Vladimir G. Maz'ya

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THEORY OF MULTIPLIERS IN SPACES OF DIFFERENTIABLE FUNCTIONS AND ITS APPLICATIONS

V. G. Maz'ya
Leningrad, USSR

"... and then the different branches of Arithmetic - Ambition, Distraction, Uglification, and Derision."
"I never heard of 'Uglification'", Alice ventured to say. "What is it?"
Lewis Carrol, "Alice's Adventures in Wonderland"

By a multiplier, acting from a functional space S_1 into another one, S_2, we mean a function which defines a linear mapping S_1 into S_2 by pointwise multiplication. Thus, with the pair of spaces S_1, S_2 we associate a third one - the space of multipliers $M(S_1 \rightarrow S_2)$*).

Multipliers appear in various problems of analysis and theory of differential and integral equations. Their usefulness can be illustrated, for example, by the following most simple observation: The Schrödinger operator

$$\Delta + \gamma(x) I : W_p^m \rightarrow W_p^{m-2}$$

is bounded if and only if $\gamma \in M(W_p^m \rightarrow W_p^{m-2})$.

In this way, it is reasonable to consider the coefficients of differential operators as multipliers. The same concerns the symbols of pseudodifferential operators. Multipliers also appear in the theory of differentiable mappings preserving the Sobolev spaces. Solutions of boundary value problems can be sought in classes of multipliers. Because of their algebraic properties, multipliers are suitable objects for a generalization of the basic facts of the calculus (theorems on superposition, on implicit functions etc.).

The aim of the present lectures is to give a survey of the theory of multipliers in pairs of Sobolev, Slobodeckiĭ, Bessel potential,

*) Since a multiplier cannot "beautify" S_1 (modulo annulling its elements on a set), the Mock Turtle's term "uglifier" is not quite senseless, either.

spaces etc. *). Regardless of the substantiality and numerous applications of this theory, it attracted relatively little attention until lately. Among first papers concerning our subject, let us mention the one due to Devinatz and Hirschman [1], 1959, about the spectrum of the operator of multiplication in the space W_2^ℓ, $2|\ell| < 1$, on the unit circumference, two papers by Hirschman [2], 1961, and [3], 1962, which also deal with multipliers in W_2^ℓ and finally, a study of multipliers in the space of Bessel potentials due to Strichartz [4], 1967.

The lectures mainly include results of the author and Mrs. T. O. Shaposhnikova obtained in the years 1979 - 1980 (see [5] - [13]) and compiled in the monograph "Multipliers in spaces of differentiable functions", which is just being printed.

For lack of space, we restrict our exposition to the formulation of results. The only exception is Section 1.1, which includes proofs. The contents of our lectures is as follows:

1. Description of spaces of multipliers
    - 1.1. Multipliers in pairs of Sobolev spaces
    - 1.2. Multipliers in pairs of Bessel potential spaces
    - 1.3. Multipliers in pairs of Slobodeckiĭ spaces
2. Some properties of multipliers
    - 2.1. On the spectrum of a multiplier in H_p^ℓ
    - 2.2. On functions of multipliers
    - 2.3. The essential norm in $M(W_p^m \to W_p^\ell)$
    - 2.4. Completely continuous multipliers
    - 2.5. Traces and extensions of multipliers in W_p^ℓ
3. Multipliers in a pair of Sobolev spaces in a domain
4. Applications of multipliers
    - 4.1. Convolution operator in a pair of weighted spaces L_2
    - 4.2. Singular integral operators with symbols from spaces of multipliers
    - 4.3. On the norm and the essential norm of a differential operator
    - 4.4. Coercive estimates of solutions of elliptic boundary value problems in spaces of multipliers
    - 4.5. Implicit Function Theorems
    - 4.6. On (p,ℓ)-diffeomorphisms
    - 4.7. On regularity of the boundary in the L_p-theory of elliptic

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*) At the same time, we omit the L_p-theory of Fourier multipliers.

boundary value problems

References

1. Description of spaces of multipliers

1.1. Multipliers in pairs of Sobolev spaces

We start with studying the spaces $M(W_p^m \to W_p^\ell)$, where W_p^k is the Sobolev space in R^n, i.e. the completion of C_0^∞ with respect to the norm $||\nabla_k u||_{L_p} + ||u||_{L_p}$ *).

Let $\gamma \in M(W_p^m \to W_p^\ell)$, $u_n \to u$ in W_p^m and $\gamma u_n \to v$ in W_p^ℓ. Then there exists a sequence of positive integers $\{n_k\}_{k\geq 1}$ such that

$$u_{n_k}(x) \to u(x) \ , \quad \gamma(x)u_{n_k}(x) \to v(x)$$

almost everywhere. Consequently, $v = \gamma u$ almost everywhere in R^n and the operator $W_p^m \ni u \to \gamma u \in W_p^\ell$ is closed. Since it is defined on the whole W_p^m, Closed Graph Theorem implies that it is bounded.

As the norm in the space $M(W_p^m \to W_p^\ell)$ we introduce the norm of the operator of multiplication:

$$||\gamma||_{M(W_p^m \to W_p^\ell)} = \sup\{||\gamma u||_{W_p^\ell} : ||u||_{W_p^m} \leq 1\} .$$

We shall write briefly MW_p^m instead of $M(W_p^m \to W_p^m)$.

By $W_{p,loc}^\ell$ we denote the space

$$\{u: u\eta \in W_p^\ell \text{ for all } \eta \in C_0^\infty\} .$$

Evidently, $M(W_p^m \to W_p^\ell) \subset W_{p,loc}^\ell$.

In what follows,

$$Q_\rho(x) = \{y \in R^n : |y-x| < \rho\} \ , \quad Q_\rho = Q_\rho(0) .$$

We introduce the space

$$W_{p,unif}^\ell = \{u : \sup_{z \in R^n} ||\eta_z u||_{W_p^\ell} < \infty\} ,$$

with $\eta_z(x) = \eta(x-z)$, $\eta \in C_0^\infty$, $\eta = 1$ on Q_1. Let

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*) If no domain is indicated in the symbol for the space, then the domain is understood to be R^n.

$$||u||_{W^{\ell}_{p,unif}} = \sup_{z \in R^n} ||\eta_z u||_{W^{\ell}_p} .$$

In precisely the same way as above we introduce the spaces S_{loc} and S_{unif} for any other functional space S which may appear in the forthcoming considerations.

It is evident that the norm in $W^{\ell}_{p,unif}$ is equivalent to the norm

$$\sup_{x \in R^n} ||\gamma;\ Q_1(x)||_{W^{\ell}_p} .$$

Let us present some auxiliary assertions, which serve as a base for the proof of a theorem on necessary and sufficient conditions for a function to belong to the space $M(W^m_p \to W^{\ell}_p)$, $p > 1$.

In the next lemma, the symbol $cap(e,W^m_p)$ stands for the capacity of a compact $e \subset R^n$ induced by the norm of the space W^m_p , that is,

$$cap(e,W^m_p) = \inf\{||u||^p_{W^m_p} : u \in C^{\infty}_0 ,\quad u \geq 1 \text{ on } e\} .$$

Replacing here W^m_p by any other functional space S which includes C^{∞}_0 , we obtain the definition of the capacity $cap(e,S)$. There is a number of papers devoted to the study and applications of such set functions (see, e.g., [14] - [16] and others).

<u>*LEMMA 1.1*</u>. *Let* $p \in (1,+\infty)$, $m = 1,2,...$ *and let* μ *be a measure in* R^n . *Then the exact constant in the inequality*

$$(1.1) \qquad \int |u|^p d\mu \leq C||u||^p_{W^m_p} , \quad u \in C^{\infty}_0 ,$$

is equivalent to the quantity

$$\sup_e \frac{\mu(e)}{cap(e,W^m_p)} ,$$

where e *is an arbitrary compact with a positive capacity* $cap(e,W^m_p)$.

For $p = 2$, $m = 1$ this lemma was established by the author in [17], 1962.

The p r o o f of Lemma 1.1 is based on the following property of the norm in W^m_p :

$$(1.2) \qquad \int_0^{\infty} cap(N_t,W^m_p)d(t^p) \leq c||u||^p_{W^m_p} ,$$

where $N_t = \{x : |u(x)| \geq t\}$. The validity of inequalities of the type (1.2) was established in the author's paper [18], where (1.2) (and even a stronger inequality, in which the role of the capacity of the set N_t was played by the capacity of a condenser $N_t \setminus N_{2t}$) was obtained only for $m = 1$ and $m = 2$. Later, Adams [19] proved (1.2) for all integers m . Inequalities analogous to (1.2) were obtained for Slobodeckiĭ and Besov spaces and for spaces of potentials (see [20] - [22]).

The estimate (1.2) being established, Lemma 1.1 can be proved very easily. By definition of Lebesgue integral we have the identity

$$\int |u|^p d\mu = \int_0^\infty \mu(N_t) d(t^p) .$$

Hence

$$\int |u|^p d\mu \leq \sup_e \frac{\mu(e)}{\operatorname{cap}(e, W_p^m)} \int_0^\infty \operatorname{cap}(N_t; W_p^m) d(t^p) ,$$

which together with (1.2) implies the desired upper bound for C .

Minimizing the right-hand side of the inequality (1.1) on the set $\{u \in C_0^\infty : u \geq 1 \text{ on } e\}$, we obtain

$$C \geq \sup_e \frac{\mu(e)}{\operatorname{cap}(e, W_p^m)} .$$

<u>LEMMA 1.2</u>. [5] *The exact constants* C_0 , C *in the inequalities*

$$\int (|\nabla_\ell u|^p + |u|^p) d\mu \leq C_0 ||u||^p_{W_p^m} ,$$

(1.3)

$$\int |u|^p d\mu \leq C ||u||^p_{W_p^{m-\ell}} ,$$

where $m > \ell$, $u \in C_0^\infty$, *are equivalent.*

P r o o f . The estimate $C_0 \leq cC$ is evident. Let us prove a converse estimate. Let $x \to \sigma$ be a smooth positive function on the half-axis $[0, \infty)$ that equals x for $x > 1$. An arbitrary function $u \in C_0^\infty$ can be written in the form

$$u = (-\Delta)^\ell [\sigma(-\Delta)]^{-\ell} u + T(-\Delta) ,$$

where T is a function from $C_0^\infty[0, \infty)$. As

$$(-\Delta)^\ell = (-1)^\ell \sum_{|\alpha| = \ell} \frac{\ell!}{\alpha!} D^{2\alpha} ,$$

we have

$$\int |u|^p d\mu \leq cC_0 (||\nabla_\ell [\sigma(-\Delta)]^{-\ell} u||^p_{W_p^m} + ||Tu||^p_{W_p^m}) .$$

By Michlin's theorem [23] on multipliers of Fourier transform in L_p, the right-hand side cannot exceed $c_1 C_0 ||u||^p_{W_p^{m-\ell}}$. The proof is complete.

This lemma implies

COROLLARY 1.1. *Let* $\gamma \in L_{p,loc}$, $p \in (1,\infty)$ *and let* u *be an arbitrary function from* C_0^∞ .

The exact constant C *in the inequality*

$$||\gamma \nabla_\ell u||_{L_p} + ||\gamma u||_{L_p} \leq C||u||_{W_p^m}$$

is equivalent to the norm $||\gamma||_{M(W_p^{m-\ell} \to L_p)}$ *).

In the following lemma we denote by γ_h the mollification of γ with radius h , that is,

$$\gamma_h(x) = h^{-n}\int K(\tfrac{x-\xi}{h})\gamma(\xi)d\xi ,$$

where $K \in C_0^\infty$, $K \geq 0$ and $||K||_{L_1} = 1$.

