\xi)$ twice differentiable functions of their arguments;

(ii) for $\eta \in R^n$ there are positive constants C_1 , C_2 such that

$$\sum_{i,j=1}^{n} \frac{\partial a_i(x,u,\xi)}{\partial \xi_j} \eta_i \eta_j \geq C_1(1 + |\xi|)^{p-2}|\eta|^2 ,$$

$$\Big\{\sum_{i,j=1}^{n}\left|\frac{\partial a_i}{\partial \xi_j}\right| + \sum_{i=1}^{n}\left|\frac{\partial a_i}{\partial u}\right|\Big\}(1 + |u| + |\xi|) + |a_0(x,u,\xi)| \leq$$

$$\leq C_2(1 + |u| + |\xi|)^{p-1} ,$$

$$\sum_{i,j=1}^{n}\left|\frac{\partial a_i}{\partial x_j}\right| + \sum_{i=1}^{n}\left|\frac{\partial a_0}{\partial \xi_i}\right| + \left|\frac{\partial a_0}{\partial u}\right| + \sum_{i=1}^{n}\left|\frac{\partial a_0}{\partial x_i}\right| \leq C_2(1 + |u| + |\xi|)^{p-1} ,$$

$$|b(x,u)| \le C_2(1 + |u|)^{p-1} ;$$

(iii) there is $R > 0$ such that for $|u| > R$,

$$\sum_{i=1}^{n} \frac{\partial^2 a_i(x,u,0)}{\partial x_i \partial u} - \frac{\partial a_0(x,u,0)}{\partial u} \le C < 0 , \quad x \in \overline{\Omega} ,$$

$$u\left[\sum_{i=1}^{n} a_i(x,u,0)\cos(\nu,x_i) + b(x,u)\right] > 0 , \quad x \in \partial\Omega .$$

THEOREM 6. *Let* (i) - (iii) *be fulfilled and let* $\partial\Omega$ *be a surface of class* $C^{2,\lambda}$, $\lambda > 0$. *Then there is* $\delta > 0$ *such that the problem* (2), (3) *has a solution belonging to* $C^{2,\delta}(\Omega)$.

P r o o f . We include the problem (2), (3) into a parametric family

$$(23) \qquad tL(u) + (1-t)\left\{\sum_{i=1}^{n} \frac{d}{dx_i}\left[\left(1 + \left|\frac{\partial u}{\partial x}\right|\right)^{p-2}\right]\frac{\partial u}{\partial x_i} - u\right\} = 0 , \quad x \in \Omega ,$$

$$(24) \qquad tBu + (1-t)\left(1 + \left|\frac{\partial u}{\partial x}\right|\right)^{p-2} \frac{\partial u}{\partial \nu} = 0 , \quad x \in \partial\Omega .$$

Analogously to [3] we can show that the generalized solution of the problem (23), (24), which belongs to $W_p^1(\Omega)$, belongs to $C^{2,\delta}(\Omega)$ as well. Then, applying the maximum principle, we easily obtain the estimate

$$(25) \qquad \max_{\Omega}|u(x)| \le M_0 .$$

for an arbitrary solution of the problem (23), (24), with a constant M_0 independent of t . Then the integral identity yields the estimate

$$(26) \qquad ||u||_{W_p^1(\Omega)} \le M_1$$

for an arbitrary solution $u(x)$ of the problem (23), (24), M_1 being a positive number.

Now let us consider the parametric family of mappings

$$A_t : B(0,M_1+1) \to [W_p^1(\Omega)]^* ,$$
$$A_t = tA + (1-t)A_0 ,$$

with the operator A defined by the identity (7) and the operator A_0 by the identity

$$\langle A_0 u,\phi\rangle = \int_{\Omega}\left\{\left(1 + \left|\frac{\partial u}{\partial x}\right|\right)^{p-2}\sum_{i=1}^{n}\frac{\partial u}{\partial x_i}\frac{\partial \phi}{\partial x_i} + u\phi\right\}dx .$$

Theorem 4 applied to the operator A_0 implies that $\mathrm{Deg}(A_0, B(0,M_1+1), 0) = 1$. From Theorem 2 and the estimate (26) we easily obtain that $\mathrm{Deg}(A, B(0,M_1+1), 0) = 1$. The assertion of Theorem 6 now follows by virtue of the nonzero rotation principle.

2. Topological characteristics of general nonlinear elliptic problems

1. In this chapter it is shown how general elliptic problems can be reduced to operator equations of the form (20) with the operator A satisfying the condition (α_0).

In what follows, Ω is a bounded domain in R^n with an infinitely differentiable boundary $\partial\Omega$, $n_0 = [\frac{n}{2}] + 1$, $m, m_1, \ldots, m_m$ are nonnegative integers, $m \geq 1$. We denote by $M(q)$ the number of mutually distinct multiindices $\alpha = (\alpha_1, \ldots, \alpha_n)$ with nonnegative integer coordinates α_i and the length $|\alpha| = \alpha_1 + \ldots + \alpha_n$ not greater than q.

Let $\ell_0 = \max(2m, m_1 + \frac{1}{2}, \ldots, m_m + \frac{1}{2})$ and assume that functions

$$F : \bar{\Omega} \times R^{M(2m)} \to R^1 ,$$
$$G_j : \bar{\Omega} \times R^{M(m_j)} \to R^1 , \quad j = 1, \ldots, m ,$$

are given, possessing continuous derivatives with respect to all their arguments up to the orders $\ell - 2m + 1$, $\ell - 2m_j + 1$, respectively, where the number ℓ satisfies the condition $\ell \geq \ell_0 + n_0$.

The functions $F(x,\xi)$, $G_j(x,\eta)$, $\xi = (\xi_\alpha : |\alpha| \leq 2m)$, $\eta = (\eta_\beta : |\beta| \leq m_j)$ will be also written in the form

$$F(x,\xi) = F(x, \xi_0, \xi_1, \ldots, \xi_{2m}) ,$$
$$G_j(x,\eta) = G_j(x, \eta_0, \eta_1, \ldots, \eta_{m_j}) ,$$

with $\xi_k = (\xi_\alpha : |\alpha| = k)$, $\eta_k = (\eta_\alpha : |\alpha| = k)$.

We shall also use the notation $D^\alpha(u) = (\frac{\partial}{\partial x_1})^{\alpha_1} \ldots (\frac{\partial}{\partial x_n})^{\alpha_n} u$ for any multiindex α, $D^k u = \{D^\alpha u : |\alpha| = k\}$.

Finally, let

$$F_\alpha(x,\xi) = \frac{\partial F(x,\xi)}{\partial \xi_\alpha} , \quad G_{j,\beta}(x,\eta) = \frac{\partial G_j(x,\eta)}{\partial \eta_\beta} . \tag{27}$$

In the present chapter we shall consider the boundary value problem

$$F(x,u,\dots,D^{2m}u) = f(x) \ , \quad x \in \Omega \ , \tag{28}$$

$$G_j(x,u,\dots,D^{m_j}u) = g_j(x) \ , \quad j = 1,\dots,m \ , \quad x \in \partial\Omega \tag{29}$$

under the following conditions:

(i) for an arbitrary function $v(x) \in C^{\ell_0,\delta}(\bar\Omega)$, $0 < \delta < 1$, the operator

$$\mathfrak{A}(v) : H^{\varepsilon_0}(\Omega) \longrightarrow H^{\ell_0}(\Omega,\partial\Omega) =$$
$$= H^{\ell_0-2m}(\Omega) \times H^{\ell_0-m_1-\frac{1}{2}}(\partial\Omega) \times \dots \times H^{\ell_0-m_m-\frac{1}{2}}(\partial\Omega)$$

defined by the identities

$$\begin{aligned} &\mathfrak{A}(v)u = \big(L(v)u, B_1(v)u, \dots, B_m(v)u\big) \ , \\ &L(v)u = \sum_{|\alpha| \leq 2m} F_\alpha(x,v,\dots,D^{2m}v)D^\alpha u \ , \\ &B_j(v)u = \sum_{|\beta| \leq m_j} G_{j,\beta}(x,v,\dots,D^{m_j}v)D^\beta u\Big|_{\partial\Omega} \end{aligned} \tag{30}$$

is elliptic and Fredholm; here $H^\ell(\Omega) = W_2^\ell(\Omega)$;

(ii) there is a function $H : \bar\Omega \times R^{M(2m-1)} \longrightarrow R^1$ of the class $C^{\ell-2m}$, such that the problem

$$\begin{aligned} &L(v)u + M(v)u = 0 \ , \quad x \in \Omega \ , \\ &B_j(v)u = 0 \ , \quad j = 1,\dots,m \ , \quad x \in \partial\Omega \ , \end{aligned} \tag{31}$$

has in $C^{\ell_0,\delta}(\bar\Omega)$ only the zero solution for an arbitrary function $v \in C^{\ell_0,\delta}(\Omega)$. Here

$$M(v)u = \sum_{|\gamma| \leq 2m-1} H_\gamma(x,v,\dots,D^{2m-1}v)D^\gamma u \ , \quad H_\gamma(x,\xi) = \frac{\partial H(x,\xi)}{\partial \xi_\gamma} \ .$$

The condition (i) expresses the ellipticity of the operator $L(v)$ and the fact that $L(v)$ and $B_j(v)$ at each point $x \in \partial\Omega$ satisfy the condition of Ya. B. Lopatinskiĭ, while the fact that $\mathfrak{A}$ is a Fredholm operator indicates that its index vanishes.

<u>REMARK 1.</u> Introducing simple modifications, we could replace the operator $M(v)$ from the condition (ii) by a totally continuous ope-

rator $\Gamma(v) : H^{\ell_0}(\Omega) \to H^{\ell_0}(\Omega,\partial\Omega)$, such that for every $v \in C^{\ell_0,\delta}(\overline{\Omega})$ the operator $\mathfrak{A}(v) + \Gamma(v)$ would represent an isomorphism of the corresponding spaces. Since the existence of such an operator $\Gamma(v)$ is obvious, the condition (ii) does not restrict the class of problems for which we below introduce the topological characteristic.

