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INITIAL VALUÉ PROBLEMS FOR ELASTOPLASTIC

AND ELASTO-VISCOPLASTIC SYSTEMS

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Introduction

In this paper we want to summarize results on initial value problems for elastoplastic and elasto-viscoplastic systems, which have been obtained in recent years by several authors. Models describing the behaviour of elastoplastic systems have been known for a long time (see e. g. KOITER [16], HODGE [12]), but there did not exist a rigorous theory of the mathematical problems associated with these models. Several attempts were made to find formulations of models more suitable for theoretical purposes (see MOREAU [23, 24], DEL PIERO [3], HAUPT [10], NGUYEN [27], HALPHEN-NGUYEN [9]). It seems that the most important results are those of NGUYEN [27] and HALPHEN-NGUYEN [9]. In order to describe the influence of the "history" on a process they introduced internal state variables and internal stresses. Following MOREAU (cf. [23]) they used the language of convex analysis. Existence results for materials without hardening effects were obtained first by DUVAUT-LIONS [4] (dynamic processes) and then by JOHNSON [15] (quasi-static processes). It was shown by MOREAU [24] that for materials without hardening it is invitable in general to use irreflexive spaces of L^∞-type or C-type (cf. also NAYROLES [25]). If some kind of hardening of the material is involved the situation is much simpler since satisfactory a priori estimates for the solutions to the problems are available under natural assumptions. This was shown for purely kinematic hardening by HALPHEN [8] and, independently, by GRÖGER [5, 6]. Models involving internal parameters were treated from the mathematical point of view by NEČAS-TRÁVNÍČEK [26] and TRÁVNÍČEK [30]. They proved that hardening effects associated with plastic work also lead to satisfactory existence-uniqueness results. Their method of proof is an approximation of elastoplastic materials by simpler materials with internal parameters, which were investigated by KRATOCHVÍL-DILLON [19], KRATOCHVÍL-NEČAS [20] and JOHN [14]. Starting from the results of NEČAS-TRÁVNÍČEK [26], the author [7] treated materials subject to a combination of kinematic hardening and other types of hardening. Let us mention that mathematical results on the behaviour of a single elastoplastic element were presented also by KRASNOSELSKI, POKROVSKI and others (see [18, 29] and the papers quoted there).

The present paper consists of seven sections. In Section 1 we introduce the necessary notation. Section 2 is devoted to the formulation of constitutive relations and of the problems treated in this paper. The constitutive relations we consider include as special cases various models of elastoplastic and elasto-viscoplastic behaviour. Following HALPHEN-NGUYEN [9] we make use of internal state variables and internal stresses in the following way: We introduce spaces of generalized deformations (resp. generalized stresses) the elements of which may be interpreted as pairs consisting of usual deformations and internal state variables (resp. usual stresses and internal stresses). Since our formulation of the constitutive relations is a rather abstract one we present some more concrete examples in Section 3.

In Section 4, which is the main part of the paper, we consider quasi-static processes with a "definite" kinematic hardening property (not excluding other types of hardening). Systems with definite kinematic hardening are distinguished by the fact that the corresponding mathematical problems can be transformed easily to standard problems of the theory of evolution equations with maximal monotone operators in Hilbert spaces. Moreover, the results for such systems are somewhat better than those known for systems without definite kinematic hardening. Section 4 is divided into three parts. In the first part we prove an existence-uniqueness result. In the second part we show that quasi-static processes depend in a certain sense continuously on the law describing the plastic or viscoplastic properties of the system. In particular, we shall see that elastoplastic materials can be approximated by viscoplastic materials. The third part of Section 4 is devoted to questions arising in connection with a discretization of time. Discrete-time problems were considered by many authors (see NGUYEN [28] and the papers quoted there). NGUYEN [28] proposed an iteration method for solving equations which result from an implicit difference scheme in time. He proved that this iteration method converges under a certain "hardening assumption", but he was not able to formulate conditions ensuring that this assumption is satisfied. Moreover, he did not answer the question whether the solutions to the discrete-time problems approximate the solution to the original problem, because he did not know any existence result for the original problem. We shall show that in the case of definite kinematic hardening the convergence of the difference method as well as the convergence of certain iteration methods can be proved by standard arguments.