LEMMA 1.3. *The following estimate hold :*

$$(1.4) \qquad ||\gamma_h||_{M(W_p^m \to W_p^\ell)} \leq ||\gamma||_{M(W_p^m \to W_p^\ell)} \leq \varliminf_{h\to 0} ||\gamma_h||_{M(W_p^m \to W_p^\ell)} .$$

P r o o f . Let $u \in C_0^\infty$. Minkowski inequality yields

$$||\nabla_{j,x}\int h^{-n}K(\xi/h)\gamma(x-\xi)u(x)d\xi||_{L_p} \leq$$

$$\leq \int h^{-n}K(\xi/h)\left(\int |\nabla_{j,y}[\gamma(y)u(y-\xi)]|^p dy\right)^{1/p} d\xi ,$$

with $j = 0,\ell$. Hence

$$||\gamma_h u||_{W_p^\ell} \leq ||\gamma||_{M(W_p^m \to W_p^\ell)} \int h^{-n}K(\xi/h)\left[\left(\int |\nabla_{m,y}\, u(y-\xi)|^p dy\right)^{1/p} +\right.$$

*) Two quantities a , b are said to be equivalent (notation: $a \sim b$), if their ratio is bounded and separated from zero by positive constants.

$$+ \left(\int |u(y-\xi)|^p dy\right)^{1/p}\Big] d\xi \leq ||\gamma||_{M(W_p^m \to W_p^\ell)} ||u||_{W_p^m} .$$

This implies the left-hand inequality in (1.4). The right-hand inequality in (1.4) follows from the relation

$$||\gamma u||_{W_p^\ell} = \underline{\lim}_{h\to 0} ||\gamma_h u||_{W_p^\ell} \leq \underline{\lim}_{h\to 0} ||\gamma_h||_{M(W_p^m \to W_p^\ell)} ||u||_{W_p^m} .$$

<u>*LEMMA 1.4*</u>. *If* $\gamma \in M(W_p^m \to W_p^\ell) \cap M(W_p^{m-\ell} \to L_p)$, $p \in (1,\infty)$, *then* $D^\alpha\gamma \in M(W_p^m \to W_p^{\ell-|\alpha|})$ *for any multiindex* α *with a positive order* $|\alpha| \leq \ell$. *We have the estimate*

$$(1.5)\qquad \begin{aligned} &||D^\alpha\gamma||_{M(W_p^m \to W_p^{\ell-|\alpha|})} \leq \\ &\leq \varepsilon||\gamma||_{M(W_p^{m-\ell} \to L_p)} + c(\varepsilon)||\gamma||_{M(W_p^m \to W_p^\ell)} , \end{aligned}$$

where ε *is an arbitrary positive number.*

P r o o f . Using the identity

$$uD^\alpha\gamma = \sum_{\alpha\geq\beta>0} c_{\alpha\beta} D^\beta(\gamma D^{\alpha-\beta}u) ,$$

with $c_{\alpha\beta}$ constants, which is easily verified by induction, we obtain

$$||uD^\alpha\gamma||_{W_p^{\ell-|\alpha|}} \leq c \sum_{\alpha\geq\beta>0} ||\gamma D^{\alpha-\beta}u||_{W_p^{\ell-|\alpha|+|\beta|}} .$$

Consequently, it suffices to prove (1.5) for $|\alpha| = 1$, $\ell \geq 1$. We have

$$(1.6)\qquad \begin{aligned} &||u\nabla\gamma||_{W_p^{\ell-1}} \leq ||u\gamma||_{W_p^\ell} + ||\gamma\nabla u||_{W_p^{\ell-1}} \leq \\ &\leq (||\gamma||_{M(W_p^m \to W_p^\ell)} + ||\gamma||_{M(W_p^{m-1} \to W_p^{\ell-1})})||u||_{W_p^m} . \end{aligned}$$

The interpolation property of the Sobolev space (see [24],[25]) implies the inequality

$$(1.7)\qquad ||\gamma||_{M(W_p^{m-j} \to W_p^{\ell-j})} \leq c||\gamma||^{(\ell-j)/\ell}_{M(W_p^m \to W_p^\ell)} ||\gamma||^{j/\ell}_{M(W_p^{m-\ell} \to L_p)} .$$

Estimating the norm $||\gamma||_{M(W_p^{m-1} \to W_p^{\ell-1})}$ in (1.6) by means of the last inequality we arrive at (1.5).

Now we are able to establish both-sided estimates for the norm in $M(W_p^m \to W_p^\ell)$, formulated in terms of the spaces $M(W_p^k \to L_p)$. Let us start with the lower bound.

LEMMA 1.5. *Let* $\gamma \in M(W_p^m \to W_p^\ell)$. *Then*

$$\|\nabla_\ell \gamma\|_{M(W_p^m \to L_p)} + \|\gamma\|_{M(W_p^{m-\ell} \to L_p)} \leq c\|\gamma\|_{M(W_p^m \to W_p^\ell)} . \tag{1.8}$$

P r o o f . First, let us assume that $\gamma \in M(W_p^{m-\ell} \to L_p)$. It is evident that

$$\begin{aligned}\|\gamma\nabla_\ell u\|_{L_p} &\leq \|\gamma\|_{M(W_p^m \to W_p^\ell)}\|u\|_{W_p^m} + c \sum_{\substack{|\alpha|+|\beta|=\ell \\ \beta\neq 0}} \|D^\alpha u D^\beta \gamma\|_{L_p} \leq \\ &\leq \Big(\|\gamma\|_{M(W_p^m \to W_p^\ell)} + c\sum_{j=1}^{\ell} \|\nabla_j\gamma\|_{M(W_p^{m-\ell+j} \to L_p)}\Big)\|u\|_{W_p^m} .\end{aligned} \tag{1.9}$$

By virtue of Lemma 1.4,

$$\|\nabla_j\gamma\|_{M(W_p^{m-\ell+j} \to L_p)} \leq \varepsilon\|\gamma\|_{M(W_p^{m-\ell} \to L_p)} + c(\varepsilon)\|\gamma\|_{M(W_p^{m-\ell+j} \to W_p^j)} .$$

Using the inequality (1.7) for estimating the last norm on the right-hand side, we conclude that

$$\|\nabla_j\gamma\|_{M(W_p^{m-\ell+j} \to L_p)} \leq \varepsilon\|\gamma\|_{M(W_p^{m-\ell} \to L_p)} + c(\varepsilon)\|\gamma\|_{M(W_p^m \to W_p^\ell)} .$$

We substitute this inequality in (1.9). Then

$$\|\gamma\nabla_\ell u\|_{L_p} \leq \Big(\varepsilon\|\gamma\|_{M(W_p^{m-\ell} \to L_p)} + c(\varepsilon)\|\gamma\|_{M(W_p^m \to W_p^\ell)}\Big)\|u\|_{W_p^m} . \tag{1.10}$$

At the same time,

$$\|\gamma u\|_{L_p} \leq \|\gamma\|_{M(W_p^m \to W_p^\ell)}\|u\|_{W_p^m} . \tag{1.11}$$

Adding (1.10), (1.11) and using Corollary 1.1, we find the estimate

$$\|\gamma\|_{M(W_p^{m-\ell} \to L_p)} \leq \varepsilon\|\gamma\|_{M(W_p^{m-\ell} \to L_p)} + c(\varepsilon)\|\gamma\|_{M(W_p^m \to W_p^\ell)} .$$

Consequently,

(1.12) $$||\gamma||_{M(W_p^{m-\ell} \to L_p)} \leq c||\gamma||_{M(W_p^m \to W_p^\ell)} .$$

It remains to dispose of the assumption $\gamma \in M(W_p^{m-\ell} \to L_p)$. Since $\gamma \in M(W_p^m \to W_p^\ell)$, we have $||\gamma\eta||_{L_p} \leq c||\eta||_{W_p^m}$, where $\eta \in C_0^\infty(Q_2(x))$, $\eta = 1$ on $Q_1(x)$, x being an arbitrary point in R^n. Consequently,

$$\sup_x ||\gamma;\ Q_1(x)||_{L_p} < \infty$$

and for every $k = 0,1,\ldots$ there exists such a constant c_h that $|\nabla_k\gamma_h| \leq c_h$ holds. As the function γ_h is bounded together with all its derivatives, we conclude that γ_h is a multiplier in W_p^k for every $k = 1,2,\ldots$ and, a fortiori, $\gamma_h \in M(W_p^{m-\ell} \to L_p)$. Hence

$$||\gamma_h||_{M(W_p^{m-\ell} \to L_p)} \leq c||\gamma_h||_{M(W_p^m \to W_p^\ell)} .$$

Lemma 1.3 makes it possible to pass here to the limit as $h \to 0$, and we obtain (1.12) for all $\gamma \in M(W_p^m \to W_p^\ell)$.

Let us estimate the first summand on the left-hand side of (1.8). We have

$$||u\nabla_\ell\gamma||_{L_p} \leq ||\gamma||_{M(W_p^m \to W_p^\ell)}||u||_{W_p^m} + c\sum_{\substack{|\alpha|+|\beta|=\ell \\ \alpha\neq 0}}||D^\alpha u D^\beta\gamma||_{L_p} \leq$$

$$\leq \Big(||\gamma||_{M(W_p^m \to W_p^\ell)} + c\sum_{j=0}^{\ell-1}||\nabla_j\gamma||_{M(W_p^{m-\ell+j} \to L_p)}\Big)||u||_{W_p^m} ,$$

which together with Lemma 1.4 and the inequality (1.12) yields

$$||u\nabla_\ell\gamma||_{L_p} \leq c\Big(||\gamma||_{M(W_p^m \to W_p^\ell)} + ||\gamma||_{M(W_p^{m-\ell} \to L_p)}\Big)||u||_{W_p^m} \leq$$

$$\leq c\ ||\gamma||_{M(W_p^m \to W_p^\ell)}||u||_{W_p^m} .$$

This immediately provides the estimate

$$||\nabla_\ell\gamma||_{M(W_p^m \to L_p)} \leq c||\gamma||_{M(W_p^m \to W_p^\ell)} .$$

Our lemma is proved.

The following lemma represents the conversion of our last result.

LEMMA 1.6. The following inequality holds:

$$\|\gamma\|_{M(W_p^m \to W_p^\ell)} \le c\big(\|\nabla_\ell \gamma\|_{M(W_p^m \to L_p)} + \|\gamma\|_{M(W_p^{m-\ell} \to L_p)}\big) . \tag{1.13}$$

P r o o f . It is sufficient to assume that the right-hand side of the inequality (1.13) is finite.

Lemma 1.5 together with the inequality (1.17) imply the estimate

$$\|\nabla_j \gamma\|_{M(W_p^{m-\ell+j} \to L_p)} \le c\|\gamma\|^{j/\ell}_{M(W_p^m \to W_p^\ell)} \|\gamma\|^{1-j/\ell}_{M(W_p^{m-\ell} \to L_p)} , \quad j = 1,\dots,\ell-1 . \tag{1.14}$$

Let $u \in C_0^\infty$. We have

$$\|\nabla_\ell(\gamma u)\|_{L_p} \le c \sum_{j=0}^{\ell} \| \, |\nabla_j \gamma| |\nabla_{\ell-j} u| \|_{L_p} \le c\big(\|\nabla_\ell \gamma\|_{M(W_p^m \to L_p)} + \|\gamma\|_{M(W_p^{m-\ell} \to L_p)} + \sum_{j=1}^{\ell-1} \|\nabla_j \gamma\|_{M(W_p^{m-\ell+j} \to L_p)}\big)\|u\|_{W_p^m} .$$

Hence and from (1.14) we obtain

$$\|\nabla_\ell(\gamma u)\|_{L_p} \le c\big(\|\nabla_\ell \gamma\|_{M(W_p^m \to L_p)} + \|\gamma\|_{M(W_p^{m-\ell} \to L_p)}\big)\|u\|_{W_p^m} .$$

Now we only have to observe that

$$\|\gamma u\|_{L_p} \le \|\gamma\|_{M(W_p^{m-\ell} \to L_p)} \|u\|_{W_p^{m-\ell}} .$$

Lemma 1.6 is proved.

Combining the formulations of Lemmas 1.5, 1.6, we obtain a result, which was established in [5].

THEOREM 1.1. Let m , ℓ *be integers,* $p \in (1,\infty)$. *A function* γ *belongs to the space* $M(W_p^m \to W_p^\ell)$ *if and only if* $\gamma \in W^\ell_{p,\mathrm{loc}}$, $\nabla_\ell \gamma \in M(W_p^m \to L_p)$ *and* $\gamma \in M(W_p^{m-\ell} \to L_p)$.

Moreover, we have the relation

$$\|\gamma\|_{M(W_p^m \to W_p^\ell)} \sim \|\nabla_\ell \gamma\|_{M(W_p^m \to L_p)} + \|\gamma\|_{M(W_p^{m-\ell} \to L_p)} .$$

It is apparent that the problem of describing the space $M(W_p^m \to L_p)$,

$p > 1$, is solved by Lemma 1.1. In particular,

$$||\gamma||_{M(W_p^m \to L_p)} \sim \sup_e \frac{||\gamma;e||_{L_p}}{[\mathrm{cap}(e,W_p^m)]^{1/p}} .$$

This relation enables us to transcribe Theorem 1.1 in another form.

<u>*THEOREM 1.2*</u>. [5] *A function* γ *belongs to the space* $M(W_p^m \to W_p^\ell)$, $p \in (1,\infty)$, *if and only if* $\gamma \in W_{p,\mathrm{loc}}^\ell$ *and any compact* $e \subset R^n$ *satisfies*

$$||\nabla_\ell \gamma;e||_{L_p}^p \leq c\ \mathrm{cap}(e,W_p^m) ,$$

$$||\gamma;e||_{L_p}^p \leq c\ \mathrm{cap}(e,W_p^{m-\ell}) .$$

Moreover, the following relation holds:

$$(1.15) \qquad ||\gamma||_{M(W_p^m \to W_p^\ell)} \sim \sup_e \left(\frac{||\nabla_\ell \gamma;e||_{L_p}}{[\mathrm{cap}(e,W_p^m)]^{1/p}} + \frac{||\gamma;e||_{L_p}}{[\mathrm{cap}(e,W_p^{m-\ell})]^{1/p}} \right) .$$

Let us point out an important special case of Theorem 1.2 with $m=\ell$.

<u>*COROLLARY 1.2*</u>. *A function* γ *belongs to the space* MW_p^ℓ , $p \in (1,\infty)$, *if and only if* $\gamma \in W_{p,\mathrm{loc}}^\ell$ *and any compact* $e \subset R^n$ *satisfies*

$$||\nabla_\ell \gamma;e||_{L_p}^p \leq c\ \mathrm{cap}(e,W_p^\ell) .$$

Moreover, the following relation holds:

$$(1.16) \qquad ||\gamma||_{MW_p^\ell} \sim \sup_e \frac{||\nabla_\ell \gamma;e||_{L_p}}{[\mathrm{cap}(e,W_p^\ell)]^{1/p}} + ||\gamma||_{L_\infty} .$$

<u>REMARK 1.1</u>. When formulating Theorem 1.2 and Corollary 1.2 we can restrict ourselves to compacts e satisfying the condition $\mathrm{diam}(e) \leq 1$.