REMARK 2. The results of the present chapter are easily obtained provided the index of the operator $\mathfrak{A}(v)$ is negative. If the index of the operator is positive then no one-to-one correspondence between the solution of the problem (28), (29) and those of the operator equation of the form (20) is established. In this case it is possible to define a finite number of nonlinear functionals $\ell_i(u)$ (their number being equal to the index) in such a way that the solutions of the operator equation $Au = 0$ subjected to the condition $\ell_i(u) = 0$ (orthogonality type conditions) are solutions of the problem (28), (29) as well.

Let D be an arbitrary bounded domain in the space $H^{\ell}(\Omega)$ with a boundary $\partial\Omega$, let $f(x)$, $g_j(x)$ be fixed elements of the spaces $H^{\ell-2m}(\Omega)$, $H^{\ell-m_j-\frac{1}{2}}(\partial\Omega)$, respectively.

We define a nonlinear operator $A_1 : H^{\ell}(\Omega) \to [H^{\ell}(\Omega)]^*$ by

$$\begin{aligned} \langle A_1 u,\phi\rangle = &\left(F(x,u,\dots,D^{2m}u) - f(x),\ L(u)\phi + M(u)\phi\right)_{\ell-2m,\Omega} + \\ &+ \sum_{j=1}^{m}\left(G_j(x,u,\dots,D^{m_j}u) - g_j(x),\ B_j(u)\phi\right)_{\ell-m_j-\frac{1}{2},\partial\Omega} , \end{aligned} \tag{32}$$

where $(\cdot,\cdot)_{\ell,\Omega}$ and $(\cdot,\cdot)_{\ell,\partial\Omega}$ are the inner products in the spaces $H^{\ell}(\Omega)$, $H^{\ell}(\partial\Omega)$, respectively.

THEOREM 7. *If the conditions* (i), (ii) *are fulfilled and* $\ell \geq \ell_0+n_0$, *then the operator* A_1 *defined by* (32) *is continuous, bounded and satisfies the condition* $(S)_+$. *If the problem* (28), (29) *has no solutions belonging to* ∂D , *then the degree* $\mathrm{Deg}(A_1,\overline{D},0)$ *of the mapping* A_1 *of the domain* $\overline{D}$ *with respect to zero of the space* $[H^{\ell}(\Omega)]^*$ *is defined.*

P r o o f . The continuity and boundedness of the operator A_1 are easily verified. Let us prove the validity of the condition $(S)_+$.

Let u_n be an arbitrary sequence weakly converging to u_0 , let

$u_n \in D$ and

(33) $$\lim_{n\to\infty} \langle A_1 u_n, u_n - u_0\rangle \leq 0 \ .$$

The weak convergence of u_n in $H^{\ell}(\Omega)$ implies the strong convergence of u_n to u_0 in $C^{\ell_0}(\Omega)$, since $\ell - \ell_0 \geq n_0 = [\frac{n}{2}] + 1$ and the corresponding imbedding operator $H^{\ell}(\Omega) \to C^{\ell_0}(\bar{\Omega})$ is compact. For $\ell_0 < j < \ell$ the Nirenberg-Gagliardo inequality yields

$$\sum_{|\alpha|=j} ||D^{\alpha}(u_n - u_0)||_{L_{2(\ell-\ell_0)/(j-\ell_0)}} \leq$$

(34) $$\leq C \sum_{|\alpha|=\ell} \left|\left|D^{\alpha}(u_n-u_0)\right|\right|_2^{\frac{j-\ell_0}{\ell-\ell_0}} \left|\left|u_n-u_0\right|\right|_{C^{\ell_0}(\Omega)}^{1-\frac{j-\ell_0}{\ell-\ell_0}} +$$

$$+ C\left|\left|u_n-u_0\right|\right|_{C^{\ell_0}(\bar{\Omega})}$$

and this implies that $u_n \to u_0$ in $W^j_{2(\ell-\ell_0)/(j-\ell_0)}(\Omega)$.

Consider $D^{\alpha}F(x,u,\ldots,D^{2m}u)$ with $|\alpha| \leq \ell-2m$. An easy computation yields

$$D^{\alpha}F(x,u,\ldots,D^{2m}u) = \sum_{|\beta|\leq 2m} F_{\beta}(x,u,\ldots,D^{2m}u)D^{\alpha+\beta}u + R_{\alpha}u \ ,$$

where $R_{\alpha}(u)$ satisfies the estimate

(35) $$|R_{\alpha}(u)| \leq C_0(M)\{\sum_{j=\ell_0+1}^{\ell-1} \left|D^j u\right|^{\frac{\ell-\ell_0}{j-\ell_0}} + 1\}$$

provided the function $u(x)$ satisfies the inequality

(36) $$\left|\left|u\right|\right|_{C^{\ell_0}(\Omega)} \leq M$$

with a constant M .

Analogously we verify that $D^{\alpha}\big(M(u)v\big) = R_{1,\alpha}(u,v)$,

(37) $$D^{\alpha}\big(L(u)v\big) = L(u)D^{\alpha}v + R_{2,\alpha}(u,v) \ ,$$

where $R_{i,\alpha}(u,v)$ satisfies the estimate

(38) $$|R_{i,\alpha}(u,v)| \leq C_0(M)\left|\left|v\right|\right|_{C^{\ell_0}(\Omega)} \{\sum_{j=\ell_0+1}^{\ell-1} \left|D^j u\right|^{\frac{\ell-\ell_0}{j-\ell_0}} + 1\} +$$

$$+ C_0(M) \sum_{i=\ell_0+1}^{\ell-1} |D^i v| \{ \sum_{j=\ell_0+1}^{\ell+2m-i} |D^j u|^{\frac{\ell-i}{j-\ell_0}} + 1\} , \quad i = 1,\ 2 ,$$

provided the function $u(x)$ satisfies the condition (36).

Using the above identities we can write

$$(39) \qquad (F(x,u_n,\dots,D^{2m}u_n)-f(x), L(u_n)(u_n-u_0)+M(u_n)(u_n-u_0))_{\ell-2m,\Omega} =$$

$$= \sum_{|\alpha|\leq \ell-2m} \int_\Omega D^\alpha[F(x,u_n,\dots,D^{2m}u_n)-f(x)]\{D^\alpha L(u_n)(u_n-u_0) +$$

$$+ D^\alpha M(u_n)(u_n-u_0)\}dx =$$

$$= \sum_{|\alpha|\leq \ell-2m} \int_\Omega [L(u_n)D^\alpha u_n + R_\alpha(u_n) - D^\alpha f][L(u_n)D^\alpha(u_n-u_0) +$$

$$+ \sum_{j=1}^{2} R_{j,\alpha}(u_n,u_n-u_0)]dx = (L(u_0)u_n, L(u_0)(u_n-u_0))_{\ell-2m,\Omega} +$$

$$+ R_1^{(n)} ,$$

where

$$R_1^{(n)} = \sum_{|\alpha|\leq \ell-2m} \int_\Omega \{[L(u_n)D^\alpha u_n - L(u_0)D^\alpha u_n - R_{2,\alpha}(u_0,u_n) + R_\alpha(u_n) - D^\alpha f]\cdot$$

$$\cdot [L(u_n)D^\alpha(u_n-u_0) + \sum_{j=1}^{2} R_{j,\alpha}(u_n,u_n-u_0)] + D^\alpha(L(u_0)u_n)\cdot$$

$$\cdot [L(u_n)D^\alpha(u_n-u_0) - L(u_0)D^\alpha(u_n-u_0) + \sum_{j=1}^{2} R_{j,\alpha}(u_n,u_n-u_0) -$$

$$-R_{2,\alpha}(u_0,u_n-u_0)]\}dx .$$

In virtue of the inequalities (35), (38) and the above mentioned strong convergence of u_n to u_0 in $C^{\ell_0}(\Omega)$ and $W^j_{2(\ell-\ell_0)/(j-\ell_0)}(\Omega)$ we easily obtain

$$(40) \qquad \lim_{n\to\infty} R_1^{(n)} = 0 .$$

Now we pass to integrals over $\partial\Omega$ in (32), denoting

$$(41) \qquad G_j(x,u_n,\dots,D^{m_j}u_n) = G_j(x,u_0,\dots,D^{m_j}u_0) +$$

$$+ \int_0^1 B_j(u_0 + t(u_n-u_0))(u_n-u_0)dt .$$

Then

$$(42) \qquad \left(G_j(x,u_n,\ldots,D^{m_j}u_n)-g_j(x),B_j(u_n)(u_n-u_0)\right)_{\ell-m_j-\frac{1}{2},\partial\Omega} =$$

$$= \left(B_j(u_0)(u_n-u_0),B_j(u_0)(u_n-u_0)\right)_{\ell-m_j-\frac{1}{2},\partial\Omega} + R_{2,j}^{(n)} ,$$

where

$$R_{2,j}^{(n)} = \left(B_j(u_0)(u_n-u_0),B_j(u_n)(u_n-u_0)-B_j(u_0)(u_n-u_0)\right)_{\ell-m_j-\frac{1}{2},\partial\Omega} +$$

$$+ \Big(G_j(x,u_0,\ldots,D^{m_j}u_0)-g_j(x)+\int_0^1 B_j(u_0+t(u_n-u_0))(u_n-u_0)dt -$$

$$- B_j(u_0)(u_n-u_0),B_j(u_n)(u_n-u_0)\Big)_{\ell-m_j-\frac{1}{2},\partial\Omega} .$$

We shall prove that

$$(43) \qquad \lim_{n\to\infty} R_{2,j}^{(n)} = 0 .$$

Taking into account the boundedness of the imbedding operator $W_2^{\ell-m_j}(\Omega) \longrightarrow W_2^{\ell-m_j-\frac{1}{2}}(\partial\Omega)$, we conclude, for example for the first summand:

$$(44) \qquad \left|\left(B_j(u_0)(u_n-u_0),B_j(u_n)(u_n-u_0)-B_j(u_0)(u_n-u_0)\right)_{\ell-m_j-\frac{1}{2},\partial\Omega}\right| \leq$$

$$\leq C||B_j(u_0)(u_n-u_0)||_{\ell-m_j,\Omega}\cdot$$

$$\cdot||B_j(u_n)(u_n-u_0)-B_j(u_0)(u_n-u_0)||_{\ell-m_j,\Omega} .$$

Further, writing the operator B_j in the form of (37), we find that the first summand on the right hand side of (44) is uniformly bounded while the second tends to zero. Analogously we show that the limit of the second summand for $R_{2,j}^{(n)}$ vanishes, which completes the proof of (43).