In Section 5 we show that dynamic processes of systems with definite kinematic hardening lead to standard initial value problems for second order evolution equations.

Section 6 is devoted to quasi-static processes of systems possessing a hardening property but not a definite kinematic hardening.

In Section 7 we present results on dynamic processes for systems with hardening different from definite kinematic hardening and for systems without any hardening. To this end we introduce a "weak formulation" of the corresponding mathematical problem. In the case of Prandtl-Reuss' equations such a formulation was used already by

DUVAUT-LIONS [4].

In this paper we do not consider models describing the influence of the history on a process by means of integral operators. Furthermore, we do not investigate models including nonlinear relations between internal state variables and internal stresses. Such models are treated by TRÁVNÍČEK [30].

1. Notation

In this section we introduce the notation needed later. Most of it is commonly used.

Let E be a Hilbert space. By E' we denote its dual and by $\langle .,. \rangle$ the dual pairing of E' and E. We shall use the same brackets for different pairs of spaces E, E'. The norm and the scalar product on E are denoted by $||.||_E$ and $(.,.)_E$, respectively. The notation id_E is used for the identity map of E. We shall frequently use notions, notation and results from convex analysis and the theory of maximal monotone operators as presented for instance by BREZIS [2] and BARBU [1]. Let us mention here only a few definitions and the corresponding notation. The indicatrix I_C of a subset C of a space E is defined by

$$I_C(e) := \begin{cases} 0 & \text{if} \quad e \in C , \\ +\infty & \text{if} \quad e \in E \setminus C . \end{cases}$$

If $\phi : E \to]-\infty, +\infty]$ is proper convex and lower semicontinuous, the multivalued subdifferential mapping $\partial\phi$ from E to E' is defined by

$$\forall e \in E : \partial\phi(e) := \{f \in E' \,|\, \forall \bar{e} \in E: \langle f, \bar{e}-e \rangle \leqq \phi(\bar{e}) - \phi(e)\}.$$

We set

$$D(\phi) := \{e \in E \,|\, \phi(e) < \infty\}, \quad D(\partial\phi) := \{e \in E \,|\, \partial\phi\ (e) \neq \emptyset\}.$$

The set $D(\phi)$ is called the effective domain of ϕ.

Let $S := [0, T]$ be a fixed finite interval of time. By $C(S;E)$

we denote the set of all continuous mappings from S into E equipped with the maximum norm. By $L^p(S;E)$, $1 \leqq p \leqq \infty$, we denote the set of all Bochner measurable mappings from S into E such that

$$\int_S ||u(t)||_E^p \, dt < \infty \quad \text{if} \quad 1 \leqq p < \infty ,$$

$$\operatorname*{ess\,sup}_{t \in S} ||u(t)||_E < \infty \text{ if } \quad p = \infty ,$$

equipped with its natural norm. If $u \in L^1(S;E)$ we denote by u', u'' the first and the second derivative of u with respect to time in the sense of distributions on $]0, T[$ with values in E. For $1 \leqq p \leqq \infty$ we define

$$W^{1,p}(S;E) := \{u \in C(S;E) | u' \in L^p(S;E)\} ,$$

$$W^{2,p}(S;E) := \{u \in C(S;E) | u'' \in L^p(S;E)\}$$

and we use the usual norms on these spaces. Instead of $W^{1,2}(S;E)$ we write $H^1(S;E)$.

If E_1, E_2 are Hilbert spaces we denote by $\mathscr{L}(E_1;E_2)$ the space of all linear continuous mappings from E_1 into E_2. If $L \in \mathscr{L}(E_1;E_2)$ we denote by L^* the adjoint operator to L and we define

$$\operatorname{Im} L := \{Le | e \in E_1\}, \quad \operatorname{Ker} L := \{e \in E_1 | Le = 0\} .$$

To any operator $L \in \mathscr{L}(E_1;E_2)$ a mapping $L_S \in \mathscr{L}(L^2(S;E_1); L^2(S;E_2))$ is associated defined by

$$\forall u \in L^2(S;E_1): (L_S u)(t) := Lu(t) \quad \text{for a. e. } t \in S .$$

Similarly, to any multivalued mapping M from E_1 into E_2 (i. e. to any subset M of $E_1 \times E_2$) a multivalued mapping M_S from $L^2(S;E_1)$ into $L^2(S;E_2)$ is associated defined by

$$M_S := \{(u_1,u_2) \in L^2(S;E_1) \times L^2(S;E_2) | (u_1(t),u_2(t)) \in M \quad \text{for a. e. } t \in S\}.$$

For the sake of simplicity we shall write L and M instead of L_S and M_S, respectively. This will not lead to misunderstandings.