If $pm > n$, $p > 1$, we can avoid the notion of capacity when describing the space $M(W_p^m \to W_p^\ell)$. Indeed, we have

<u>*THEOREM 1.3*</u>. *If* $pm > n$, $p \in (1,\infty)$, *then* $M(W_p^m \to W_p^\ell) = W_{p,\mathrm{unif}}^\ell$.

P r o o f . The inequalities

$$\mathrm{cap}(e,W_p^m) \leq c , \quad \mathrm{cap}(e,W_p^{m-\ell}) \leq c$$

hold provided $\operatorname{diam}(e) \leq 1$, and thus (1.15) implies

$$||\gamma||_{M(W_p^m \to W_p^\ell)} \geq c||\gamma||_{W^\ell_{p,\mathrm{unif}}} .$$

In this way, $W^\ell_{p,\mathrm{unif}} \subset M(W_p^m \to W_p^\ell)$.

We shall show that the converse inclusion holds as well provided $pm > n$. To this aim we need the following known estimates of the capacity:

(1.17) $\quad \operatorname{cap}(e,W_p^k) \geq c$ provided $pk > n$ and $e \neq \emptyset$,

(1.18) $\quad \operatorname{cap}(e,W_p^k) \geq c(\operatorname{mes}_n e)^{1-pk/n}$ provided $pk < n$,

(1.19) $\quad \operatorname{cap}(e,W_p^k) \geq c\left(\log(2^n/\operatorname{mes}_n e)\right)^{1-p}$ provided $pk = n$ and $\operatorname{diam}(e) \leq 1$.

By virtue of (1.17), we have

$$\sup_{\{e;\operatorname{diam}(e)\leq 1\}} \frac{||\nabla_\ell\gamma;e||_{L_p}}{[\operatorname{cap}(e,W_p^m)]^{1/p}} \leq c \sup_{x\in R^n} ||\nabla_\ell\gamma;Q_1(x)||_{L_p} .$$

Analogously, for $p(m-\ell) > n$ we obtain

$$\sup_{\{e:\operatorname{diam}(e)\leq 1\}} \frac{||\gamma;e||_{L_p}}{[\operatorname{cap}(e,W_p^{m-\ell})]^{1/p}} \leq c \sup_{x\in R^n} ||\gamma;Q_1(x)||_{L_p} .$$

The estimates (1.18) and (1.19) imply that for $p(m-\ell) \leq n$ the left-hand side of the last inequality does not exceed $c \sup_{x\in R^n} ||\gamma;Q_1(x)||_{L_q}$, where

$$q = n/(m-\ell) \quad \text{for } p(m-\ell) < n ,$$
$$q > p \quad \text{for } p(m-\ell) = n .$$

Now, by noticing that $W_p^\ell(Q_1) \subset L_q(Q_1)$ for $pm > n$ we complete the proof.

We have deduced the identity $M(W_p^m \to W_p^\ell) = W^\ell_{p,\mathrm{unif}}$ by means of Theorem 1.2. Nonetheless, it is easy to establish it directly.

The capacity is not necessary for the description of the space $M(W_1^m \to W_1^\ell)$, either. The following assertion, which was proved by the author in [26], represents the analogue of Lemma 1.1 for $p = 1$.

LEMMA 1.6. Let m *and* ℓ *be integers,* $m \geq \ell \geq 0$. *The exact constant in the inequality*

$$\int|u|d\mu \leq C||u||_{W_1^m} , \quad u \in C_0^\infty$$

is equivalent to the quantity

$$\sup_{x\in R^n, r\in(0,1)} r^{m-n}\mu(Q_r(x)) .$$

<u>*THEOREM 1.4*</u>. (i) *If* $m \geq n$, $m \geq \ell$, *then*

$$||\gamma||_{M(W_1^m \to W_1^\ell)} \sim \sup_{x\in R^n} ||\gamma;Q_1(x)||_{W_1^\ell} .$$

(ii) *If* $\ell < n$, *then*

$$||\gamma||_{MW_1^\ell} \sim \sup_{x\in R^n, r\in(0,1)} r^{\ell-n}||\nabla_\ell\gamma;Q_r(x)||_{L_1} + ||\gamma||_{L_\infty} .$$

(iii) *If* $\ell < m < n$, *then*

$$||\gamma||_{M(W_1^m \to W_1^\ell)} \sim \sup_{x\in R^n, r>0} r^{m-n}||\nabla_\ell\gamma;Q_r(x)||_{L_1} + \sup_{x\in R^n} ||\gamma;Q_1(x)||_{L_1} .$$

Provided $mp \leq n$, $p > 1$, we can give upper and lower bounds for the norm in $M(W_p^m \longrightarrow W_p^\ell)$, which do not coincide but, on the other hand, do not involve the capacity. Theorem 1.2 together with the estimate of capacity of a ball immediately yields

<u>*COROLLARY 1.3*</u>. *The following estimates hold:*

$$||\gamma||_{M(W_p^m\to W_p^\ell)} \geq \begin{cases} c \sup\limits_{x\in R^n, r\in(0,1)} r^{m-n/p}\left(||\nabla_\ell\gamma;Q_r(x)||_{L_p} + r^{-\ell}||\gamma;Q_r(x)||_{L_p}\right), & \textit{if}\ pm < n ,\ p > 1 , \\ c \sup\limits_{x\in R^n, r\in(0,1)} (\log 2/r)^{(p-1)/p}\left(||\nabla_\ell\gamma;Q_r(x)||_{L_p} + r^{-\ell}||\gamma;Q_r(x)||_{L_p}\right), & \textit{if}\ pm = n ,\ p > 1 . \end{cases}$$

On the other hand, Theorem 1.2, Remark 1.1 and the estimates (1.18), (1.19) imply

<u>*COROLLARY 1.4*</u>. *The following estimates hold:*

$$||\gamma||_{M(W_p^m\to W_p^\ell)} \leq \begin{cases} c\left(\sup\limits_{\{e:\mathrm{diam}(e)\leq 1\}} \dfrac{||\nabla_\ell\gamma;e||_{L_p}}{(\mathrm{mes}_n e)^{1/p-m/n}} + \sup\limits_{x\in R^n}||\gamma;Q_1(x)||_{L_p}\right), & \textit{if}\ pm<n,\ p>1,\ \ell<m , \\ c\left(\sup\limits_{\{e:\mathrm{diam}(e)\leq 1\}} \left(\log(2^n/\mathrm{mes}_n e)\right)^{(p-1)/p}||\nabla_\ell\gamma;e||_{L_p} + \sup\limits_{x\in R^n}||\gamma;Q_1(x)||_{L_p}\right), & \textit{if}\ pm=n,\ p>1,\ \ell<m . \end{cases}$$

If $m = \ell$ *, then these estimates are valid after replacing* $\sup_{x \in R^n} ||\gamma;Q_1(x)||_{L_p}$ *by* $||\gamma||_{L_\infty}$.

Sometimes, Corollaries 1.3 and 1.4 enable us to easily verify conditions for inclusion of individual functions in the space $M(W_p^m \to W_p^\ell)$. Let us give two examples of this type.

<u>EXAMPLE 1.1</u>. Let $\mu > 0$ and

$$\gamma(x) = \eta(x) \exp(i|x|^{-\mu}) ,$$

where $\eta \in C_0^\infty$, $\eta(0) = 1$. Evidently,

$$|\nabla_\ell \gamma(x)| \sim |x|^{-\ell(\mu+1)}$$

when $x \to 0$. Consequently,

$$\gamma \in W_p^\ell \Longleftrightarrow n > p\ell(\mu+1) .$$

By Theorem 1.3 the same inequality is both a necessary and sufficient condition for γ to belong to the space $M(W_p^m \to W_p^\ell)$ for $pm > n$.

Let us assume that $mp < n$. Then

$$||\nabla_\ell \gamma;Q_r||_{L_p} \sim || \; |x|^{-\ell(\mu+1)};Q_r||_{L_p}$$

and for $m < \ell(\mu+1)$,

$$\lim_{r\to 0} r^{m-n/p}||\nabla_\ell \gamma;Q_r||_{L_p} = \infty .$$

According to Corollary 1.3 this means that $\gamma \notin M(W_p^m \to W_p^\ell)$ for $m < \ell(\mu+1)$. If $m \geq \ell(\mu+1)$, then

$$||\nabla_\ell \gamma;e||_{L_p} \leq c|| \; |x|^{-\ell(\mu+1)};e||_{L_p} \leq (\text{mes}_n e)^{-\ell(\mu+1)+n/p}$$

for an arbitrary compact e , $\text{diam}(e) \leq 1$.
This together with Corollary 1.4 implies that $\gamma \in M(W_p^m \to W_p^\ell)$. Hence for $mp < n$,

$$\gamma \in M(W_p^m \to W_p^\ell) \Longleftrightarrow m \geq \ell(\mu+1) .$$

In this same way we verify that

$$\gamma \in M(W_p^m \to W_p^\ell) \Longleftrightarrow m > \ell(\mu+1)$$

for $mp = n$.

<u>EXAMPLE 1.2</u>. Let $\mu, \nu > 0$, $\eta \in C_0^\infty(Q_1)$, $\eta(0) = 1$ and

$$\gamma(x) = \eta(x)\left(\log|x|^{-1}\right)^{-\nu} \exp\left(i\left(\log|x|^{-1}\right)^{\mu}\right) .$$

Evidently,

$$|\nabla_\ell \gamma(x)| \sim c|x|^{-\ell}\left(\log|x|^{-1}\right)^{\ell(\mu-1)-\nu} .$$

By an analogous argument as in Example 1.1, we obtain from the last relation and from Corollaries 1.3 and 1.4 that

$$\gamma \in W_p^{\ell} \Leftrightarrow \ell(\mu-1) < \nu-1/p ,$$

$$\gamma \in MW_p^{\ell} \Leftrightarrow \ell(\mu-1) \leq \nu-1$$

provided $\ell p = n$.

1.2. Multipliers in pairs of Bessel potential spaces

For an arbitrary real μ we set

$$\Lambda^{\mu} = (-\Delta+1)^{\mu/2} = F^{-1}(1+|\xi|^2)^{\mu/2}F ,$$

where F is the Fourier transform in R^n .

We introduce a space H_p^m $(1<p<\infty,\ m\geq 0)$, which is obtained by completing the space C_0^{∞} with respect to the norm

$$||u||_{H_p^m} = ||\Lambda^m u||_{L_p} .$$

If m is an integer, then $H_p^m = W_p^m$. It is well known (cf. [27]) that $u \in H_p^m$ if and only if $u = \Lambda^{-m}f$, where $f \in L_p$. In other words, each element of the space H_p^m is the Bessel potential with a density belonging to L_p .

Let $(S_m u)(x) = |\nabla_m u(x)|$ for an integer $m > 0$ and

$$(S_m u)(x) = \Big(\int_0^{\infty} \Big[\int_{Q_1} |\nabla_{[m]} u(x+\theta y) - \nabla_{[m]} u(x)|d\theta\Big]^2 y^{-1-2\{m\}}dy\Big)^{1/2}$$

provided $m > 0$ is non-integer.

According to Strichartz's theorem [4], we have

$$||\Lambda^m u||_{L_p} \sim ||S_m u||_{L_p} + ||u||_{L_p} .$$

The following theorem providing a characterization of the space $M(H_p^m \to H_p^{\ell})$ is proved in [11].

<u>*THEOREM 1.5*</u>. *A function* γ *belongs to the space* $M(H_p^m \to H_p^{\ell})$, $p \in (1,\infty)$, *if and only if* $\gamma \in M_{p,loc}^{\ell}$ *and for any compact* $e \subset R^n$,

$$||S_{\ell}\gamma;e||_{L_p}^p \leq c\ \mathrm{cap}(e,H_p^m) ,$$

$$||\gamma;e||_{L_p}^p \leq c\ \mathrm{cap}(e,H_p^{m-\ell})$$

holds.

Further, we have the relation

$$||\gamma||_{M(H_p^m \to H_p^\ell)} \sim \sup_e \left(\frac{||S_\ell \gamma; e||_{L_p}}{[\mathrm{cap}(e, H_p^m)]^{1/p}} + \frac{||\gamma; e||_{L_p}}{[\mathrm{cap}(e, H_p^{m-\ell})]^{1/p}} \right).$$

On the right-hand side, the restriction diam(e) ≤ 1 *may be added.*

In particular, for $m = \ell$ *we have*

$$||\gamma||_{MH_p^\ell} \sim \sup_{\{e:\mathrm{diam}(e) \leq 1\}} \frac{||S_\ell \gamma; e||_{L_p}}{[\mathrm{cap}(e, H_p^\ell)]^{1/p}} + ||\gamma||_{L_\infty} .$$

Another equivalence relation for the norm in the space $M(H_p^m \to H_p^\ell)$ *is*

$$||\gamma||_{M(H_p^m \to H_p^\ell)} \sim$$

$$\sim \sup_{\{e:\mathrm{diam}(e) \leq 1\}} \left(\frac{||\gamma; e||_{L_{pm/(m-\ell)}}}{[\mathrm{cap}(e, H_p^m)]^{(m-\ell)/mp}} + \frac{||S_\ell \gamma; e||_{L_p}}{[\mathrm{cap}(e, H_p^m)]^{1/p}} \right) .$$

This immediately implies that *for* $pm > n$, $p \in (1, \infty)$, *the space* $M(H_p^m \to H_p^\ell)$ *coincides with* $H_{p,\mathrm{unif}}^\ell$. The identity $MH_p^\ell = H_{p,\mathrm{unif}}^\ell$ was established by Strichartz [4].