The above considerations imply

$$(45) \qquad \langle A_1(u_n),u_n-u_0\rangle = ||L(u_0)(u_n-u_0)||^2_{\ell-2m,\Omega} +$$

$$+ \sum_{j=1}^{m} ||B_j(u_0)(u_n-u_0)||^2_{\ell-m_j-\frac{1}{2},\partial\Omega} + R^{(n)} ,$$

where

$$R^{(n)} = \left(L(u_0)u_0,L(u_0)(u_n-u_0)\right)_{\ell-2m,\Omega} + R_1^{(n)} + \sum_{j=1}^{m} R_{2,j}^{(n)} \quad \text{and}$$

$$R^{(n)} \to 0 \quad \text{for} \quad n \to \infty .$$

Making use of the apriori estimates for linear elliptic operators [17], we obtain

(46) $$||u_n - u_0||^2_{\ell,\Omega} \leq C\{||L(u_0)(u_n-u_0)||^2_{\ell-2m,\Omega} + \sum_{j=1}^{m} ||B_j(u_0)(u_n-u_0)||^2_{\ell-m_j-\frac{1}{2},\partial\Omega} + ||u_n-u_0||^2_{0,\Omega}\} .$$

Now (33), (45) and (46) imply the strong convergence of u_n to u_0 in $H^{\ell}(\Omega)$, which completes the proof of the theorem.

<u>REMARK 3.</u> The coefficients of the operators $L(u_0)$, $B_j(u_0)$ are not sufficiently smooth to allow for immediate application of the results of [17] when establishing the inequality (46). In this case it is necessary to obtain more precise estimates, based on the form of the operators $L(u_0)$, $B_j(u_0)$, on the inclusion of u_0 in the space $H^{\ell}(\Omega)$ and on the Nirenberg-Gagliardo inequality. We are not going into details since the procedure is straightforward.

<u>2.</u> The degree of a mapping A_1 introduced in Theorem 7 enables us to apply topological methods when studying the problem (28), (29). In particular, these methods are based on the investigation of a family of parametric problems and, in presence of apriori estimates, on the possibility of a homotopy between the problem (28), (29) and another simpler and more special problem of the same type.

We restrict ourselves to the formulation of a single one of the possible consequences.

<u>THEOREM 8.</u> Let $\tilde{F} : [0,1] \times \bar{\Omega} \times R^{M(2m)} \to R^1$, $\tilde{G}_j : [0,1] \times \bar{\Omega} \times R^{M(m_j)} \to R^1$, $j = 1,\dots,m$ *be continuous mappings. Assume that for each* $t \in [0,1]$ *, the functions* $F_t(x,\xi) = \tilde{F}(t,x,\xi)$, $G_{j,t}(x,\eta) = \tilde{G}_j(t,x,\eta)$ *satisfy the conditions* (i), (ii) *of Sec. 1 and, moreover,*

(a) *there exists a positive function* $K : R^1_+ \to R^1$ *such that for* $t \in [0,1]$, $u \in H^{\ell}(\Omega)$ *, the relations*

(47) $$F_t(x,u,\dots,D^{2m}u) = t\, f(x) \quad , \quad x \in \Omega ,$$
$$G_{j,t}(x,u,\dots,D^{m_j}u) = t\, g_j(x) \quad , \quad j = 1,\dots,m \quad , \quad x \in \partial\Omega ,$$

imply the estimate

$$||u||_{\ell,\Omega} \leq K(||f||_{\ell-2m,\Omega} + \sum_{j=1}^{m} ||g_j||_{\ell-m_j-\frac{1}{2},\partial\Omega}) ;$$

(b) $F_0(x,-\xi) = -F_0(x,\xi)$, $\tilde{G}_j(x,-\eta) = -\tilde{G}_j(x,\eta)$.

Then the problem (47) *has at least one solution in* $H^{\ell}(\Omega)$ *for arbitrary* $f \in H^{\ell-2m}(\Omega)$, $g_j \in H^{\ell-m_j-\frac{1}{2}}(\partial\Omega)$.

The p r o o f of the theorem follows from Theorems 5 and 3, provided that for fixed f , g_j , we replace D by the ball $B(0,R)$ in $H^{\ell}(\Omega)$ of the radius

$$R = K\left(||f||_{\ell-2m,\Omega} + \sum_{j=1}^{m} ||g_j||_{\ell-m_j-\frac{1}{2},\partial\Omega}\right) + 1 .$$

The operators A_t corresponding to the problem (47) are introduced according to the identity (32).

REMARK 4. The condition (a) of Theorem 8 may be weakened by requiring an apriori estimate of the problem (47) in $C^{\ell_0+\alpha}(\Omega)$.

3. The definition of the operator A_1 introduced in Sec. 1 requires sufficient smoothness of the functions $F(x,\xi)$, $G_j(x,\eta)$. It is possible to weaken the conditions concerning smoothness of these functions by considering the corresponding operators in the space $W_p^{\ell_0+1}(\Omega)$ for $p > n$.

Let the functions $F : \bar{\Omega} \times R^{M(2m)} \to R^1$, $G_j : \bar{\Omega} \times R^{M(m_j)} \to R^1$, $j = 1,\dots,m$, belong to the spaces C^{ℓ_0+2-2m} , $C^{\ell_0+2-m_j}$, respectively, and let the conditions (i), (ii) from Sec. 1 be satisfied for them with some δ , $0 < \delta < 1-\frac{n}{p}$.

Let D be an arbitrary bounded domain in the space $W_p^{\ell_0+1}(\Omega)$ with a boundary $\partial\Omega$, let $f(x)$, $g_j(x)$ be functions from the spaces $W_p^{\ell_0-2m+1}(\Omega)$, $B_p^{\ell_0-m_j-\frac{1}{p}+1}(\partial\Omega)$, where $B_p^s(\partial\Omega)$ denotes a Besov space. Under these conditions the operator $\mathfrak{A}$ defined above can be considered as an operator from $W_p^{\ell_0+1}(\Omega)$ into $W_p^{\ell_0+1}(\Omega,\partial\Omega)$:

$$W_p^{\ell_0+1}(\Omega,\partial\Omega) = W_p^{\ell_0-2m+1}(\Omega) \times \prod_{j=1}^{m} B_p^{\ell_0+1-m_j-\frac{1}{p}}(\partial\Omega) .$$

Introduce a nonlinear operator $A_2 : W_p^{\ell_0+1}(\Omega) \to [W_p^{\ell_0+1}(\Omega)]^*$ by the identity

$$(48)\qquad \langle A_2u,\phi\rangle = \sum_{|\alpha|\leq \ell_0+1-2m}\int_\Omega \psi_p\{D^\alpha[F(x,u,\ldots,D^{2m}u)-f]\}D^\alpha[L(u)\phi + M(u)\phi]dx +$$

$$+\sum_{j=1}^{m}\sum_{|\beta|=\ell_0-m_j}\int_{\partial\Omega}\int_{\partial\Omega}\psi_p\{D_x^\beta[G_j(x,\ldots,D_x^{m_j}u(x))-g_j(x)] -$$

$$- D_y^\beta[G_j(y,\ldots,D_y^{m_j}u(y))-g_j(y)]\}[D_x^\beta B_j(u)\phi(x)-D_y^\beta B_j(u)\phi(y)]\cdot$$

$$\cdot\frac{1}{|x-y|^{n+p-2}}d_xSd_yS +$$

$$+\sum_{j=1}^{m}\int_{\partial\Omega}\psi_p\{G_j(x,\ldots,D^{m_j}u(x))-g_j(x)\}B_j(u)\phi dS ,$$

where

$$\psi_p(t) = t + |t|^{p-2}t .$$

THEOREM 9. *The operator* $A_2 : W_p^{\ell_0+1}(\Omega)\to[W_p^{\ell_0+1}(\Omega)]^*$ *defined by* (48) *is continuous, bounded and satisfies the condition* $(S)_+$ *for* $p > n$. *If the problem* (28), (29) *has no solutions belonging to* ∂D, *then the degree* $\mathrm{Deg}(A_2,\bar{D},0)$ *of the mapping* A_2 *of the set* $\bar{D}$ *with respect to zero of the space* $[W_p^{\ell_0+1}(\Omega)]^*$ *is defined.*

P r o o f proceeds analogously to that of Theorem 7.

Notice that, if the condition (ii) is satisfied, the solutions of the problem (28), (29) coincide with those of the operator equation $A_2u = 0$ in the same way as was the case in Sec. 1.

REMARK 5. We can assume that the operator $\mathfrak{A}(v)$ is elliptic not for all smooth functions v but only for those which are solutions of the problem (28), (29). Then a construction analogous to that given above yields operators A_1, A_2 corresponding to the differential problem (28), (29). These operators satisfy the condition (α_0) and consequently, we can in this case define the degree of the mapping.

3. Nonlinear Dirichlet problem and coercive estimates for pairs of linear elliptic operators

1. In Chap. 2 we introduced the degree of a mapping, correspon-

ding to general nonlinear elliptic boundary value problems, by constructing the mappings A_1 , A_2 . When applying the theory of degree to individual problems it is desirable to have, as far as possible, a simpler representation of the mappings in question. A simpler construction of the mappings A can be obtained in the case of nonlinear Dirichlet problem; it is based on important coercive estimates for pairs of linear operators.