Let $G \subset \mathbb{R}^N$ be a domain. By ∂G we denote the boundary of G. By $L^2(G; \mathbb{R}^N)$ $(H^1(G; \mathbb{R}^N))$ we denote the Hilbert space of all mappings from G into $\mathbb{R}^N$ the coordinates of which belong to $L^2(G)$ $(H^1(G))$ (equipped with its natural scalar product). In the same way we shall use the notation $L^2(\partial G; \mathbb{R}^N)$.

2. Constitutive relations; problems

We assume that any mechanical system is associated with a Hilbert space Y the elements of which are called generalized deformations of the system. The space dual to Y will be denoted by Z . Its elements are to be regarded as generalized stresses. Examples of such spaces will be given in the next section. Let $S := [0, T]$ be a fixed finite interval of time. Mappings from S into a Hilbert space will be called processes; for example a mapping from S into Y is called a deformation process. The behaviour of a system is governed by a constitutive relation, i. e. by a relation between its deformation processes and its stress processes.

We shall call a mechanical system plastic if its deformation processes y and its stress processes z are related in the following way:

(2.1) $$y'(t) \in \partial I_C(z(t)) \qquad \text{for a. e. } t \in S$$

where C denotes a convex closed nonempty subset of Z .

If $y \in \partial I_C(z)$ then $\lambda y \in \partial I_C(z)$ for $\lambda \geqq 0$. Therefore the constitutive relation (2.1) is rate independent, i. e. invariant with respect to any (absolutely continuous) transformation of the time scale.

We shall call a mechanical system viscous if its deformation processes y and its stress processes z are related in the following way:

(2.2) $$z(t) = M\, y'(t) \qquad \text{for a. e. } t \in S$$

where $M : Y \to Z$ denotes a Lipschitzian and strongly monotone potential operator.

If the behaviour of a mechanical system can be explained by the parallel action of a plasticity law and a viscosity law then we have instead of (2.1) or (2.2)

(2.3) $$z(t) \in (\partial I_C^* + M)y'(t) \quad \text{for a. e. } t \in S$$

where I_C^* denotes the conjugate to the functional I_C. Media characterized by a relation of the type (2.3) are sometimes called Bingham-Media. The serial action of a plasticity law and a viscosity law leads to a relation of the type

(2.4) $$y'(t) \in (\partial I_C + M^{-1})z(t) \quad \text{for a. e. } t \in S .$$

Generalizing (2.3) and (2.4) we shall consider constitutive relations of the form

(2.5) $$y'(t) \in \partial\phi(z(t)) \quad \text{for a. e. } t \in S$$

where

(2.6) $\phi : Z \to]-\infty, +\infty]$ is proper convex and lower semicontinuous.

The parallel or serial action of laws of this type leads (under mild conditions on the functionals involved) again to a law of the same type. Let us mention that KRASNOSELSKI [18] investigated laws which are obtained by superposition of a family of "elementary" laws.

In this paper we shall deal with constitutive relations describing the serial superposition of an elasticity law and a law which represents the parallel action of a law of the type (2.5),(2.6) and another elasticity law. Such constitutive relations can be written as follows (cf. Figure 1):

(2.7) $$\begin{cases} y(t) = e(t) + p(t),\ z(t) = q(t) + r(t),\ z(t) = Ae(t), \\ r(t) = Bp(t),\ p'(t) \in \partial\phi(q(t)) \quad \text{for a. e. } t \in S \end{cases}$$

where ϕ satisfies (2.6) and A, B represent linear elasticity laws. We suppose that

(2.8) $A, B \in \mathcal{L}(Y; Z)$ are symmetric and positive.

From (2.7) it follows that $p'(t) \in \partial\phi(z(t) - Bp(t))$. If $\phi = I_C$, this implies $z(t) \in C + Bp(t)$, i. e. the set of possible stresses moves according to the plastic deformation p. Therefore we say that B determines a kinematic hardening property of the system. If B is positive definite we shall speak of definite kinematic hardening.