We can also prove one-sided estimates for the norm in $M(H_p^m \to H_p^\ell)$ which do not involve capacity and are analogous to those formulated in Corollaries 1.3 and 1.4. The upper bounds yield various sufficient conditions for functions to belong to the class $M(H_p^m \to H_p^\ell)$, formulated in terms of well known functional spaces. Let us present two theorems of this type.

THEOREM 1.6. (i) *If* $\ell p < n$ *and* $\gamma \in H_{n/\ell,\mathrm{unif}}^\ell \cap L_\infty$, *then* $\gamma \in MH_p^\ell$ *and the estimate*

$$||\gamma||_{MH_p^\ell} \leq c\left(||\gamma||_{H_{n/\ell,\mathrm{unif}}^\ell} + ||\gamma||_{L_\infty} \right)$$

holds.

(ii) *If* $mp < n$, $\ell < m$ *and* $\gamma \in H_{n/m,\mathrm{unif}}^\ell$ *then* $\gamma \in M(H_p^m \to H_p^\ell)$ *and*

$$||\gamma||_{M(H_p^m \to H_p^\ell)} \leq c||\gamma||_{H_{n/m,\mathrm{unif}}^\ell} .$$

In the next assertion, $B_{q,\infty}^\mu$ is a space of S. M. Nikol'skiĭ, which consists of functions in R^n with a finite norm

$$\sup_{h\in R^n} |h|^{-\{\mu\}}||\Delta_h \nabla_{[\mu]} v||_{L_q} + ||v||_{W_q^{[\mu]}} ,$$

where $\Delta_h v(x) = v(x+h) - v(x)$.

THEOREM 1.7. [13] *Let* $q \geqq p$, $\{\ell\} > 0$.

(i) *If* $n/q > \ell$, $\{n/q\} > 0$ *and* $\gamma \in B^{n/q}_{q,\infty,unif} \cap L_\infty$, *then* $\gamma \in MH^\ell_p$. *The following inequality holds:*

$$||\gamma||_{MH^\ell_p} \leqq$$
$$\leqq c\left(\sup_{x\in R^n, h\in Q_1} |h|^{-\{n/q\}}||\Delta_h \nabla_{[n/q]}\gamma; Q_1(x)||_{L_q} + ||\gamma||_{L_\infty}\right) .$$

(ii) *If* $n/q > m$, $\mu = n/q-m+\ell$, $\{\mu\} > 0$ *and* $\gamma \in B^\mu_{q,\infty,unif}$, *then* $\gamma \in M(H^m_p \to H^\ell_p)$ *and the following inequality holds:*

$$||\gamma||_{M(H^m_p\to H^\ell_p)} \leqq$$
$$\leqq c\left(\sup_{x\in R^n, h\in Q_1} |h|^{-\{\mu\}}||\Delta_h \nabla_{[\mu]}\gamma; Q_1(x)||_{L_q} + \sup_{x\in R^n} ||\gamma; Q_1(x)||_{L_q}\right) .$$

Hirschman [3] obtained the following sufficient condition for γ to belong to the class MW^ℓ_2 on a unit circumference $\mathcal{C}$: γ is bounded and has a finite q-variation $Var_q(\gamma)$ for some q , $2 < q < 1/\ell$.

Here the q-variation is understood to be the quantity

$$\text{(1.20)} \qquad Var_q(\gamma) = \sup \left(\sum_{j=0}^{m-1} |\gamma(t_{j+1})-\gamma(t_j)|^q\right)^{1/q} ,$$

the supremum being taken over all partitions of the circumference $\mathcal{C}$ by points t_j .

Theorem 1.7 immediately yields a sufficient condition for a function to belong to the class $MH^\ell_p(R^1)$, which for $p = 2$ coincides (after replacing R^1 by $\mathcal{C}$) with Hirschman's condition.

Let us introduce the local q-variation of a function γ given on R^1 by (1.20) with the supremum being taken over all choices of a finite number of points $t_0 < t_1 < \dots < t_m$ considered in an arbitrary interval σ of unit length. Since evidently

$$\int_{\sigma} |\gamma(t+h)-\gamma(t)|^q dt \leq c|h|\,[\mathrm{Var}_q(\gamma)]^q$$

we arrive at

COROLLARY 1.5. Let $n = 1$, $q \geq p$ *and* $\ell q < 1$. *If* $\gamma \in L_\infty$ *and* $\mathrm{Var}_q(\gamma) < \infty$, *then* $\gamma \in MH_p^\ell$ *and the estimate*

$$||\gamma||_{MH_p^\ell} \leq c(||\gamma||_{L_\infty} + \mathrm{Var}_q(\gamma)) .$$

holds.

1.3. Multipliers in pairs of Slobodeckiĭ spaces

We introduce the function

$$(D_{p,\ell}u)(x) = \left(\int |\nabla_{[\ell]}u(x+h) - \nabla_{[\ell]}u(x)|^p |h|^{-n-p\{\ell\}} dh\right)^{1/p} ,$$

where $p \in (1,\infty)$ and $\{\ell\} > 0$. The space of functions with a finite norm $||D_{p,\ell}u||_{L_p} + ||u||_{L_p}$ is called the Slobodeckiĭ space and denoted by W_p^ℓ .

The next theorem gives a characterization of the space $M(W_p^m \to W_p^\ell)$ with $\{m\} > 0$, $\{\ell\} > 0$, $p \in (1,\infty)$.

THEOREM 1.8. [10] *A function* γ *belongs to the space* $M(W_p^m \to W_p^\ell)$ (m *and* ℓ *non-integers,* $m \geq \ell$, $1 < p < \infty$) *if and only if* $\gamma \in$ $\in W_{p,loc}^\ell$ *and for every compact* $e \subset R^n$,

$$||D_{p,\ell}\gamma;e||_{L_p}^p \leq \text{const } \mathrm{cap}(e,W_p^m) , \quad ||\gamma;e||_{L_p}^p \leq \text{const } \mathrm{cap}(e,W_p^{m-\ell}) .$$

Further, we have the relation

$$||\gamma||_{M(W_p^m \to W_p^\ell)} \sim \sup_e \left\{\frac{||D_{p,\ell}\gamma;e||_{L_p}}{[\mathrm{cap}(e,W_p^m)]^{1/p}} + \frac{||\gamma;e||_{L_p}}{[\mathrm{cap}(e,W_p^{m-\ell})]^{1/p}}\right\} .$$

Here again we can restrict ourselves to compacts satisfying the condition $\mathrm{diam}(e) \leq 1$.

From this result we can easily deduce that $M(W_p^m \to W_p^\ell) = W_{p,unif}^\ell$ *provided* m , ℓ *are non-integers,* $p \in (1,\infty)$ *and* $pm > n$.

The next result deals with the case $p = 1$. We shall use the norm $|||\cdot;Q_r|||_{W_1^\ell}$, which is defined by

$$|||u;Q_r|||_{W_1^\ell} = r^{-\ell}||u;Q_r||_{L_1} + ||\nabla_\ell u;Q_r||_{L_1}$$

for ℓ integer and by

$$|||u;Q_r|||_{W_1^\ell} = r^{-\ell}||u;Q_r||_{L_1} + \int_{Q_r}\int_{Q_r} |\nabla_{[\ell]}u(x)-\nabla_{[\ell]}u(y)| \frac{dxdy}{|x-y|^{n+\{\ell\}}}$$

for ℓ non-integer.

THEOREM 1.9. [10] *A function* γ *belongs to the space* $M(W_1^m \to W_1^\ell)$, $m \geq \ell \geq 0$, *if and only if* $\gamma \in W_{1,loc}^\ell$ *and*

$$|||\gamma;Q_r(x)|||_{W_1^\ell} \leq \text{const } r^{-m+n}$$

holds for any ball $Q_r(x)$, $0 < r < 1$. *Further, we have*

$$||\gamma||_{M(W_1^m\to W_1^\ell)} \sim \sup_{x\in R^n, r\in(0,1)} r^{m-n}|||\gamma;Q_r(x)|||_{W_1^\ell} .$$

For $m \geq n$ *the last relation is equivalent to*

$$||\gamma||_{M(W_1^m\to W_1^\ell)} \sim \sup_{x\in R^n} |||\gamma;Q_1(x)|||_{W_1^\ell} \sim ||\gamma||_{W_{1,unif}^\ell} .$$

Let us collect some embeddings representing sufficient conditions for a function to belong to the space $M(W_p^m \to W_p^\ell)$.

THEOREM 1.10. *Let* $p \in (1,\infty)$ *and* $\{m\}, \{\ell\} > 0$.

(i) *If* $\mu = n/q - \ell$ *and* $\{\mu\} > 0$, *then*

$$L_\infty \cap B_{q,\infty,unif}^\mu \subset MW_p^\ell .$$

(ii) *If* $n/q > m > \ell$, $\mu = n/q - m + \ell$ *and* $\{\mu\} > 0$, *then*

$$B_{q,\infty,unif}^\mu \subset M(W_p^m \to W_p^\ell) .$$

(iii) *If* $p\ell < n$ *and* $p \geq 2$, *then*

$$L_\infty \cap H_{n/\ell,unif}^\ell \subset MW_p^\ell .$$

(iv) *If* $m > \ell$, $\{m\} > 0$, $\{\ell\} > 0$, $pm < n$ *and* $p \geq 2$, *then*

$$H_{n/m,unif}^\ell \subset M(W_p^m \to W_p^\ell) .$$

The embeddings (iii) and (iv) fail if $p < 2$.

(v) *If* $q \in [n/\ell,\infty)$ *for* $p\ell < n$ *or* $q \in (p,\infty]$ *for* $p\ell = n$, *then*

$$L_\infty \cap B_{q,p,unif}^\ell \subset MW_p^\ell .$$

(vi) *If* $m > \ell$, $q \in [n/m,\infty]$ *for* $pm < n$ *or* $q \in (p,\infty)$ *for* pm = n , *then*

$$B^{\ell}_{q,p,\mathrm{unif}} \subset M(W^m_p \to W^{\ell}_p) \ .$$

The symbol $B^s_{q,p}$ in (v) and (vi) stands for the Besov space which consists of functions with a finite norm

$$\left(\int \|\Delta_h \nabla_{[s]} u\|^p_{L_q} |h|^{-n-p\{s\}} dh\right)^{1/p} + \|u\|_{W^{[s]}_q},$$

where $\{s\} > 0$, $q, p \geq 1$.

The assertion (i) implies the following result, which is analogous to Corollary 1.5.

<u>*COROLLARY 1.6*</u>. *Let* $n = 1$, $q \geq p$ *and* $\ell q < 1$. *If* $\gamma \in L_\infty$ *and* $\mathrm{Var}_q(\gamma) < \infty$, *then* $\gamma \in MW^{\ell}_p$ *and the following estimate holds:*

$$\|\gamma\|_{MW^{\ell}_p} \leq c(\|\gamma\|_{L_\infty} + \mathrm{Var}_q(\gamma)) \ .$$

Putting $q = \infty$ in (v) and (vi), we obtain a simple criterion for a function γ to belong to the class MW^{ℓ}_p (and hence, a fortiori, to $M(W^m_p \to W^{\ell}_p)$), in terms of the modulus of continuity ω of the vector function $\nabla_{[\ell]}\gamma$:

$$\int_0^1 \left[\frac{\omega(t)}{t^{\{\ell\}+1/p}}\right]^p dt < \infty \ .$$

By means of lacunary trigonometrical series it is not difficult to prove that in a certain sense even this rough sufficient condition cannot be improved.

To amend points (i), (ii) of Theorem 1.10 we present the following result concerning the case $pm = n$.

<u>*THEOREM 1.11*</u>. [13] *Let* $\{m\}$, $\{\ell\} > 0$, $p > 1$ *and*

$$\langle\gamma\rangle = \sup_{y \in R^n} \sup_{h \in Q_{1/2}} |h|^{-\{\ell\}} \log(1/|h|) \|\Delta_h \nabla_{[\ell]}\gamma; Q_1(y)\|_{L_p} \ .$$

1) *If* $\ell p = n$, $\gamma \in L_\infty$ *and* $\langle\gamma\rangle < \infty$, *then* $\gamma \in MW^{\ell}_p$ *and the inequality*

$$\|\gamma\|_{MW^{\ell}_p} \leq c(\langle\gamma\rangle + \|\gamma\|_{L_\infty})$$

holds.

2) *If* $mp = n$, $\gamma \in L_{p,\mathrm{unif}}$ *and* $\langle\gamma\rangle < \infty$, *then* $\gamma \in$

$\in M(W_p^m \to W_p^\ell)$ *for* $\ell < m$. *Further, the inequality*

$$||\gamma||_{M(W_p^m \to W_p^\ell)} \leq c(\langle\gamma\rangle + ||\gamma||_{L_{p,unif}})$$

holds.