We shall be interested in inequalities of the type

$$(49) \qquad (Lu,Mu)_\ell \geq c_1 ||u||^2_{2m+\ell} - c_2 ||u||^2_0 \quad .$$

for functions $u(x) \in W_2^{2m+\ell}(\Omega) \cap \overset{o}{W}{}_2^m(\Omega)$, that is, satisfying the boundary value conditions

$$(50) \qquad D^\alpha u(x) = 0 \ , \quad |\alpha| \leq m-1 \ , \quad x \in \partial\Omega \ .$$

The symbols $(.,.)_\ell$, $||\cdot||_\ell$ in (49) stand for the inner product and the norm in $W_2^\ell(\Omega)$, respectively, while L , M are linear differential operators of the order $2m$ with sufficiently smooth coefficients. It is easy to see that a necessary condition for the inequality (49) to be valid with positive constants c_1 , c_2 independent of u is that the operators L , M be elliptic. This follows from the fact that the ellipticity of the operator L is necessary for the validity of the inequality (49) in the case $L = M$ (see [17]).

In [17], the estimate (49) is proved provided $L = M$. For $\ell = 0$ the inequality (49) was proved in the papers by P. E. Sobolevskiĭ and O. A. Ladyženskaya for arbitrary elliptic operators L , M of the second order $(m = 1)$.

For $\ell > 0$ or $m > 1$ the inequality (49) need not hold. Examples of linear elliptic operators L , M of the fourth order $(m = 2)$, such that the inequality (49) fails to hold for $\ell = 0$, are given in [14]. In the same paper the reader will find examples of second order operators, for which (49) is not valid for $\ell = 1$, as well as an example of operators L , M with complex-valued coefficients, for which the analogue of the inequality (49) does not hold in the complex case.

The above consideration shows that to make the validity of the estimate (49) possible it is necessary to have a certain relation between the operators L , M . Among the most general and important possible approaches let us point out the following one: for a given family of operators L , prove the existence and present a construction of an operator M such that the estimate (49) holds for the operator M and for any operator L from the given family. It turned out that this formulation of the problem admits its solution, which was presented in [14].

Let Ω be a bounded domain in R^n with a boundary $\partial\Omega$ of the class C^∞. A positive number A is called a constant of ellipticity of a linear operator

$$L(x,D) = \sum_{|\alpha|\leq 2m} a_\alpha(x)D^\alpha , \quad x \in \overline{\Omega} \subset R^n$$

if for $x \in \overline{\Omega}$, $\xi = (\xi_1,\ldots,\xi_n) \in R^n$ the inequality

$$\mathrm{Re} \sum_{|\alpha|=2m} a_\alpha(x)\xi^\alpha \geq A|\xi|^{2m}$$

holds. The coefficients $a_\alpha(x)$ of the operator $L(x,D)$ are considered to be complex-valued.

For a nonnegative integer ℓ and $0 < \lambda < 1$ let us denote by $Z^{\ell,\lambda}_{2m}(A,B,\Omega)$ the family of linear regularly elliptic operators $L(x,D)$ of the 2m-th order with a single constant of ellipticity A and with coefficients $a_\alpha(x)$ of the class $C^{\ell,\lambda}(\overline{\Omega})$, which satisfy the conditions

$$||a_\alpha(x)||_{C^{\ell,\lambda}(\overline{\Omega})} \leq B , \quad |\alpha| \leq 2m .$$

<u>THEOREM 10.</u> *For arbitrary positive numbers* A , B , λ *there are positive constants* C_1 , C_2 , *depending only on* A , B , λ , m , Ω , *and a linear operator* $M(x,D) = \sum_{|\alpha|\leq 2m} b_\alpha(x)D^\alpha$ *with infinitely differentiable real coefficients* $b_\alpha(x)$, *such that* $M(x,D) \in Z^{0,\lambda}_{2m}(1,C_2,\Omega)$ *and for* $L(x,D) \in Z^{0,\lambda}_{2m}(A,B,\Omega)$ *the inequality*

$$(51) \qquad \mathrm{Re} \int_\Omega L(x,D)u\cdot\overline{M(x,D)u}\, dx \geq C_1||u||^2_{2m} - C_2||u||^2_0$$

holds for any $u(x) \in W^{2m}_2(\Omega) \cap \overset{o}{W}{}^m_2(\Omega)$.

P r o o f of the apriori estimate (51) and the construction of the operator M(x,D) proceeds by means of the corresponding local considerations and their putting together by means of the partition of unity. Since the concluding phase of the proof is more standard and is given in [14], we restrict ourselves to establishing the apriori estimate in the model case.

First, let us consider functions of one variable x, which are defined on the half-axis $R^1_+ = \{x \in R^1 : x \geqq 0\}$. By $P_{2m}(A,B,R^1_+)$ we denote the family of ordinary differential operators $P(D) = \sum_{j=0}^{2m} p_j D^j$, $D = \frac{1}{i}\frac{d}{dx}$, with constant complex coefficients p_j and satisfying for $\xi \in R^1$ the conditions

$$\operatorname{Re} P(\xi) \geqq A(1+|\xi|^2)^m , \quad |p_j| \leqq B , \quad j = 0,\dots,2m .$$

LEMMA 5. There are positive numbers q, k, $q \geqq 1$, $k \leqq A$ *depending only on* A, B, m *and such that for* $P(D) \in P_{2m}(A,B,R^1_+)$, $Q(D) = q^{2m} + D^{2m}$ *and for any function* $u(x) \in W_2^{2m}(R^1_+) \cap \overset{o}{W}{}_2^m(R^1_+)$ *the inequality*

$$(52) \qquad \operatorname{Re} \int_{R^1_+} Pu\overline{Qu}dx \geqq k \int_{R^1_+} \sum_{j=0}^{2m} |D^j u|^2 dx$$

holds.

P r o o f . In order to verify the estimate (52) we only have to notice that

$$(53) \qquad \operatorname{Re} \int_{R^1_+} Pu\cdot\overline{D^{2m}u}\, dx \geqq k_1 \int_{R^1_+} |D^{2m}u|^2 dx - k_2 \int_{R^1_+} \sum_{j=0}^{2m-1} |D^j u|^2 dx ,$$

$$\operatorname{Re} \int_{R^1_+} Pu\,\overline{u}\, dx \geqq k_3 \int_{R^1_+} \sum_{j=0}^{m} |D^j u|^2 dx$$

holds with some constants k_1, k_2, k_3 depending only on A, B. The first inequality in (53) follows from the Cauchy inequality. In order to obtain the second inequality we set

$$\int_{R^1_+} Pu\,\overline{u}\, dx = \sum_{j=0}^{m} p_j \int_{R^1} D^j \tilde{u}\, \overline{\tilde{u}}\, dx + \sum_{j=m+1}^{2m} p_j \int_{R^1} D^m \tilde{u}\, \overline{D^{j-m}\tilde{u}}\, dx ,$$

where $\tilde{u}(x)$ is the extension of $u(x)$ to R^1 by zero outside R^1_+. Then the second inequality in (53) is a consequence of the Parseval identity and of the condition imposed on $\operatorname{Re} P(\xi)$.

Now the inequality (52) follows from (53) by the interpolation inequality and by a proper choice of q.

Lemma 5 enables us to prove in a simple way the corresponding assertion for the homogeneous elliptic operator with constant coefficients and functions defined in $R^n_+ = \{x = (x_1,\dots,x_n) \in R^n : x_n \geq 0\}$. For $n \geq 2$ we denote by $P_{2m}(A,B,R^m_+)$ the family of differential operators $P(D) = \sum_{|\alpha|=2m} p_\alpha D^\alpha$ with constant complex coefficients p_α satisfying the conditions

$$\operatorname{Re} P(\xi) \geq A|\xi|^{2m}, \quad |p_\alpha| \leq B, \quad |\alpha| = 2m \quad \text{for} \quad \xi \in R^n .$$

LEMMA 6. For any function $u(x) \in W_2^{2m}(R^n_+) \cap \overset{o}{W}{}_2^m(R^n_+)$ *and* $P(D) \in P_{2m}(A,B,R^n_+)$, $Q(D) = q^{2m}\left[\sum_{j=1}^{n-1} D_j^2\right]^m + D_n^{2m}$, *the inequality*

$$\operatorname{Re} \int_{R^n_+} P(D)u\,\overline{Q(D)u}\,dx \geq k \int_{R^n_+} \sum_{|\alpha|=2m} |D^\alpha u|^2 dx \tag{54}$$

holds with the constants k, q *defined in Lemma 5.*

Lemma 6 is a consequence of Lemma 5 and the Parseval identity.

The rest of the proof of Theorem 10 proceeds on the basis of the model inequality (54) by means of the partition of unity.

THEOREM 11. For arbitrary positive numbers A, B *a nonnegative integer* ℓ *and* λ, $0 < \lambda < 1$, *there are infinitely differentiable real-valued functions* $C_{\alpha\beta}(x)$, $|\alpha|, |\beta| \leq \ell$, $C_{\alpha\beta}(x) = C_{\beta\alpha}(x)$, *and positive constants* k_1, k_2 *depending only on* A, B, m, ℓ, λ, Ω, *such that for* $L(x,D) \in Z^{\ell,\lambda}_{2m}(A,B,\Omega)$, $u(x) \in W_2^{2m+\ell}(\Omega) \cap \overset{o}{W}{}_2^m(\Omega)$ *the inequality*

$$\operatorname{Re}[Lu,Mu]_\ell \geq k_1\|u\|^2_{2m+\ell} - k_2\|u\|^2_0 \tag{55}$$

holds, where the operator M *was introduced in Theorem 10, and*

$$[v_1,v_2]_\ell = \sum_{|\alpha|,|\beta|\leq\ell} \int_\Omega C_{\alpha\beta}(x) D^\alpha v_1 \cdot \overline{D^\beta v_2}\,dx , \tag{56}$$

$$k_1\|v\|^2_\ell \leq [v,v]_\ell \leq k_2\|v\|^2_\ell \tag{57}$$

hold for any $v(x)$, $v_1(x)$, $v_2(x) \in W_2^\ell(\Omega)$.

In the beginning let us point out that in order to obtain the validity of the inequality of the form (49), it was of advantage to introduce a special scalar product $[\cdot,\cdot]_\ell$ in the space $W_2^\ell(\Omega)$.

The counterexamples given in [14] demonstrate the importance of the special choice of the scalar product.

Theorem 11 will be first proved for the case of differential operators on the half-axis and then it will be established in the usual way, by employing the partition of unity, for the general case.

<u>LEMMA 7</u>. *For arbitrary positive numbers* A , B *and a nonnegative integer* ℓ *there are positive numbers* ρ , K , $\rho > 1$, *depending only on* A , B , ℓ , m *and such that for* $P(D) \in P_{2m}(A,B,R^1_+)$, $u(x) \in W_2^{2m+\ell}(R^1_+) \cap \overset{o}{W}{}_2^m(R^1_+)$ *the inequality*

$$\text{(58)} \qquad \operatorname{Re} \int_{R^1_+} \{D^\ell Pu \cdot \overline{D^\ell Qu} + \rho Pu \cdot \overline{Qu}\} dx \geqq K \int_{R^1_+} \sum_{j=0}^{2m+\ell} |D^j u|^2 dx$$

holds with the operator $Q(D)$ *introduced in Lemma 5.*

In order to prove the inequality it suffices to notice that

$$\operatorname{Re} \int_{R^1_+} D^\ell Pu \cdot \overline{D^\ell Qu}\, dx \geqq K' \int_{R^1_+} |D^{2m+\ell} u|^2 dx - K'' \int_{R^1_+} \sum_{j=\ell}^{2m+\ell-1} |D^j u|^2 dx$$

holds with constants K' , K'' depending only on A , B , and then to use the estimate (52) and the interpolation inequality.

Applying the Parseval identity to Lemma 7, we obtain the estimate

$$\operatorname{Re} \sum_{|\alpha|=\ell} \int_{R^n_+} C_\alpha(\rho) D^\alpha P(D) u \cdot \overline{D^\alpha Q(D) u}\, dx \geqq K \int_{R^n_+} \sum_{|\alpha|=2m+\ell} |D^\alpha u|^2 dx$$

for any function $u(x) \in W_2^{2m+\ell}(R^n_+) \cap \overset{o}{W}{}_2^m(R^n_+)$ and an operator $P(D) \in P_{2m}(A,B,R^n_+)$, where $Q(D)$ is the operator introduced in Lemma 6 and $C_\alpha(\rho)$ is defined as follows:

$$C_\alpha(\rho) = \frac{\ell!}{\alpha_1! \ldots \alpha_{n-1}!} \qquad \text{for } \alpha_n = 0 ,$$

$$C_\alpha(\rho) = 0 \qquad \text{for } 0 < \alpha_n < \ell ,$$

$$C_\alpha(\rho) = \rho \qquad \text{for } \alpha_n = \ell .$$

<u>**REMARK 6.**</u> Theorem 11 still holds if the family of operators

$Z^{\ell,\lambda}_{2m}(A,B,\Omega)$ is for $\ell > \frac{n}{2}$ replaced by the family $\tilde{Z}^{\ell}_{2m}(A,B,\Omega)$ that consists of linear regularly elliptic operators $L(x,D)$ of the $2m$-th order with a single constant of ellipticity A and with the coefficients $a_\alpha(x)$ of the class $W^{\ell}_2(\Omega)$, satisfying the conditions

$$||a_\alpha(x)||_{W^{\ell}_2(\Omega)} \leq B \ , \quad |\alpha| \leq 2m \ .$$

To verify the assertion of the remark it suffices to estimate the subordinate terms when applying the partition of unity, using the Hölder inequality and the imbedding theorem for the Sobolev spaces.

2. Now we shall show a simple way of reducing the Dirichlet problem to an operator equation with an operator satisfying the condition $(S)_+$ on the basis of Theorem 11. For the sake of simplicity we will consider the case of the homogeneous boundary value problem.

Let a function $F : \overline{\Omega} \times R^{M(2m)} \to R^1$ have continuous derivatives up to an order $\ell+1$, $\ell \geq [\frac{n}{2}] + 1$, and let us assume that there is a nonnegative nonincreasing function $\nu : R^1_+ \to R^1$ such that

$$\sum_{|\alpha|=2m} F_\alpha(x,\xi)\eta^\alpha \geq \nu(|\xi|)\cdot|\eta|^{2m}$$

holds provided $x \in \overline{\Omega}$, $\xi \in R^{M(2m)}$, $\eta \in R^n$. (We keep here the notation from Chap. 2, Sec. 1.)

We shall discuss the solvability in $X = W^{2m+\ell}_2(\Omega) \cap \overset{o}{W}{}^m_2(\Omega)$ of the problem

(59) $$F(x,u,\dots,D^{2m}u) = f(x) \ , \quad x \in \Omega \ ,$$

(60) $$D^\alpha u = 0 \ , \quad |\alpha| \leq m-1 \ , \qquad x \in \partial\Omega \ .$$

Let D be an arbitrary bounded domain in the space X and assume that the problem (59), (60) has no solution belonging to ∂D .

Consider the family $Z = \{L(v) : v \in \overline{D}\}$, the operator $L(v)$ being defined in Chap. 2, Sec. 1. In virtue of Theorem 11 and Remark 6 we can construct a linear uniformly elliptic operator $M(x,D)$ of the order $2m$, with infinitely differentiable in $\overline{\Omega}$ coefficients, and a scalar product $[\cdot,\cdot]_\ell$ in the space $W^{\ell}_2(\Omega)$ so that there are constants C_1 , C_2 such that

(61) $$[L(v)u,Mu]_\ell \geq C_1||u||^2_{2m+\ell} - C_2||u||^2_0 \quad \text{for} \quad v \in \overline{D} \ ,$$

$$(-1)^m \int_\Omega Mu\cdot u\,dx \geq C_1||u||^2_m \ .$$

We define a nonlinear operator $A_3 : \overline{D} \longrightarrow X^*$ by

(62) $$\langle A_3 u, \phi \rangle = [F(x,u,\dots,D^{2m}u) - f(x), M\phi]_\ell \ .$$

THEOREM 12. The operator $A_3 : \overline{D} \longrightarrow X^*$ *defined by* (62) *is continuous, bounded and satisfies the condition* $(S)_+$. *If the problem* (59), (60) *has no solution belonging to* ∂D , *then* $\mathrm{Deg}(A_3, \overline{D}, 0)$ *is defined.*

P r o o f is analogous to that of Theorem 7. If $u_n \in \overline{D}$ and u_n weakly converges to u_0 in X , then analogously to (45) we can obtain

$$\langle A_3 u_n, u_n - u_0 \rangle = [L(u_0)(u_n - u_0), M(u_n - u_0)]_\ell + \tilde{R}^{(n)} \ ,$$

where $\tilde{R}^{(n)} \to 0$ with $n \longrightarrow \infty$. The strong convergence of u_n to u_0 then follows from (61), provided $\lim\limits_{n\to\infty} \langle A_3 u_n, u_n - u_0 \rangle \leqq 0$.

REMARK 7. If we study the problem (59), (60) that is elliptic only on the solutions, we have to take the family $\{L(v),\ v \in N_D\}$ for Z when constructing the operator M . Here N_D is the set of solutions of the problem (59), (60) which belong to $\overline{D}$.

4. Solvability of nonlinear boundary value problems

In the present chapter we apply the topological methods developed above to three distinct nonlinear boundary value problems: the Dirichlet problem for the Monge-Ampere equation and for the general nonlinear equation of the thin layer, and the Neumann problem for the quasilinear second order elliptic equation.

The common feature of all the problems considered is that they differ from the well-known classes of quasilinear divergent equations in the Sobolev spaces. Another common feature is the essential nonlinearity of the problems, as well as the fact that for all these problems it is possible to find apriori estimates and establish the existence theorem for the classical solution.

1. Let us study the problem of existence of a solution regular in a closed domain, of the Dirichlet problem

(63) $$u_{xx}u_{yy} - u_{xy}^2 = \phi(x,y,u,\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}) \ , \quad (x,y) \in \Omega \ ,$$

$$(64) \qquad u\big|_{\partial\Omega} = g(x,y) \ .$$

Here Ω is a circle of radius R and centre at the origin of the coordinate system. We assume $g(x,y) \in C^{4,\lambda}(\partial\Omega)$, $\phi(x,y,u,p,q)$ is a positive function of a class $C^{2,\lambda}(G)$, $0 < \lambda < 1$, where $G = \{(x,y,u,p,q) : (x,y) \in \overline{\Omega}, \ (u,p,q) \in R^3\}$, and g , ϕ satisfy the conditions

(i) $\phi(x,y,u,p,q) < \Phi(x^2+y^2)f(p^2+q^2)$

provided $u \leqq m = \max\limits_{(x,y)\in\partial\Omega} g(x,y)$;

$$\text{(ii)} \quad \iint_\Omega \Phi(x^2+y^2)dxdy < \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \inf_{(\xi-p)^2+(\eta-q)^2<M_k} f^{-1}(\xi^2+\eta^2)dpdq \ ,$$

M_k being the lower winding of the curve *) determined by the condition (64).

The existence of a solution, regular in the closed domain $\overline{\Omega}$, of the problem (63), (64) is proved in [18] by the Newton-Kantorovič method under the conditions (i), (ii) and the additional assumption

$$(65) \qquad \frac{\partial\phi(x,y,u,p,q)}{\partial u} \geqq 0 \ .$$

Making use of the apriori estimates of a solution of the problem (63), (64), given in [18], as well as of the topological method developed in [12], the author with A. E. Šiškov proved in [19] the classical solvability of the problem (63), (64), assuming only (i), (ii) but without the assumption (65).

Let us denote by $H_4^+(\Omega)$ the family of functions belonging to $W_2^4(\Omega)$ and satisfying the conditions

$$u_{xx}u_{yy} - u_{xy}^2 > 0 \ , \quad u_{xx} > 0 \ , \quad (x,y) \in \overline{\Omega} \ .$$

THEOREM 13 [19]. *Let* $g(x,y) \in C^{4,\lambda}(\partial\Omega)$, $\phi(x,y,u,p,q) \in C^{2,\lambda}(G)$ *and let the conditions* (i), (ii) *be fulfilled. Then the problem* (63), (64) *has at least one solution in* $H_4^+(\Omega)$.