Since the norm of a function in the space $B^s_{2,\infty}$ is equivalent to the norm

$$\sup_{R>1} R^s ||Fu; Q_{2R} \setminus Q_R||_{L_2} + ||u||_{L_2} ,$$

where F is the Fourier transform (see Triebel [25]), the points (i), (ii) of Theorem 1.10 imply

COROLLARY 1.7. 1) *Let* $1 < p \leq 2$, $n/2 > \ell$, $\{\ell\} > 0$. *If* $\gamma \in L_\infty$ *and* $(F\gamma)(\xi) = O((1+|\xi|)^{-n})$, *then* $\gamma \in MW_p^\ell$.

2) *Let* $1 < p < 2$, $n/2 > m$, $\{m\}, \{\ell\} > 0$, $m > \ell$. *If* $(F\gamma)(\xi) = O((1+|\xi|)^{m-\ell-n})$, *then* $\gamma \in M(W_p^m \to W_p^\ell)$.

It is easily seen that the norm

$$\sup_{y \in R^n} \sup_{h \in Q_{1/2}} |h|^{-\{\ell\}} \log(1/|h|) ||\Delta_h \nabla_{[\ell]} u; Q_1(y)||_{L_2} + ||u||_{L_2}$$

is equivalent to the norm

$$\sup_{R>2} R^\ell \log R \, ||Fu; Q_{2R} \setminus Q_R||_{L_2} + ||u||_{L_2} .$$

Hence and from Theorem 1.1 we obtain

COROLLARY 1.8. *If* $2\ell = n$, $\gamma \in L_\infty$ *and*

$$F\gamma(\xi) = O(|\xi|^{-n} (\log |\xi|)^{-1}) ,$$

for $|\xi| \geq 2$, *then* $\gamma \in MW_2^\ell$.

2. Some properties of multipliers

2.1. On the spectrum of a multiplier in H_p^ℓ

Let us start with the following simple property of multipliers in H_p^ℓ .

LEMMA 2.1. *The following estimate is valid:*

$$||\gamma||_{L_\infty} \leq ||\gamma||_{MH_p^\ell} . \tag{2.1}$$

P r o o f . For any $N = 1,2,...$ and an arbitrary function u from C_0^∞ we have

$$||\gamma^N u||_{L_p} \leq ||\gamma^N u||_{H_p^\ell}^{1/N} \leq ||\gamma||_{MH_p^\ell} ||u||_{H_p^\ell}^{1/N} .$$

Passing to the limit for $N \to \infty$, we conclude (2.1).

We shall need another lemma on the composition of a function of one variable and a multiplier. We will consider this problem once more in Theorem 2.2.

<u>LEMMA 2.2</u>. *Let* $\gamma \in MH_p^\ell$ *and let* σ *be a segment on the real axis such that* $\gamma(x) \in \sigma$ *for a.e.* $x \in R^n$ *. Further, let* $f \in C^{[\ell],1}(\sigma)$ *. Then* $f(\gamma) \in MH_p^\ell$ *and we have the estimate*

$$||f(\gamma)||_{MH_p^\ell} \leq c \sum_{j=0}^{k} ||f^{(j)};\sigma||_{L_\infty} ||\gamma||_{MH_p^\ell}^j ,$$

where $k = \ell+1$ *provided* ℓ *is integer and* $k = [\ell]+1$ *provided* $\{\ell\} > 0$.

P r o o f . Let us consider the less trivial case, $\{\ell\} > 0$. Let $\ell \in (0,1)$. Then

$$||uf(\gamma)||_{H_p^\ell} \leq c\Big(||S_\ell(uf(\gamma))||_{L_p} + ||uf(\gamma)||_{L_p}\Big) .$$

Since

$$S_\ell(uf(\gamma)) \leq |u|S_\ell f(\gamma) + ||f(\gamma)||_{L_\infty} S_\ell u \leq$$

$$\leq |u| \; ||f';\sigma||_{L_\infty} S_\ell \gamma + ||f(\gamma)||_{L_\infty} S_\ell u ,$$

we have

$$||uf(\gamma)||_{H_p^\ell} \leq c(||f'||_{L_\infty} ||S_\ell \gamma||_{M(H_p^\ell \to L_p)} + ||f(\gamma)||_{L_\infty})||u||_{H_p^\ell} .$$

This together with Theorem 1.5 implies the estimate

$$||f(\gamma)||_{MH_p^\ell} \leq c(||f'||_{L_\infty} ||\gamma||_{MH_p^\ell} + ||f(\gamma)||_{L_\infty}) .$$

Now it only remains to proceed by induction on $[\ell]$.

From Lemmas 2.1 and 2.2 we immediately conclude

<u>COROLLARY 2.1</u>. *If* $\gamma \in MH_p^\ell$ *and* $||\gamma^{-1}||_{L_\infty} < \infty$ *, then* $\gamma^{-1} \in MH_p^\ell$ *and we have the estimate*

$$||\gamma^{-1}||_{MH_p^\ell} \leq c||\gamma^{-1}||_{L_\infty}^{k+1}||\gamma||_{MH_p^\ell}^{k},$$

where k *is the same number as in Lemma 2.2.*

We shall say that a complex number λ belongs to the spectrum of the multiplier $\gamma \in MH_p^\ell$, if the operator of multiplication by $\gamma-\lambda$ has no bounded inverse.

Taking into account the embedding of MH_p^ℓ into L_∞, we immediately obtain from Corollary 2.1

COROLLARY 2.2. A number λ *belongs to the spectrum of a multiplier* $\gamma \in MH_p^\ell$ *if and only if* $(\gamma-\lambda)^{-1} \notin L_\infty$ *or, equivalently, if for any positive number* ε *the set* $\{x: |\gamma(x)-\lambda| < \varepsilon\}$ *has a positive* n-*dimensional measure.*

For $p = 2$, $2\ell < 1$ this result was obtained in [1].

A number λ is called an eigenvalue of a multiplier $\gamma \in MH_p^\ell$, if there exists a nonzero element $u \in H_p^\ell$ such that $(\gamma-\lambda)u = 0$.

It is clear that the set of eigenvalues is contained in the spectrum. Let us introduce a condition which is necessary and sufficient for λ to belong to the set of eigenvalues of a multiplier.

We shall need some definitions.

Let $\overset{o}{H}{}_p^\ell(Q_\delta)$ be the completion of the space $C_0^\infty(Q_\delta)$ (Q_δ being the open ball) with respect to the norm of the space H_p^ℓ.

A Borel set E, $E \subset Q_{\delta/2}$, is said to be an H_p^ℓ-nonessential subset of the ball $Q_{\delta/2}$, $p\ell \leq n$, if

$$\operatorname{cap}\{E, \overset{o}{H}{}_p^\ell(Q_\delta)\} \leq c_0 \delta^{n-p\ell},$$

where c_0 is a small positive constant that depends only on n, p, ℓ. If $p\ell > n$, then we define that the only H_p^ℓ-nonessential set is the empty set.

Let A be a Borel subset of R^n. The lowest upper bound of all numbers δ for which the set

$$\{Q_{\delta/2}(x): Q_{\delta/2}(x) \setminus A \text{ is a } H_p^\ell\text{-nonessential subset of } Q_{\delta/2}(x)\}$$

is nonempty, will be called the H_p^ℓ-inner diameter of A and denoted by $d(A;H_p^\ell)$.

Obviously, for $p\ell > n$ this definition leads to the usual inner diameter of the set A.

We have

THEOREM 2.1. A number λ *is an eigenvalue of a multiplier* $\gamma \in MH_p^\ell$ *if and only if the* H_p^ℓ*-inner diameter of the set* $\{x: \gamma(x) = \lambda\}$ *is positive. (If* $p\ell > n$ *then this means that the set* $\{x: \gamma(x) = \lambda\}$ *possesses interior points.)*

2.2. On functions of multipliers

According to Hirschman [2], the composition $\phi(\gamma)$ of a function $\phi \in C^{0,\rho}$, $\rho \in (0,1]$ and a multiplier γ in the space W_2^ℓ , $\ell \in (0,1)$, represents a multiplier in W_2^r , where $r \in (0,\ell\rho)$ provided $\rho < 1$ and $r = \ell$ for $\rho = 1$.

Let us give a generalization of this result, which was obtained in [10].

THEOREM 2.2. Let $\gamma \in M(W_p^m \to W_p^\ell)$, $n \geq \ell$, $0 < \ell < 1$, $p > 1$. *Further, let* ϕ *be a function defined on* R^1 *if* $\text{Im}\,\gamma = 0$, *or on* C^1 *if* γ *is a complex-valued function. Assume that* $\phi(0) = 0$ *and that*

$$|\phi(t+\tau) - \phi(t)| \leq A|\tau|^\rho$$

with $\rho \in (0,1]$.

Then $\phi(\gamma) \in M(W_p^{m-\ell+r} \to W_p^r)$ *with* $r \in (0,\ell\rho)$ *provided* $\rho < 1$, *and with* $r = \ell$ *provided* $\rho = 1$. *The following estimate holds:*

$$||\phi(\gamma)||_{M(W_p^{m-\ell+r}\to W_p^r)} \leq cA(||\gamma||^\rho_{M(W_p^m\to W_p^\ell)} + ||\gamma||_{M(W_p^m\to W_p^\ell)}).$$

2.3. The essential norm in $M(W_p^m \to W_p^\ell)$

Let $p \geq 1$ and let both m and ℓ be simultaneously either integers or non-integers, $m \geq \ell \geq 0$.

Let us denote by

$$\text{ess}\,||\gamma||_{M(W_p^m\to W_p^\ell)}$$

the essential norm of the operator of multiplication for the function $\gamma \in M(W_p^m \to W_p^\ell)$, that is, the number

$$\inf_{\{T\}} ||\gamma - T||_{W_p^m\to W_p^\ell} ,$$

where $\{T\}$ is the family of all completely continuous operators $W_p^m \to W_p^\ell$.

The following theorem gives both-sides estimates of the essential norm (see [10]).

THEOREM 2.3. Let $\gamma \in M(W_p^m \to W_p^\ell)$, $m \geq \ell \geq 0$ *, and let both* m *and* ℓ *be simultaneously either integers or non-integers.*

(i) *If* $p > 1$ *and* $mp \leq n$ *, then*

$$\text{ess}\,||\gamma||_{M(W_p^m \to W_p^\ell)} \sim \lim_{\delta\to 0} \sup_{\{e:\text{diam}(e)\leq\delta\}} \left(\frac{||\gamma;e||_{L_p}}{[\text{cap}(e,W_p^{m-\ell})]^{1/p}} + \frac{||D_{p,\ell}\gamma;e||_{L_p}}{[\text{cap}(e,W_p^m)]^{1/p}}\right) +$$

$$+ \lim_{r\to\infty} \sup_{\{e\subset R^n\setminus Q_r:\text{diam}(e)\leq 1\}} \left(\frac{||\gamma;e||_{L_p}}{[\text{cap}(e,W_p^{m-\ell})]^{1/p}} + \frac{||D_{p,\ell}\gamma;e||_{L_p}}{[\text{cap}(e,W_p^m)]^{1/p}}\right).$$

In particular,

$$\text{ess}\,||\gamma||_{MW_p^\ell} \sim ||\gamma||_{L_\infty} + \lim_{\delta\to 0} \sup_{\{e:\text{diam}(e)\leq\delta\}} \frac{||D_{p,\ell}\gamma;e||_{L_p}}{[\text{cap}(e,W_p^\ell)]^{1/p}} +$$

$$+ \lim_{r\to\infty} \sup_{\{e\subset R^n\setminus Q_r:\text{diam}(e)\leq 1\}} \frac{||D_{p,\ell}\gamma;e||_{L_p}}{[\text{cap}(e,W_p^\ell)]^{1/p}} .$$

(ii) *If* $m < n$ *, then*

$$\text{ess}\,||\gamma||_{M(W_1^m\to W_1^\ell)} \sim$$

$$\sim \lim_{\delta\to 0} \delta^{m-n} \sup_{x\in R^n} \left(\delta^{-\ell}||\gamma;Q_\delta(x)||_{L_1} + ||D_{1,\ell}\gamma;Q_\delta(x)||_{L_1}\right) +$$

$$+ \varlimsup_{|x|\to\infty} \sup_{r\in(0,1)} r^{m-n}\left(r^{-\ell}||\gamma;Q_r(x)||_{L_1} + ||D_{1,\ell}\gamma;Q_r(x)||_{L_1}\right) .$$

In particular,

$$\text{ess}\,||\gamma||_{MW_1^\ell} \sim ||\gamma||_{L_\infty} + \varlimsup_{\delta\to 0} \sup_{x\in R^n} \delta^{\ell-n}||D_{1,\ell}\gamma;Q_\delta(x)||_{L_1} +$$

$$+ \varlimsup_{|x|\to\infty} \sup_{r\in(0,1)} r^{m-n}||D_{1,\ell}\gamma;Q_r(x)||_{L_1} .$$

(iii) *If* $mp > n$, $p > 1$ *or* $m \geq n$, $p = 1$ *, then*

$$\text{ess}\,||\gamma||_{M(W_p^m\to W_p^\ell)} \sim \varlimsup_{|x|\to\infty} ||\gamma;Q_1(x)||_{W_p^\ell} \quad \textit{for}\ m > \ell ,$$

$$\text{ess}\,||\gamma||_{MW_p^\ell} \sim ||\gamma||_{L_\infty} + \varlimsup_{|x|\to\infty} ||\gamma;Q_1(x)||_{W_p^\ell} .$$

2.4. Completely continuous multipliers

Let us denote by $\overset{\circ}{M}(W_p^m \to W_p^\ell)$, $m \geqq \ell$, the family of such functions γ that the operator of multiplication by γ is completely continuous as an operator from W_p^m into W_p^ℓ .