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*) The lower winding of the curve $z = \phi(x,y)$, $(x,y) \in \partial\Omega$ is the number

$$\sup_{(x_0,y_0)\in\partial\Omega} [\inf\{p^2+q^2; \ (p,q) \in R^2 \text{ such that the plane } z = p(x-x_0) + q(y-y_0) + \phi(x_0,y_0) \text{ touches the curve from below}\}] .$$

(see [18], p. 110).

REMARK 8. It follows from the imbedding theorems in the Sobolev spaces and from [17] that the solution whose existence is asserted in Theorem 13, belongs to $C^{4,\lambda}(\bar{\Omega})$.

We will show how to reduce the problem (63), (64) to a nonlinear operator equation. It follows from [18] and [17] that under the assumptions of Theorem 13, solutions of the problem

$$u_{xx}u_{yy} - u_{xy}^2 = \tau\phi(x,y,u,\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}) + (1-\tau)\Phi(x^2+y^2)f(|\nabla u|^2) , \quad (x,y) \in \Omega , \tag{66}$$

$$u\big|_{\partial\Omega} = \tau g(x,y) , \tag{67}$$

which belong to $H_4^+(\Omega)$, satisfy the apriori estimate $||u||_4 \leq k$ with a certain positive constant k , provided $0 \leq \tau \leq 1$. Here and in what follows $||\cdot||_\ell$ stands for the norm in $W_2^\ell(\Omega)$.

Let $h(x,y) \in W_2^4(\Omega)$ be a harmonic function in Ω satisfying the boundary value condition (64). In $X = W_2^4(\Omega) \cap \overset{\circ}{W}{}_2^1(\Omega)$ let us define a domain D by

$$D = \{v = u-\tau h : u \in H_4^+(\Omega), \ ||u||_4 < k + 1, \ 0 \leq \tau \leq 1\} .$$

For $v \in D$ we consider the parametric family of differential equations

$$\begin{aligned} F_\tau\left(x,y,v,\frac{\partial v}{\partial x},\frac{\partial v}{\partial y},\frac{\partial^2 v}{\partial x^2},\frac{\partial^2 v}{\partial x\partial y},\frac{\partial^2 v}{\partial y^2}\right) &= (v_{xx}+\tau h_{xx})(v_{yy}+\tau h_{yy}) - \\ &- (v_{xy}+\tau h_{xy})^2 - \tau\phi\left(x,y,v+\tau h,\frac{\partial v}{\partial x} +\tau\frac{\partial h}{\partial x}, \frac{\partial v}{\partial y} +\tau\frac{\partial h}{\partial y}\right) - \\ &- (1-\tau)\Phi(x^2+y^2)f\left(|\nabla(v+\tau h)|^2\right) = 0 . \end{aligned} \tag{68}$$

By $N_{\bar{D}}$ we denote the set of solutions of the equation (68) which belong to $\bar{D}$. Then $N_{\bar{D}} \cap \partial D = \emptyset$.

Let $L_\tau(v)$ be a linear differential operator constructed for the function F_τ from (68) in an analogous way as the operator $L(v)$ was constructed for the function $F(x,\xi)$ in Chap. 2, Sec. 1. Consider the family of differential operators

$$Z = \{L_\tau(v) : v \in N_{\bar{D}} , \ \tau \in [0,1]\} .$$

According to Theorem 11 we can construct a linear elliptic operator M of the second order and an inner product $[\ ,\ ]$ in $W_2^2(\Omega)$ so that

$$[L_\tau(v)u,Mu] \geq C_1||u||_4^2 - C_2||u||_0^2$$

and

$$-(Mu,u) \geqq C_1 ||u||_1^2 .$$

Analogously to (62) we now define the family of nonlinear operators $A_\tau : \overline{D} \longrightarrow X^*$

$$(69) \qquad \langle A_\tau v,\phi\rangle = \left[F_\tau\left(x,y,v,\frac{\partial v}{\partial x},\frac{\partial v}{\partial y},\frac{\partial^2 v}{\partial x^2},\frac{\partial^2 v}{\partial x \partial y},\frac{\partial^2 v}{\partial y^2}\right),M\phi\right] .$$

Analogously to Theorem 12 it is verified that A_τ are bounded demicontinuous operators and that the family of operators A_τ satisfies the condition $(\alpha_0^{(\tau)})$. Since the equation $A_\tau u = 0$ has no solution with $u \in \partial D$, we have $\mathrm{Deg}(A_1,\overline{D},0) = \mathrm{Deg}(A_0,\overline{D},0)$ by Theorem 1. For $\tau = 0$ the equation (68) has a unique solution in $\overline{D}$. This solution represents a non-degenerate critical point of the field $A_0(v)$ (in the terminology of [8]). Hence $\mathrm{Deg}(A_0,\overline{D},0) \neq 0$ and the solvability of the problem (63), (64) is a consequence of Theorem 5.

REMARK 9. Using a topological method developed in [12], A. E. Šiškov obtained apriori estimates and solvability for a more general boundary value problem, which results by adding a quasilinear second order elliptic operator to the left hand side of (63).

2. In this section we establish the existence of a classical solution of the Dirichlet problem for the general nonlinear elliptic equation in a narrow strip. By $\{\Omega_h,\ 0 < h \leqq 1\}$ we denote the family of domains in R^n with infinitely differentiable boundaries, such that

(a) $\Omega_{h_1} \subset \Omega_{h_2}$ for $h_1 < h_2$;

(b) there are open coverings $\{U_i : i = 1,\dots,I\}$ of the set $\overline{\Omega}_1$ and diffeomorphisms $\phi_i : U_i \cap \overline{\Omega}_1 \longrightarrow R^n$ of the class C^∞ satisfying $\phi_i(U_i \cap \overline{\Omega}_h) = S_h = \{x \in R^n : |x'| < 1,\ 0 \leqq x_n \leqq h\}$, $x = (x',x_n)$, $x' = (x_1,\dots,x_{n-1})$.

Let $\{\psi_i(x)\}$, $i = 1,\dots,I$ be a partition of unity subordinate to the covering.

For nonnegative integers m, ℓ, k and an arbitrary $p > 1$ we denote by $W_p^{2m,\ell,k}(\Omega_h)$ the closure of the set of functions infinitely differentiable in Ω_h, with respect to the norm

$$||u||^p_{W_p^{2m,\ell,k}(\Omega)} = \sum_{i=1}^{I} ||\psi_i(\phi_i^{-1}(y))u(\phi_i^{-1}(y))||^p_{W_p^{2m,\ell,k}(S_h)} ,$$

where

$$||v(y)||^p_{W_p^{2m,\ell,k}(S_h)} =$$

$$= \sum_{|\alpha|\leq 2m} \int_{S_h} \{ \sum_{j=0}^{k} |D_n^j D^\alpha v(y)|^p + \sum_{|\beta|\leq\ell}' |D^\beta D^\alpha v(y)|^p \} dy .$$

Here we use the usual multiindex notation; $\sum'$ indicates the summation over all multiindices with the last coordinate zero.

The space $C^{s,r,\lambda}(\Omega_h)$ for nonnegative integers s , r and $\lambda \in [0,1]$ consists of functions defined in Ω_h which have the finite norm

$$||u(x)||_{C^{s,r,\lambda}(\Omega_h)} = \max_i ||\psi_i(\phi_i^{-1}(y))u(\phi_i^{-1}(y))||_{C^{s,r,\lambda}(S_h)} ,$$

where

$$||v(y)||_{C^{s,r,\lambda}(S_h)} = \sum_{|\beta|\leq s} \sum_{|\alpha|\leq r}' ||D^\beta D^\alpha v(y)||_{C^{0,\lambda}(S_h)}$$

and $||\cdot||_{C^{0,\lambda}}$ is the current norm in the space of functions satisfying the Hölder condition with the exponent λ .

In order to obtain apriori estimates of solutions of both linear and nonlinear equations in the domains Ω_h we shall need some inequalities which are connected with the imbedding and interpolation of the corresponding Sobolev spaces. We shall give only two of such estimates, in which the letter k denotes a constant depending only on n , m , ℓ and of the constants characterizing the differential properties of the functions ϕ_i , ψ_i , which are assumed to be fixed.

<u>*LEMMA 8.*</u> *An arbitrary function* $u(x) \in W_2^{2m,\ell,0}(\Omega_h) \cap \overset{o}{W}{}_2^m(\Omega_h)$ *with* $\ell \geq n+1$, $0 < \lambda < \frac{1}{2}$ *satisfies the estimate*

$$(70) \qquad ||u(x)||_{C^{2m-1,2,\lambda}(\Omega_h)} \leq k\, h^{\frac{1}{2}-\lambda} \cdot ||u(x)||_{W_2^{2m,\ell,0}(\Omega_h)} .$$

<u>*LEMMA 9.*</u> *An arbitrary function* $u(x) \in W_2^{2m,\ell,0}(\Omega_h) \cap \overset{o}{W}{}_2^m(\Omega_h) \cap \cap W_{2\ell}^{2m,1,0}(\Omega_h)$ *satisfies estimate*

$$||u(x)||^{\frac{2\ell}{r}}_{W^{2m,r,0}_{2\ell/r}(\Omega_h)} \leq \varepsilon ||u(x)||^2_{W^{2m,\ell,0}_2(\Omega_h)} + \\ + k\varepsilon^{-\frac{\ell(r-1)}{\ell-r}} ||u(x)||^{2\ell}_{W^{2m,1,0}_{2\ell}(\Omega_h)}$$

with an arbitrary positive ε *and* $1 < r \leq \ell-1$.

Further, we give a coercive estimate for pairs of linear elliptic operators in S_h . This estimate differs from those of Chap. 3 by its uniformity for $h \in (0,1]$ and, on the other hand, by a special choice of the function space whose elements are involved in the estimate.