Evidently, $\gamma \in \overset{\circ}{M}(W_p^m \to W_p^\ell)$ if and only if

$$\text{ess}\ ||\gamma||_{M(W_p^m \to W_p^\ell)} = 0 .$$

Consequently, Theorem 2.3 implies the following necessary and sufficient conditions for a function $\gamma \in M(W_p^m \to W_p^\ell)$ to belong to the class $\overset{\circ}{M}(W_p^m \to W_p^\ell)$.

COROLLARY 2.3. (i) *If* $pm \leqq n$, $p > 1$, *then* $\gamma \in \overset{\circ}{M}(W_p^m \to W_p^\ell)$ *if and only if*

$$\lim_{\delta\to 0} \sup_{\{e:\text{diam}(e)\leqq\delta\}} \left(\frac{||\gamma;e||_{L_p}}{[\text{cap}(e,W_p^{m-\ell})]^{1/p}} + \frac{||D_{p,\ell}\gamma;e||_{L_p}}{[\text{cap}(e,W_p^m)]^{1/p}}\right) = 0 ,$$

$$\lim_{r\to\infty} \sup_{\{e\subset R^n\setminus Q_r:\text{diam}(e)\leqq 1\}} \left(\frac{||\gamma;e||_{L_p}}{[\text{cap}(e,W_p^{m-\ell})]^{1/p}} + \frac{||D_{p,\ell}\gamma;e||_{L_p}}{[\text{cap}(e,W_p^m)]^{1/p}}\right) = 0 .$$

(ii) *If* $pm > n$, $p \geq 1$ *or* $m = n$, $p = 1$, *then* $\gamma \in \overset{\circ}{M}(W_p^m \to W_p^\ell)$ *if and only if* $\gamma \in W_{p,\text{unif}}^\ell$ *and*

(2.2) $$\lim_{|x|\to\infty} ||\gamma;Q_1(x)||_{W_p^\ell} = 0 .$$

(iii) *If* $m < n$, *then the inclusion* $\gamma \in \overset{\circ}{M}(W_p^m \to W_p^\ell)$ *is valid if and only if the identity*

$$\lim_{\delta\to 0} \delta^{m-n} \sup_{x\in R^n} |||\gamma;Q_\delta(x)|||_{W_1^\ell} = 0$$

is valid simultaneously with (2.2).

The next theorem offers still another characterization of the space $\overset{\circ}{M}(W_p^m \to W_p^\ell)$.

THEOREM 2.4. The space $\overset{\circ}{M}(W_p^m \to W_p^\ell)$ *is the completion of* C_0^∞ *with respect to the norm of the space* $M(W_p^m \to W_p^\ell)$.

For the case $m = \ell$ we have the following result, which strengthens Lemma 2.1.

THEOREM 2.5. *For* $\ell > 0$, $1 \leq p < \infty$, *the following estimate is valid:*

$$||\gamma||_{L_\infty} \leq \text{ess}\, ||\gamma||_{MW_p^\ell} .$$

In accordance with Theorem 2.4, let us denote by $\overset{o}{M}W_p^\ell$ the completion of the space C_0^∞ with respect to the norm of MW_p^ℓ. The following theorem, together with Theorem 2.5, shows that the essential norm in $\overset{o}{M}W_p^\ell$ is equivalent to the norm in L_∞ .

THEOREM 2.6. *If* $\gamma \in \overset{o}{M}W_p^\ell$, $\ell \geq 0$, $p \geq 1$, *then*

$$\text{ess}\, ||\gamma||_{MW_p^\ell} \leq c||\gamma||_{L_\infty}$$

is valid.

2.5. Traces and extensions of multipliers in W_p^ℓ

Let $R^{n+m} = \{z = (x,y): x \in R^n, y \in R^m\}$ and let $W_{p,\beta}^k(R^{n+m})$ be the completion of the space $C_0^\infty(R^{n+m})$ with respect to the norm

$$\left(\int_{R^{n+m}} |y|^{p\beta}(|\nabla_k U|^p + |U|^p)dz\right)^{1/p} .$$

As is well known [28], [22], the space $W_p^\ell(R^n)$ for non-integer ℓ represents the space of traces on R^n of functions from $W_{p,\beta}^k(R^{n+m})$, where $\beta = k-\ell-m/p$. Moreover, $W_p^\ell(R^n)$ is the space of traces on R^n of functions from $W_p^{\ell+m/p}(R^{n+m})$.

We will formulate two theorems which demonstrate that an analogous situation occurs for the corresponding spaces of multipliers.

Following Stein [27], we introduce an operator of extension of functions defined on R^n to the space R^{n+m} by means of the identity

$$(T\gamma)(x,y) = \int\zeta(t)\gamma(x+|y|t)dt , \tag{2.3}$$

where the function ζ is subjected to the conditions

$$\int(1+|x|)^\ell \sum_{j=0}^{k} \sup_{\partial Q_{|x|}} |\nabla_j\zeta|(1+|x|)^j dx = C < \infty , \tag{2.4}$$

$$\int\zeta(x)dx = 1 , \tag{2.5}$$

$$\int x^\alpha\zeta(x)dx = 0 , \quad 0 < |\alpha| \leq [\ell] .$$

THEOREM 2.7. [6] *Let* $\{\ell\} > 0$, $\ell < k$, *where* k *is an integer,* $\Gamma \in MW^{k}_{p,\beta}(R^{n+m})$, $\beta = k-\ell-m/n$ *and* $\gamma(x) = \Gamma(x,0)$. *Then we have the estimates*

$$c_1C^{-1}||T\gamma;R^{n+m}||_{MW^k_{p,\beta}} \leqq ||\gamma;R^n||_{MW^\ell_p} \leqq ||\Gamma;R^{n+m}||_{MW^k_{p,\beta}} .$$

An assertion analogous to Theorem 2.7 is valid even for the space of multipliers $MW^k_{p,\beta}(R^{n+1}_+)$, where $R^{n+1}_+ = \{z = (x,y): x \in R^n, y > 0\}$ while $W^k_{p,\beta}(R^{n+1}_+)$ is the completion of $C^\infty_0(\overline{R^{n+1}_+})$ with respect to the norm

$$\left(\int_{R^{n+1}_+} y^{p\beta}|\nabla_{k,z}U|^p dz\right)^{1/p} + ||U;R^{n+1}_+||_{L_p} .$$

THEOREM 2.8. [6] (i) *Let* $\{\ell\} > 0$, $\Gamma \in MW^k_{p,\beta}(R^{n+1}_+)$, $\beta = k-\ell-1/p$, $k > \ell$ *and* $\gamma(x) = \Gamma(x,0)$. *Then*

$$||\gamma;R^n||_{MW^\ell_p} \leqq c||\Gamma;R^{n+1}_+||_{MW^k_{p,\beta}} .$$

(ii) *Let* $\{\ell\} > 0$, $p \geqq 1$ *and* $\nabla_s\gamma \in MW^\ell_p(R^n)$. *Further, let* $T\gamma$ *be the extension of* γ *to* R^{n+1}_+ , *defined by the formula* (2.3), *where the function* ζ *is subjected only to the conditions* (2.4), (2.5). *Then*

$$||\nabla_s(T\gamma);\ R^{n+1}_+||_{MW^k_{p,\beta}} \leqq cC||\nabla_s\gamma;\ R^n||_{MW^\ell_p} ,$$

where $k > \ell$ *and* $\beta = k-\ell-1/p$.

THEOREM 2.9. *Let* $\{\ell\} > 0$, $1 \leqq p < \infty$, $\Gamma \in MW^{\ell+m/p}_p(R^{n+m})$, $\gamma(x) = \Gamma(x,0)$. *Then we have the estimates*

$$c_1C^{-1}||T\gamma;R^{n+m}||_{MW^{\ell+m/p}_p} \leqq ||\gamma;R^n||_{MW^\ell_p} \leqq c_2||\Gamma;R^{n+m}||_{MW^{\ell+m/p}_p} .$$

3. Multipliers in a pair of Sobolev spaces in a domain

Let Ω be a bounded domain of class $C^{0,1}$, m and ℓ integers, $m \geqq \ell \geqq 0$, $p \geqq 1$. We will formulate a theorem expressing the possibility of extension of multipliers from Ω to R^n . The symbol E will denote the operator of extension of E. M. Stein (see [27], Chap. 6), which performs the extension $W^k_p(\Omega) \to W^k_p(R^n)$.

The following result is proved in [7].

THEOREM 3.1. *Let* $\gamma \in M(W_p^m(\Omega) \to W_p^\ell(\Omega))$, $1 \le p < \infty$. *Then* $E\gamma \in M(W_p^m(R^n) \to W_p^\ell(R^n))$ *and the inequality*

$$||E\gamma;R^n||_{M(W_p^m \to W_p^\ell)} \le c||\gamma;\Omega||_{M(W_p^m \to W_p^\ell)}$$

holds.

Hence and from Theorems 1.2 - 1.4 we obtain the following equivalent norms in the space $M(W_p^m(\Omega) \to W_p^\ell(\Omega))$.

THEOREM 3.2. [7] (i) *If* $p > 1$, $mp \le n$, *then*

$$||\gamma;\Omega||_{M(W_p^m \to W_p^\ell)} \sim \sup_{e \subset \Omega} \left(\frac{||\nabla_\ell \gamma;e||_{L_p}}{[\mathrm{cap}(e,W_p^m(R^n))]^{1/p}} + \frac{||\gamma;e||_{L_p}}{[\mathrm{cap}(e,W_p^{m-\ell}(R^n))]^{1/p}} \right) .$$

(ii) *If* $p > 1$, $mp > n$ *or* $p = 1$, $m \ge n$, *then*

$$||\gamma;\Omega||_{M(W_p^m \to W_p^\ell)} \sim ||\gamma;\Omega||_{W_p^\ell} .$$

(iii) *If* $m < n$, *then*

$$||\gamma;\Omega||_{M(W_1^m \to W_1^\ell)} \sim \sup_{z \in \Omega;\rho \in (0,1)} \rho^{m-n} \sum_{j=0}^{\ell} \rho^{j-\ell} ||\nabla_j \gamma;Q_\rho(z) \cap \Omega||_{L_1} .$$

Let us formulate a theorem on the essential norm of functions $\gamma \in M(W_p^m(\Omega) \to W_p^\ell(\Omega))$, where m and ℓ are integers, Ω is a bounded domain of class $C^{0,1}$.

THEOREM 3.3. (i) *If* $p > 1$ *and* $mp \le n$, *then*

$$\mathrm{ess}\, ||\gamma;\Omega||_{M(W_p^m \to W_p^\ell)} \sim \lim_{\delta \to 0} \sup_{\{e \subset \Omega:\mathrm{diam}(e)<\delta\}} \left(\frac{||\gamma;e||_{L_p}}{[\mathrm{cap}(e,W_p^{m-\ell})]^{1/p}} + \frac{||\nabla_\ell \gamma;e||_{L_p}}{[\mathrm{cap}(e,W_p^m)]^{1/p}} \right) .$$

In particular,

$$\mathrm{ess}\, ||\gamma;\Omega||_{MW_p^\ell} \sim ||\gamma;\Omega||_{L_\infty} + \lim_{\delta \to 0} \sup_{\{e \subset \Omega:\mathrm{diam}(e)<\delta\}} \frac{||\nabla_\ell \gamma;e||_{L_p}}{[\mathrm{cap}(e,W_p^\ell)]^{1/p}} .$$

(ii) *If* $m < n$, *then*

$$\mathrm{ess}\, \|\gamma;\Omega\|_{M(W_1^m \to W_1^\ell)} \sim \overline{\lim_{\delta\to 0}}\, \delta^{m-n} \sup_{z\in\Omega} \left(\delta^{-\ell}\|\gamma;Q_\delta(z)\cap\Omega\|_{L_1} + \right.$$

$$\left. + \|\nabla_\ell \gamma;Q_\delta(z)\cap\Omega\|_{L_1}\right) .$$

In particular,

$$\mathrm{ess}\, \|\gamma;\Omega\|_{MW_1^\ell} \sim \|\gamma;\Omega\|_{L_\infty} + \overline{\lim_{\delta\to 0}}\, \delta^{\ell-n} \sup_{z\in\Omega} \|\nabla_\ell\gamma;Q_\delta(z)\cap\Omega\|_{L_1} .$$

(iii) *If* $mp > n$, $p > 1$ *or* $m \geq n$, $p = 1$, *then*

$$\mathrm{ess}\, \|\gamma;\Omega\|_{M(W_p^m \to W_p^\ell)} = 0 \quad \textit{for} \quad m > \ell \quad \textit{and}$$

$$\mathrm{ess}\, \|\gamma;\Omega\|_{MW_p^\ell} \sim \|\gamma;\Omega\|_{L_\infty} \quad \textit{for} \quad m = \ell .$$

This theorem immediately yields

<u>*COROLLARY 3.1.*</u> *A function* $\gamma \in M(W_p^m(\Omega) \to W_p^\ell(\Omega))$, $m > \ell$, *belongs to the subspace* $\overset{\circ}{M}(W_p^m(\Omega) \longrightarrow W_p^\ell(\Omega))$ *of completely continuous multipliers if and only if*

$$\lim_{\delta\to 0} \sup_{\{e\subset\Omega:\mathrm{diam}(e)\leq\delta\}} \left(\frac{\|\gamma;e\|_{L_p}}{[\mathrm{cap}(e,W_p^{m-\ell})]^{1/p}} + \frac{\|\nabla_\ell\gamma;e\|_{L_p}}{[\mathrm{cap}(e,W_p^m)]^{1/p}}\right) = 0$$

provided $p > 1$ *and* $mp \leq n$;

$$\lim_{\delta\to 0} \delta^{m-n} \sup_{z\in\Omega} \left(\delta^{-\ell}\|\gamma;Q_\delta(z)\cap\Omega\|_{L_1} + \|\nabla_\ell\gamma;Q_\delta(z)\cap\Omega\|_{L_1}\right) = 0$$

provided $m < n$. *Finally,* $\overset{\circ}{M}(W_p^m(\Omega) \longrightarrow W_p^\ell(\Omega)) = M(W_p^m(\Omega) \longrightarrow W_p^\ell(\Omega))$ *provided either* $mp > n$, $p > 1$ *or* $m \geq n$, $p = 1$.

<u>4. Applications of multipliers</u>

<u>4.1. Convolution operator in a pair of weighted spaces L_2</u>

Let $K : u \longrightarrow k * u$ be the convolution operator with a kernel k . The results of the preceding section can be regarded as theorems on properties of K considered as an operator from $L_2((1+|x|^2)^{m/2})$ into $L_2((1+|x|^2)^{\ell/2})$, $m \geq \ell \geq 0$, with

$$\|u\|_{L_2((1+|x|^2)^{r/2})} = \left(\int |u|^2(1+|x|^2)^r dx\right)^{1/2} .$$

Let us give a simple example. The operator K is continuous if and only if its symbol, that is, the Fourier transform Fk , belongs to the space $M(W_2^m \to W_2^\ell)$. According to Theorem 1.8, this is equivalent to $Fk \in W^2_{2,loc}$ and

$$\int_e |Fk|^2 dx \leq \text{const cap}(e, W_2^{m-\ell}) , \quad \int_e |D_{2,\ell}(Fk)|^2 dx \leq \text{const cap}(e, W_2^m)$$

for all compacts e in R^n .

Moreover,

$$||K|| \sim \sup_e \left(\frac{||Fk;e||_{L_2}}{[\text{cap}(e,W_2^{m-\ell})]^{1/2}} + \frac{||D_{2,\ell}(Fk);e||_{L_2}}{[\text{cap}(e,W_2^m)]^{1/2}} \right) .$$

If $2m > n$, then

$$||K|| \sim \sup_{x \in R^n} \left(||Fk;Q_1(x)||_{L_2} + ||D_{2,\ell}(Fk);Q_1(x)||_{L_2} \right) \sim$$
$$\sim ||Fk||_{W^\ell_{2,unif}} .$$

For $p = 2$ the results of Sec. 2.3 give both-sided estimates of the essential norm, as well as conditions of the complete continuity of the operator K .

Theorem 2.2 describes properties of functions of an operator K mapping $L_2((1+|x|^2)^{m/2})$ continuously into $L_2((1+|x|^2)^{\ell/2})$. In particular, let $0 < \ell < 1$ and let ϕ be a complex valued function of the complex variable, $\phi(0) = 0$. By $\phi(K)$ let us denote the convolution operator with the symbol $\phi(Fk)$. If the function ϕ satisfies the uniform Lipschitz condition, then the operator $\phi(K)$ is continuous in the same pair of spaces as the operator K .

Replacing the Lipschitz condition by a weaker one, $|\phi(t+\tau) - \phi(t)| \leq A|\tau|^\rho$, where $|\tau| < 1$ and $\rho \in (0,1)$, we obtain continuity of the operator

$$\phi(K): L_2(R^n;(1+|x|^2)^{(m-\ell+r)/2}) \to L_2(R^n;(1+|x|^2)^{r/2}) ,$$

where $r \in (0,\ell\rho)$.

According to Corollary 2.2, a number λ belongs to the spectrum of an operator K , which is continuous in $L_2((1+|x|^2)^{\ell/2})$, if and only if $(Fk-\lambda)^{-1} \notin L_\infty$.

By virtue of Theorem 2.1, λ is an eigenvalue of the same operator if and only if the $(2,\ell)$-inner diameter of the set $\{\xi: (Fk)(\xi) = \lambda\}$ is positive.

4.2. Singular integral operators with symbols from spaces of multipliers

Assertions formulated in this section (cf. [9]) show the usefulness of the spaces MW_p^ℓ and $\overset{\circ}{MW}{}_p^\ell$ for developing the calculus of singular integral operators acting in the space W_p^ℓ $(1 < p < \infty,\ \ell = 1,2,\ldots)$. (The basic facts of the theory of such operators are found in the monographs [23], [27], [30].)

Let us introduce the space $C^\infty(MW_p^\ell, \partial Q_1)$ of infinitely differentiable functions on the sphere ∂Q_1 with values in MW_p^ℓ.

In the same way we introduce the space $C^\infty(\overset{\circ}{MW}{}_p^\ell, \partial Q_1)$. In what follows, A, B, C stand for singular integral operators in R^n with symbols $a(x,\theta)$, $b(x,\theta)$, $c(x,\theta)$, where $x \in R^n$, $\theta \in \partial Q_1$.

<u>THEOREM 4.1</u>. *Let* AB *be a singular operator with a symbol* ab, $A \circ B$ *– the composition of operators* A, B.

If $a \in C^\infty(MW_p^\ell, \partial Q_1)$ *and there is such a function* $b_\infty \in C^\infty(\partial Q_1)$ *that* $b - b_\infty \in C^\infty(\overset{\circ}{MW}{}_p^\ell, \partial Q_1)$, *then the operator* $AB - A \circ B$ *is completely continuous in* W_p^ℓ.

The next theorem gives condition for the operator $AB - A \circ B$ to have order -1 in W_p^ℓ.

<u>THEOREM 4.2</u>. *If* $a \in C^\infty(MW_p^{\ell+1}, \partial Q_1)$ *and* $\nabla_x b \in C^\infty(MW_p^\ell, \partial Q_1)$, *then the operator* $AB - A \circ B$ *maps* W_p^ℓ *continuously into* $W_p^{\ell+1}$. *Here* AB *is a singular operator with the symbol* ab, *while* $A \circ B$ *is the composition of operators.*

In the conclusion of this section we give two immediate consequences of Theorems 4.1 and 4.2, concerning the regularization of a singular integral operator.

<u>COROLLARY 4.1</u>. *Let there exist a function* $a_\infty \in C^\infty(\partial Q_1)$ *such that* $a - a_\infty \in C^\infty(\overset{\circ}{MW}{}_p^\ell, \partial Q_1)$. *Further, let* $c = 1/a \in L_\infty(R^n \times \partial Q_1)$. *Then* $c \in C^\infty(MW_p^\ell, \partial Q_1)$ *and* $c - c_\infty \in C^\infty(\overset{\circ}{MW}{}_p^\ell, \partial Q_1)$, *where* $c_\infty = 1/a_\infty$. *Moreover, the operators* $A \circ C - I$ *and* $C \circ A - I$ *are completely continuous in* W_p^ℓ.

<u>COROLLARY 4.2</u>. *Let* $a \in L_\infty(R^n \times \partial Q_1)$ *and* $\nabla_x a \in C^\infty(MW_p^\ell, \partial Q_1)$.

Further, let $c = 1/a \in L_\infty(R^n \times \partial Q_1)$ *. Then* $\nabla_x c \in C^\infty(MW_p^\ell, \partial Q_1)$ *and the operators* $A \circ C - I$ *and* $C \circ A - I$ *map* W_p^ℓ *continuously into* $W_p^{\ell+1}$.

4.3. On the norm and the essential norm of a differential operator

Probably the simplest application of the space $M(W_p^m \to W_p^\ell)$ to the theory of differential operators is that given in the following assertion.

PROPOSITION 4.1. *The operator*

$$P(x,D_x)u = \sum_{|\alpha| \leq k} a_\alpha(x) D_x^\alpha u \ , \quad x \in R^n \ , \tag{4.1}$$

represents a continuous mapping $W_p^h \to W_p^{h-k}$, $h \geq k$, *if* $a_\alpha \in M(W_p^{h-|\alpha|} \to W_p^{h-k})$ *for any multiindex* α *. We have the estimate*

$$||P||_{W_p^h \to W_p^{h-k}} \leq c \sum_{|\alpha| \leq k} ||a_\alpha||_{M(W_p^{h-|\alpha|} \to W_p^{h-k})} \ .$$

For some values of p , h , k , even the converse estimate holds. Namely, we have

PROPOSITION 4.2. *If* $p = 1$ *or* $p(h-k) > n$, $p > 1$, *then the following relation is valid:*

$$||P||_{W_p^h \to W_p^{h-k}} \sim \sum_{|\alpha| \leq k} ||a_\alpha||_{M(W_p^{h-|\alpha|} \to W_p^{h-k})} \ .$$

The essential norm of the operator P possesses analogous properties:

PROPOSITION 4.3. (i) *We have the estimate*

$$\text{ess } ||P||_{W_p^h \to W_p^{h-k}} \leq c \sum_{|\alpha| \leq k} \text{ess } ||a_\alpha||_{M(W_p^{h-|\alpha|} \to W_p^{h-k})} \ .$$

(ii) *If* $p = 1$ *or* $p(h-k) > n$, $p > 1$ *and if* P *maps* W_p^h *continuously into* W_p^{h-k} , *then the following relation holds:*

$$\text{ess } ||P||_{W_p^h \to W_p^{h-k}} \sim \sum_{|\alpha| \leq k} \text{ess } ||a_\alpha||_{M(W_p^{h-|\alpha|} \to W_p^{h-k})} \ .$$

Finally, let us mention that the estimate

$$||P_0;\ R^n \times S^{n-1}||_{L_\infty} \leq \text{ess } ||P||_{W_p^h \to W_p^{h-k}}$$

holds, where P_0 *is the principal homogeneous part of the operator* P .

4.4. Coercive estimates of solutions of elliptic boundary value problems in spaces of multipliers

It is well known that the solutions of elliptic boundary value problems satisfy coercive estimates in Sobolev spaces. It appears that similar estimates are valid even for norms in classes of multipliers acting on a Sobolev space.

In the half-space $R_+^{n+1} = \{(x,y): x \in R^n, y \geq 0\}$ let us consider the operator of a boundary value problem $\{P,P_1,\ldots,P_k\}$, where P is a differential operator of order $2k$ while P_j are the operators of boundary conditions, induced by differential operators of orders k_j . We assume the coefficients of operators P , P_j to be constant and such that the operators generate an elliptic boundary value problem.

THEOREM 4.4. [8] *Let* $\gamma \in W^h_{p,loc}(R_+^{n+1}) \cap L_\infty(R_+^{n+1})$, *where* h *is an integer,* $h \geq 2k$. *Further, let* $P\gamma \in M(W_p^h(R_+^{n+1}) \to W_p^{h-2k}(R_+^{n+1}))$, $P_j\gamma|_{y=0} \in M(W_p^{h-1/p}(R^n) \to W_p^{h-k_j-1/p}(R^n))$. *Then* $\gamma \in MW_p^h(R_+^{n+1})$ *and*

$$||\gamma;R_+^{n+1}||_{MW_p^h} \leq c(||P\gamma;R_+^{n+1}||_{M(W_p^h\to W_p^{h-2k})} + \sum_{j=1}^{k}||P_j\gamma;R^n||_{M(W_p^{h-1/p} \to W_p^{h-k_j-1/p})} + ||\gamma;R_+^{n+1}||_{L_\infty}) .$$

Notice that the norm of the function γ in $L_\infty(R_+^{n+1})$ cannot be omitted on the right-hand side, even in the case when $\ker\{P,P_j\} = 0$.

Theorem 2.8 is the basis for the following theorem on the first boundary value problem:

$$(4.2) \qquad P(D)u = 0 \quad \text{for} \quad y \geq 0, \quad \partial^j u/\partial y^j = \phi_j \quad \text{for} \quad y = 0, \; 0 \leq j \leq k-1 .$$

THEOREM 4.5. [6] *Let* P *be a homogeneous differential elliptic operator of order* $2k$ *with constant coefficients. If*

$$\nabla_{k-1-j}\phi_j \in MW_p^{\ell}(R^n) , \quad 0 < \ell < 1 , \quad 1 \leq p < \infty ,$$

then there exists one and only one solution of the problem (4.2), *such that* $\nabla_{k-1}u \in MW^r_{p,r-\ell-1/p}(R_+^{n+1})$, $r \geq 1$. *This solution satisfies the estimate*

$$||\nabla_{k-1}u;R_+^{n+1}||_{MW^r_{p,r-\ell-1/p}} \leq c\sum_{j=0}^{k-1}||\nabla_{k-1-j}\phi_j;R^n||_{MW^\ell_p}.$$

Let us present a theorem of the same character for the elliptic operator

$$u \longrightarrow Lu = -\sum_{i,j=1}^{n}\frac{\partial}{\partial x_i}\left(a_{ij}\frac{\partial u}{\partial x_j}\right)$$

in an arbitrary bounded domain $\Omega \subset R^n$ with coefficients from $L_\infty(\Omega)$. Let us assume that the matrix of coefficients is symmetric and positive definite.