The next lemma in which we keep the notation from Chap. 3 for $Z^{\ell,\lambda}_{2m}(A,B,\Omega)$ is of crucial importance for the estimates of solutions of the nonlinear problem as well as for the proof of the existence theorem.

LEMMA 10. *There are constants* C_1 , C_2 *depending only on* A , B , m , n , ℓ , λ , *such that for* $q > C_1$, $\rho > qC_1$, *an arbitrary operator* $L(x,D) \in Z^{1,\lambda}_{2m}(A,B,S_h)$ *and an arbitrary function* $u(x) \in$ $\in W^{2m,\ell,1}_2(S_h) \cap \overset{\circ}{W}{}^m_2(S_h)$ *that vanishes for* $|x'|$ *close to one, the estimate*

$$(71)\qquad \int_{S_h} \{L(x,D)D'_n u\cdot M(D)D'_n u + \rho \sum_{|\alpha|\leq\ell}' L(x,D)D^\alpha u\cdot M(D)D^\alpha u\}dx \geq$$

$$\geq \frac{A}{2}\int_{S_h} |D^{2m+1}_n u|^2 dx + \frac{\rho A}{2}\int_{S_h}\sum_{|\alpha|\leq\ell}' |D^\alpha D^{2m}_n u|^2 dx +$$

$$+ C_2\rho q \sum_{|\alpha|\leq\ell}' \sum_{|\beta|=m} \sum_{|\gamma|=m}' \int_{S_h} |D^{\alpha+\beta+\gamma}u|^2 dx -$$

$$- C(\rho,q)\int_{S_h} \sum_{|\alpha|\leq 2m} |D^\alpha u|^2 dx ,$$

holds, where $M(D) = D^{2m}_n + q[D^2_1+\ldots+D^2_{n-1}]^m$ *and* $C(\rho,q)$ *is a constant depending only on* A , B , m , n , λ , ρ , q .

In the case of a homogeneous operator $L(x,D)$ with constant coefficients, the proof of Lemma 10 is close to that of Lemma 7, only it is based on uniform (with respect to h) interpolation inequalities.

In what follows we fix the number ℓ , $\ell \geq n+1$. We will consider the solvability in $W_2^{2m,\ell,1}(\Omega_h)$ of the nonlinear Dirichlet problem

(72) $$F(x,u,\dots,D^{2m}u) = f(x) \ , \quad x \in \Omega_h \ ,$$

(73) $$D^\alpha u = 0 \ , \quad x \in \partial\Omega_h \ , \quad |\alpha| \leq m-1 \ .$$

The function $F(x,\xi)$ is assumed to be defined on $\bar{\Omega}_1 \times R^{M(2m)}$, to have continuous derivatives up to the order ℓ and to satisfy the uniform ellipticity condition

(74) $$\sum_{|\alpha|=2m} F_\alpha(x,\xi)\eta^\alpha \geq \nu|\eta|^{2m}$$

for $\eta \in R^n$ with a positive constant ν .

We shall assume that $F(x,0) = 0$, the function $f(x)$ belongs to the space $W_2^{0,\ell,1}(\Omega_1) \cap C^{1,\lambda}(\bar{\Omega}_1)$ and

$$||f||_{W_2^{0,\ell,1}(\Omega_1)} + ||f||_{C^{1,\lambda}(\bar{\Omega}_1)} \leq R \ , \quad ||F(x,\xi)||_{C^\ell(\bar{\Omega}_1 \times B_t)} \leq g(t) \ ,$$

where $B_t = \{\xi \in M^{M(2m)} : |\xi| \leq t\}$ and $g(t)$ is a nondecreasing positive function.

We include the problem (72), (73) in the parametric family of problems of the same type

(75) $$tF(x,u,\dots,D^{2m}u) + (1-t)L_0u = tf(x) \ , \quad x \in \Omega_h \ ,$$

(76) $$D^\alpha u = 0 \ , \quad x \in \partial\Omega_h \ , \quad |\alpha| \leq m-1 \ ,$$

where $L_0(D) = \sum_{|\alpha|=2m} a_\alpha D^\alpha$ is a fixed elliptic operator with a constant of ellipticity ν .

<u>*THEOREM 14.*</u> *Let* $u(x)$ *be an arbitrary solution of the problem* (75), (76) *satisfying*

(77) $$||u(x)||_{C^{2m-1,2,\lambda}} \leq 1 \ .$$

Then there is a constant N *depending on* m , n , ℓ , ν , R , meas Ω_1 *and on the functions* $g(t)$, ϕ_i , ψ_i , *such that*

(78) $$||u(x)||_{W_2^{2m,\ell,1}(\Omega_h)} \leq N \ .$$

P r o o f of the theorem proceeds first by establishing an estimate $||u(x)||_{C^{2m}(\Omega_h)} \leq C$ with a constant C depending only on the known parameters, and then by proving the estimate (78) via Lemmas 8 - 10.

The proof of the existence theorem is based on the application of the methods from Chap. 1, with the choice of $X = W_2^{2m,\ell,1}(\Omega_h) \cap \cap \overset{o}{W}{}_2^m(\Omega_h)$, $D = \{u \in X : ||u||_{W_2^{2m,\ell,1}(\Omega_h)} \leq N+1\}$, where the number N is defined in Theorem 14. For $x \in \bigcup_i \cap \bar{\Omega}_h$ we pass from the variable x to the new variable y by the substitution $x = \phi_i^{-1}(y)$ and denote by F_i, L_i the differential operators resulting from F, L_0 by the above change of variables. Consider the family of linear operators of the type

$$L_t(y,D_y) = t \sum_{|\alpha| \leq 2m} F_{i,\alpha}(y,u_i,\ldots,D_y^{2m}u_i)D_y^\alpha + (1-t)L_i(y,D_y)$$

for $u(x) \in D$, $u_i(y) = u\left(\phi_i^{-1}(y)\right)$.

It is easily verified that the set of such operators is contained in the family $Z_{2m}^{1,\lambda}(A,B,S_h)$ for certain values of A, B and hence according to Lemma 10 the set considered can be associated with an operator $M(D)$ and constants ρ, q such that (71) holds. In what follows, the numbers ρ, q are assumed to be chosen in the just mentioned way.

We define a family of nonlinear operators $A_t : D \to X^*$ by the identity

$$(79) \quad \langle A_t u,v\rangle = \sum_{i=1}^{I} \int_{S_h} \psi_i^2(y)\{D_n^1[tF_i(y,u_i,\ldots,D_y^{2m}u_i)L_iu_i - tf_i(y)]D_n'Mv_i + $$
$$+ \rho \sum_{|\alpha| \leq \ell}{}' D^\alpha[tF_i(y,u_i,\ldots,D_y^{2m}u_i) + (1-t)L_iu_i - tf_i(y)]D^\alpha Mv_i\}dy .$$

LEMMA 11. The family of operators A_t defined by (79) satisfies the condition $(\alpha_0^{(t)})$.

The assertion follows from Lemma 10 and is verified analogously as the corresponding assertions of Chap. 2.

We choose a number $h_0 > 0$ so that

$$(80) \qquad k h_0^{\frac{1}{2}-\lambda}(N+1) \leq 1 ,$$

where the numbers k, N are defined in accordance with the inequalities (70), (78).

THEOREM 15. Let $0 < h \leq h_0$ *, where* h_0 *is defined by the inequality* (80). *Then the problem* (72), (73) *has at least one solution, which belongs to* $W_2^{2m,\ell,1}(\Omega_h)$.

P r o o f . By virtue of Theorem 5, in order to prove Theorem 15 it suffices to verify that

(a) the equation $A_t u = 0$ has no solution $u \in \partial D$;

(b) $\mathrm{Deg}(A_0, \bar{D}, 0) \neq 0$.

We shall verify (a) by contradiction. Let $u \in \partial D$ and $A_t u = 0$. Then u is a solution of the problem (75), (76). Since $||u||_{W_2^{2m,\ell,1}(\Omega_h)} = N+1$, the function $u(x)$ in virtue of Lemma 8 and the inequality (80) satisfies the inequality (77) and consequently, by Theorem 14, the inequality (78) as well. The last inequality contradicts the condition $u \in \partial D$.

The validity of (b) follows, for instance, from Theorem 3. Thus the assertions (a), (b) and hence Theorem 15 are proved.

3. In conclusion we will present one result concerning the solvability of the model nonlinear Neumann problem. The result was obtain by A. E. Šiškov [20]. Consider the problem

$$(81)\qquad Lu \equiv \left(1 + \left|\frac{\partial u}{\partial x}\right|^2\right)\frac{\partial^2 u}{\partial x^2} + 2\frac{\partial u}{\partial x}\frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial x \partial y} + \left(1 + \left(\frac{\partial u}{\partial y}\right)^2\right)\frac{\partial^2 u}{\partial y^2} -$$

$$- a\left(u, \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2\right) - f(x,y) = 0 , \quad (x,y) \in \Omega ,$$

$$(82)\qquad Bu \equiv \frac{\partial u}{\partial n} \exp\left[\frac{1}{2}\left(1 + \left|\frac{\partial u}{\partial x}\right|^2 + \left|\frac{\partial u}{\partial y}\right|^2\right)\right] - b(x,y,u) = 0 ,$$

$$(x,y) \in \partial\Omega .$$

Here Ω is a circle $x^2+y^2 \leq R^2$, n the interior normal to $\partial\Omega$, $a(u,t)$, $f(x,y)$, $b(x,y,u)$ functions of the class $C^{1,\alpha}$ with respect to all their arguments.

The interest in problems of the type (81), (82) is caused by the fact that in the general case of non-divergent equations, for such problems no apriori estimates of solution are known, and hence no general theory has been developed for such problems. In particular, as concerns the very problem (81), (82), the condition of uniform ellipticity is not valid and, secondly, when attempting to reduce the problem to the divergent form, we obtain equations not covered by the theory developed in [3].

Let us include the problem (81), (82) in the parametric family of problems of the same type

$$(83) \qquad t(\Delta u - C_1 u) + (1-t)Lu = 0 \ ,$$

$$(84) \qquad t\left[\frac{\partial u}{\partial n} - C_1 u\right] + (1-t)Bu\Big|_{\partial\Omega} = 0 \ ,$$

where C_1 is a positive constant, Δ the Laplace operator, $t \in$ $\in [0,1]$.

Apriori estimates for the problem (83), (84) are given in the following lemma.

LEMMA 12. Let there exist positive constants C *and* R , R_1 *such that*

(a) $[a(u,0,0) + f(x,y)]u > 0$ for $|u| > R$;

(b) $\dfrac{\partial b(x,y,u)}{\partial u} \geqq C$;

(c) $\dfrac{\partial a(u,t)}{\partial u} \geqq C$ *for* $t > R_1$.

Then every solution $u(x,y)$ *of the problem* (83), (84) *satisfies the estimate*

$$||u||_{C^1(\Omega)} \leqq M$$

with a constant M *depending only on the known parameters.*

The estimate of $u(x,y)$ in C^1 enables us to obtain (by the methods from [3]) an apriori estimate of the solution in $C^{2,\alpha}(\Omega)$ and, applying the general scheme from Sec. 2, we easily obtain the existence theorem.

THEOREM 16. For any $p > n$, *the problem* (81), (82) *has at least one solution belonging to* $W_p^2(\Omega)$, *provided the assumptions of Lemma 12 are fulfilled.*

5. Topological characteristics of general nonlinear parabolic boundary value problems

We shall only sketch the method of reducing general nonlinear parabolic problems to operator equations satisfying the condition (α_0). We will consider only equations solved with respect to the time derivative, even if this is no essential restriction.

In what follows we shall deal with the problem

(85) $$\frac{\partial u}{\partial t} - F(x,t,u,\dots,D_x^{2m}u) = f(x,t) \ , \quad (x,t) \in Q = \Omega \times (0,T_0) \ ,$$

(86) $$G_j(x,t,u,\dots,D_x^{m_j}u) = g_j(x,t) \ , \quad j = 1,\dots,m \ , \quad (x,t) \in S = \partial\Omega \times (0,T_0) \ ,$$

(87) $$u\big|_{t=0} = 0 \ , \quad x \in \Omega \ ,$$

where Ω is a bounded domain in R^n with an infinitely differentiable boundary $\partial\Omega$. Let $\ell_0 = \max\{2m,m_1,\dots,m_m\}$. The methods developed above enable us to investigate the solvability of the problem (85) - (87) in the space $W_p^{2mk,k}(Q)$ with $(2mk-\ell_0)p > n+2m$ (the definition of this space is given e.g. in [21]). We restrict our considerations in this chapter to the Hilbert case, that is, $p = 2$, as being the simplest possible. Analogously to Chap. 2 we can introduce operators corresponding to the problem (85) - (87) with an arbitrary p , which in particular enables us to weaken the assumptions concerning the smoothness of the functions F , G_j and the order of the compatibility.

We introduce the following assumptions:

(i) The functions $F(x,\xi)$, $G_j(y,t,\eta)$, $j = 1,\dots,m$, are defined for $(x,t) \in \overline{Q}$, $(y,t) \in S$, $\xi \in R^{M(2m)}$, $\eta \in R^{M(m_j)}$ and belong to the classes $C^{\ell-2m+1}$, $C^{\ell-m_j+1}$, respectively, with some k and $2m\ell > \ell_0+m+\frac{n}{2}$. The functions $f(x,t)$, $g(y,t)$ belong respectively to the spaces $W_2^{2m(\ell-1),\ell-1}(Q)$, $W_2^{2m\ell-m_j-\frac{1}{2},(1/2m)\cdot(2m\ell-m_j-\frac{1}{2})}(S)$.

(ii) For any function $v \in W_2^{2m\ell,\ell}(Q)$ the operator

(88) $$\Lambda(v) = \frac{\partial}{\partial t} - \sum_{|\alpha|\leq 2m} F_\alpha(x,t,v,\dots,D_x^{2m}v)D^\alpha$$

is parabolic; $\Lambda(v)$ and the boundary value operators

(89) $$\Gamma_j(v) \doteq \sum_{|\beta|\leq m_j} G_{j,\beta}(x,t,v,\dots,D_x^{m_j}v)D^\beta \ , \quad j = 1,\dots,m$$

satisfy the "complementarity condition" (see e.g. [21], §9, Chap. VII), which represents an analogue of the Ya. B. Lopatinskiĭ condition for the parabolic case.

(iii) For $x \in \overline{\Omega}$, $y \in \partial\Omega$ the conditions

(90) $$F(x,t,0,\dots,0) = 0 \ , \quad G_j(y,t,0,\dots,0) = 0 \ , \quad t \in [0,T] \ ,$$

$$(91) \qquad \frac{\partial^i}{\partial t^i} f(x,t)\Big|_{t=0} = 0 \ , \quad \frac{\partial^i}{\partial t^i} g_j(y,t) = 0 \ , \quad i \leqq \ell-1$$

are satisfied.

The conditions (90), (91) guarantee the compatibility of the data of the problem (85) - (87). If (90), (91) are fulfilled, then the solvability of the problem (85) - (87) can be considered in the space

$$\overset{\circ}{W}{}_2^{2m\ell,\ell}(Q) = \left\{u \in W_2^{2m\ell,\ell}(Q) : \frac{\partial^i u}{\partial t^i}\Big|_{t=0} = 0 \ , \quad i \leqq \ell-1\right\} .$$

Let us introduce the operator $A : \overset{\circ}{W}{}_2^{2m\ell,\ell}(Q) \to [\overset{\circ}{W}{}_2^{2m\ell,\ell}(Q)]^*$ by the identity

$$(92) \qquad \langle Au,\phi\rangle = \left(\frac{\partial u}{\partial t} - F(x,t,u,\dots,D_x^{2m}u) - f(x,t),\Lambda(u)\phi\right)_{2m(\ell-1),Q} +$$
$$+ \sum_{j=1}^{m} \left(G_j(x,t,u,\dots,D_x^{m_j}u) - g_j(x,t),\Gamma_j(u)\phi\right)_{2m\ell-m_j-\frac{1}{2},S} ,$$

where the symbols $(.,.)_{\ell,Q}$ and $(.,.)_{\ell,S}$ denote the scalar products in the spaces $W_2^{\ell,\frac{\ell}{2m}}(Q)$, $W_2^{\ell,\frac{\ell}{2m}}(S)$, respectively, while the operators $\Lambda(u)$, $\Gamma_j(u)$ are defined by the identities (88), (89).

Employing the apriori estimates of solutions of linear parabolic problems [21] we establish

<u>*THEOREM 17.*</u> *Assume that the conditions* (i) - (iii) *are fulfilled and that the numbers* $\frac{1}{2m}(2m\ell-m_j-\frac{1}{2}) - \frac{1}{2}$ *are nonintegers. Then the operator* A *defined by* (92) *is continuous, bounded and satisfies the condition* $(S)_+$.

From the result of Chap. 1 and from Theorem 17 it follows that the degree $\mathrm{Deg}(A,\overline{D},0)$ is determined for an arbitrary bounded domain D in $\overset{\circ}{W}{}_2^{2m\ell,\ell}(Q)$ such that there are no solutions of the problem (85) - (87) on ∂D . By applying the theorems on solvability of operator equations given in Chap. 1 it is possible to obtain an analogous assertion for the problem (85) - (87). In particular, it is easy to formulate the existence theorem for the problem (85) - (87) provided a certain apriori estimate is available.

Let us mention only one of the results that holds, generally speaking, only in the parabolic case, and that follows from Theorem

I.7 in [10] on the invariance of the domain for locally one-to-one mappings satisfying the condition (α_0).

THEOREM 18. *The set of* $(f, g_1, \ldots, g_m)$ *for which the problem* (85) - (87) *has a solution under the assumptions of Theorem 17, is open in*

$$\overset{\circ}{W}{}_2^{2m(\ell-1),\ell-1}(Q) \times \prod_{j=1}^{m} \overset{\circ}{W}{}_2^{2m\ell-m_j-\frac{1}{2},\ell-\frac{1}{2m}(m_j+\frac{1}{2})}(S) .$$

For the Dirichlet boundary value problem the construction of the operator A can be simplified analogously to the elliptic case from Chap. 3. We restrict ourselves to the case of homogeneous boundary value problems

$$(93) \qquad D_x^\alpha u(x,t) = 0 , \quad |\alpha| \le m-1 , \quad (x,t) \in S .$$

We denote by X the subspace of the space $\overset{\circ}{W}{}_2^{2m\ell,\ell}(Q)$ formed by the functions satisfying the condition (93).

THEOREM 19. *Assume that a function* $F(x,t,\xi)$ *satisfies the conditions that follow from* (i) - (iii) *for an integer* ℓ *such that* $2m\ell > \frac{n}{2} + 1$. *Let* $f(x,t)$ *satisfy the conditions* (91) *and let* $\frac{1}{2m}(2m\ell-m_j-\frac{1}{2}) - \frac{1}{2}$ *be noninteger. Then for an arbitrary bounded domain* D *in* X *there exists a linear parabolic operator*

$$P = \frac{\partial}{\partial t} - \sum_{|\alpha|\le 2m} m_\alpha(x,t)D^\alpha , \quad (x,t) \in \overline{Q}$$

with infinitely differentiable coefficients, such that the operator $A : D \to X^*$ *given by*

$$\langle Au,\phi\rangle = \left[\frac{\partial u}{\partial t} - F(x,t,u,\ldots,D_x^{2m}u) - f(x,t), P\phi\right]_{2m(\ell-1),Q}$$

is continuous, bounded and satisfies the condition $(S)_+$. *Here* $[\cdot,\cdot]_{2m(1-\ell),Q}$ *is a certain inner product in* $W_2^{2m(\ell-1),\ell-1}(Q)$.

The existence of the operator P is guaranteed by the results of the author and A. E. Šiškov. An explicit formula can be given for the operator P as well as for the inner product $[\cdot,\cdot]_{2m(\ell-1),Q}$.

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