Let us consider the Dirichlet problem

$$Lu = 0 \quad \text{in} \quad \Omega\ , \quad u-g \in \overset{o}{W}{}^1_2(\Omega)\ ,$$

where $g \in W^1_2(\Omega)$. This problem is uniquely solvable.

THEOREM 4.6. [8] *If* $g \in MW^1_2(\Omega)$, *then* $u \in MW^1_2(\Omega)$. *Moreover,* $u-g \in M(W^1_2(\Omega) \longrightarrow \overset{o}{W}{}^1_2(\Omega))$ *and we have the estimate*

$$||u;\Omega||_{MW^1_2} \leq c||g;\Omega||_{MW^1_2}.$$

REMARK 4.1. Let Ω be a bounded domain with a boundary of class $C^{0,1}$. By Theorem 2.8, $MW^{1/2}_2(R^{n-1})$ is the space of traces on R^{n-1} of functions from $MW^1_2(R^n_+)$. Hence it easily follows that any function ϕ from the space $MW^{1/2}_2(\partial\Omega)$ has a extension g on Ω from $MW^1_2(\Omega)$, such that

$$||g;\Omega||_{MW^1_2} \sim ||\phi;\partial\Omega||_{MW^{1/2}_2}.$$

This together with Theorem 4.6 implies the unique solvability in the space $MW^1_2(\Omega)$ of the Dirichlet problem

$$Lu = 0 \quad \text{in} \quad \Omega\ , \quad u|_{\partial\Omega} = \phi \in MW^{1/2}_2(\partial\Omega)\ .$$

4.5. Implicit Function Theorems

The next assertion, formulated in terms of multipliers, represents an analogue of the classical Implicit Function Theorem.

THEOREM 4.7. [7] *Let* $G = \{(x,y)\colon x \in R^{n-1},\ y > \phi(x)\}$, *where* ϕ *satisfies the uniform Lipschitz condition in* R^{n-1} . *Further, let* u *be a function in* G *satisfying the following conditions:*

(i) grad $u \in MW_p^{\ell-1}(G)$, *where* ℓ *is an integer,* $\ell \geq 2$,

(ii) $u(x,\phi(x)+0) = 0$,

(iii) $\inf(\partial u/\partial y)(x,\phi(x)+0) > 0$.

Then grad $\phi \in MW_p^{\ell-1-1/p}(R^{n-1})$.

Close to this result is the next theorem on implicit mappings.

THEOREM 4.8. [7] *Let* ℓ *and* s *be integers,* $n > s > n-(\ell-1)p \geq 0$. *Further, let* $x \in R^s$, $y \in R^{n-s}$, $z = (x,y)$ *and let* $u : R^n \to R^{n-s}$, $\phi : R^s \to R^{n-s}$ *be mappings satisfying the conditions*

(i) $u'_z \in MW_p^{\ell-1}(R^n)$,

(ii) $u(x,\phi(x)) = 0$ *for almost every* $x \in R^s$,

(iii) *the matrix* $[u'_y(x,\phi(x))]^{-1}$ *exists and its norm is uniformly bounded.*

Then $\phi'_x \in MW_p^{\ell-1-(n-s)/p}$.

Local variants of Theorems 4.7, 4.8 are valid as well.

4.6. On (p,ℓ)-diffeomorphisms

Let U be an open subset of the space R^n . In the present section we study the space $W_p^{\ell}(U)$ not only for $\ell = 0,1,\ldots$ but for $\ell > 0$ non-integer as well. In this latter case,

$$||u;U||_{W_p^{\ell}} = ||u;U||_{W_p^{[\ell]}} + \sum_{j=0}^{[\ell]}\left(\int_U\int_U |\nabla_j u(x)-\nabla_j u(y)|^p \; |x-y|^{-n-p\{\ell\}}dxdy\right)^{1/p} .$$

Together with U we consider an open set $V \subset R^n$ and introduce a Lipschitzian mapping $\kappa : U \longrightarrow V$ such that the determinant $\det \kappa'$ has a constant sign and is separated from zero. If the elements of the Jacobi matrix κ' belong to the space of multipliers $MW_p^{\ell-1}(U)$, $p \geq 1$, $\ell \geq 1$, then by definition the mapping κ is a diffeomorphism.

We give a theorem on properties of the (p,ℓ)-diffeomorphisms.

THEOREM 4.9. [7] (i) *Let* $u \in W_p^{\ell}(V)$, $\ell \geq 1$ *and let* $\kappa : U \longrightarrow V$ *be a* (p,ℓ)*-diffeomorphism. Then* $u \circ \kappa \in W_p^{\ell}(U)$ *and we have the estimate*

$$||u \circ \kappa;\ U||_{W^\ell_p} \leq c||u;V||_{W^\ell_p} .$$

(ii) *If* κ *is a* (p,ℓ)*-diffeomorphism, then* κ^{-1} *is a* (p,ℓ)*-diffeomorphism as well.*

(iii) *Let* $\gamma \in MW^\ell_p(V)$, $\ell \geq 1$ *and let* κ *be a* (p,ℓ)*-diffeomorphism. Then* $\gamma \circ \kappa \in M^\ell_p(U)$ *and we have the estimate*

$$||\gamma \circ \kappa;\ U||_{MW^\ell_p} \leq c||\gamma;V||_{MW^\ell_p} .$$

(iv) *Let* U , V *and* W *be open subsets of* R^n , *let* $\kappa_1 : U \longrightarrow V$ *and* $\kappa_2 : V \longrightarrow W$ *be* (p,ℓ)*-diffeomorphisms. Then their composition* $\kappa_2 \circ \kappa_1 : U \longrightarrow W$ *is a* (p,ℓ)*-diffeomorphism as well.*

Let P be a differential operator of the form (4.1) on U , κ a (p,ℓ)-diffeomorphism $U \longrightarrow V$, $\ell \geq k$, and let Q be a differential operator on V introduced by the identity $Q(u \circ \kappa^{-1}) = (Pu) \circ \kappa^{-1}$. By virtue of Theorem 4.9, (i), (ii), the operator Q maps $W^\ell_p(V)$ continuously into $W^{\ell-k}_p(V)$ if and only if the operator P maps $W^\ell_p(U)$ continuously into $W^{\ell-k}_p(U)$.

Denote by $O^{\ell,k}_{p,loc}(U)$ the class of operators of the form (4.2) satisfying $p_\alpha \in M(W^{\ell-|\alpha|}_{p,loc}(U) \longrightarrow W^{\ell-k}_{p,loc}(U))$ for any multiindex α , $|\alpha| \leq k$.

<u>*PROPOSITION 4.4*</u>. *An operator* P *belongs to the class* $O^{\ell,k}_{p,loc}(U)$ *if and only if* $Q \in O^{\ell,k}_{p,loc}(V)$.

By virtue of Theorem 4.1, the condition $Q \in O^{\ell,k}_{p,loc}(U)$ is sufficient for the operator P to map $W^\ell_{p,loc}(U)$ into $W^{\ell-k}_{p,loc}(U)$. In each of the cases $p = 1$ and $p(\ell-k) > n$, the inclusion $P \in O^{\ell,k}_{p,loc}(U)$ represents a necessary condition as well (see Proposition 4.2).

In terms of (p,ℓ)-diffeomorphisms we can in a standard manner define the class of n-dimensional "(p,ℓ)-manifolds", both with or without boundary.

Let ℓ be an integer, $\ell \geq 2$, and $\mathfrak{M}$ a (p,ℓ)-manifold. If $p(\ell-1) \leq n$ we add the assumption that the (p,ℓ)-structure on $\mathfrak{M}$ is of class C^1 . Then Theorem 4.7 on implicit functions yields that the

(p,ℓ)-structure on $\mathfrak{M}$ induces the $(p,\ell-1/p)$-structure on $\partial\mathfrak{M}$.

As an example of a (p,ℓ)-manifold with boundary we can consider a domain in R^n with a compact closure and a boundary having a local explicit description by a Lipschitzian function with a gradient from $MW_p^{\ell-1-1/p}(R^{n-1})$.

The class of (p,ℓ)-manifolds is well suited for developing on them the L_p-theory of elliptic boundary problems. Without caring about full generality, we shall show in the next section that this is indeed the case with the boundary value problems in a subdomain of R^n.

4.7. On regularity of the boundary in the L_p-theory of elliptic boundary value problems

This section deals with the application of the theory of multipliers no elliptic boundary value problems in domains with "non-regular boundaries".

We consider the operator $\{P,P_1,\ldots,P_h\}$ of a general elliptic boundary value problem with smooth coefficients in a bounded domain $\Omega \subset R^n$. We assume that $\operatorname{ord} P = 2h \leqq \ell$, $\operatorname{ord} P_j = k_j < \ell$, $1 < p < \infty$.

It is well known that, provided the boundary is sufficiently smooth,

$$\{P;P_j\} : W_p^{\ell}(\Omega) \longrightarrow W_p^{\ell-2h}(\Omega) \times \sum_{j=1}^{h} W_p^{\ell-k_j-1/p}(\partial\Omega) \tag{4.3}$$

is of Fredholm type, that is, it has a finite index and a closed range. In particular, we have the following apriori estimate for all $u \in W_p^{\ell}(\Omega)$:

$$\begin{aligned}||u;\Omega||_{W_p^{\ell}} \leqq c\Big(||Pu;\Omega||_{W_p^{\ell-2h}} &+ \\ + \sum_{j=1}^{h}||P_ju;\partial\Omega||_{W_p^{\ell-k_j-1/p}} &+ ||u;\Omega||_{L_1}\Big),\end{aligned} \tag{4.4}$$

where the last norm on the right-hand side can be omitted provided we have uniqueness (see [25]).

The proof of these assertions of the "elliptic L_p-theory" is based on an investigation of a boundary value problem with constant coefficients in R_+^n, and on a subsequent localization of the original problem by means of a partition of unity and a local mapping of the domain onto a half-space.

The smoothness of coefficients (and, consequently, of the solution) of the resulting boundary value problem in R^n_+ is determined by the smoothness of the surface $\partial\Omega$.

We will characterize the boundary of the domain in terms of spaces of multipliers. Then, using the above mentioned technique of localization of the boundary value problem, we can apply theorems on traces of multipliers on the boundary (cf. Sec. 2.5). This approach enables us to weaken the well known requirements on the domain Ω , which guarantee validity of the L_p-theory.

Let Ω be a bounded domain of class $C^{0,1}$, that is, for every point of the boundary $\partial\Omega$ there is a neighbourhood in which Ω can be described (in a certain Cartesian system of coordinates) by an inequality $y > \phi(x)$ with a Lipschitzian function ϕ .

If $\phi \in W_p^{\ell-1/p}(R^{n-1})$, then by definition, $\Omega \in W_p^{\ell-1/p}$.

Further, let us formulate the requirement on the domain Ω , which in what follows will be called the condition $N_p^{\ell-1/p}$, $p(\ell-1) \leq n$: For every point $O \in \partial\Omega$ there exists a neighbourhood V and a domain $G = \{(x,y): x \in R^{n-1}, y > \phi(x)\}$ such that $V \cap \Omega = V \cap G$ and

$$||\nabla\phi||_{MW_p^{\ell-1-1/p}(R^{n-1})} \leq \delta .$$

Here δ is a constant, which depends on values at the point O of the coefficients of the principal homogeneous parts of the operators $P, P_1, \ldots P_h$ in the system of coordinates (x,y) . For $\ell = 1$, the role of the last inequality is played by the estimate $||\nabla\phi;R^{n-1}||_{L_\infty} \leq \delta$.

The following result was established in [12].

<u>*THEOREM 4.10*</u>. *Let a domain* Ω *satisfy the condition* $N_p^{\ell-1/p}$ *for* $p(\ell-1) \leq n$ *and belong to the class* $W_p^{\ell-1/p}$ *for* $p(\ell-1) > n$. *Then* (4.3) *is a Fredholm operator.*

It can be shown that the condition $N_p^{\ell-1/p}$ is equivalent to the inequality

$$\lim_{\varepsilon\to 0}\left(\sup_{e\subset Q_\varepsilon}\frac{||D_{p,\ell-1/p}(\phi,Q_\varepsilon);Q_\varepsilon||_{L_p}}{[\operatorname{cap}(e,W_p^{\ell-1-1/p}(R^{n-1}))]^{1/p}} + ||\nabla\phi;Q_\varepsilon||_{L_\infty}\right) \leq \delta_0 ,$$

where δ_0 is a sufficiently small constant, and

$$D_{p,\ell-1/p}(\phi;Q_\varepsilon) = \left(\int_{Q_\varepsilon} |\nabla_{\ell-1}\phi(x+h)-\nabla_{\ell-1}\phi(x)|^p |h|^{-n+2-p}\, dh\right)^{1/p} .$$

Hence we easily find that the condition $N_p^{\ell-1/p}$ follows from the convergence of one of the integrals

$$\int_{R^{n-1}} \left[(D_{p,\ell-1/p}\phi)(x)\right]^{\frac{p(n-1)}{p(\ell-1)-1}} dx \qquad \text{for } p(\ell-1) < n ,$$

$$\int_{R^{n-1}} [(D_{p,\ell-1/p}\phi)(x)]^p [\log_+ (D_{p,\ell-1/p}\phi)(x)]^{p-1}\, dx \quad \text{for } p(\ell-1) = n .$$

Notice that for $p(\ell-1) > n$, the condition from Theorem 4.10 has the form

$$\int_{R^{n-1}} [(D_{p,\ell-1/p}\phi)(x)]^p\, dx < \infty .$$

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