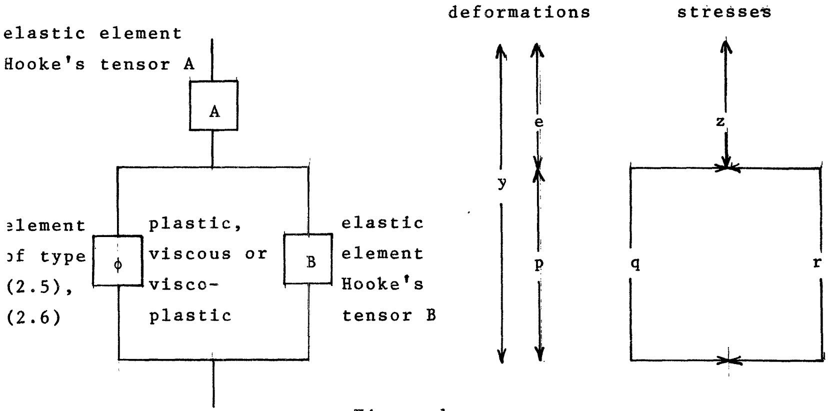

Figure 1

The relation between y and z given by (2.7) could have been generated also by the parallel action of an elasticity law and a law representing the serial superposition of a law of the type (2.5), (2.6) and another elasticity law. Thus the choice of the law (2.7) is not as arbitrary with respect to parallel and serial superpositions laws as it might seem.

Let us emphasize the fact that (2.7) includes elastoplasticity laws as well as viscoelasticity laws and laws of elasto-viscoplastic behaviour.

Now we shall formulate problems describing quasi-static or dynamic processes of a system governed by a law of the type (2.7). We suppose that the system is associated with a Hilbert space U of displacements and that the deformation process $y : S \to Y$ corresponding to a displacement process $u : S \to U$ is given by

(2.9) $$y(t) = Ku(t) \qquad \text{for a. e.} \quad t \in S$$

where

(2.10) $$K \in \mathscr{L}(U;Y), \qquad \forall u \in U: \ ||Ku||_Y \geq k_o||u||_U, \ k_o > 0 .$$

(We shall consider small deformations only.) Concrete realizations of U and K will be given in the next section.

Instead of (2.9) we could assume that

$$y(t) = Ku(t) + g(t) \qquad \text{for a. e.} \quad t \in S$$

where $g : S \to Y$ is a given function representing internal deformation sources. This would require only technical modifications of the results and proofs presented in this paper.

The space U' dual to U is to be regarded as a space of forces. If a force $f \in U'$ acts on a system the generalized stress of which is $z \in Z$ then the work corresponding to a virtual displacement $h \in U$ is given by

$$W = \langle z, Kh\rangle - \langle f, h\rangle = \langle K^*z - f, h\rangle .$$

Therefore the principle of virtual work yields that the condition of quasi-static equilibrium may be written as follows:

(2.11) $$K^*z(t) = f(t) \qquad \text{for a. e.} \quad t \in S$$

where f denotes a given force process. The corresponding condition of dynamic equilibrium is

(2.12) $$\rho u''(t) + K^*z(t) = f(t) \qquad \text{for a. e.} \quad t \in S$$

where ρ and u'' denote the density and the acceleration of the system, respectively. In order that (2.12) make sense we suppose that U is densely and continuously imbedded into U'. Besides U and U' we shall need later the interpolation space $H := [U, U']_{1/2}$ (see LIONS-MAGENES [21]) the elements of which may be considered as special forces or as "generalized" displacements. For the sake of simplicity we assume ρ to be constant.

We have to supplement the relations (2.7), (2.9) and either (2.11) or (2.12) by reasonable initial conditions. In order to obtain precise formulations of the problems we are interested in we shall impose some regularity conditions on the data and on the functions we are looking for.

Problem 1 (Quasi-static processes). We are given

$$(2.13)\quad \begin{cases} f \in H^1(S;U'),\ u_o \in U,\ p_o \in Y,\ q_o \in D(\phi) \ \textit{such that}\ K^* z_o = f(0) \\ \textit{and}\ \ q_o = z_o - Bp_o \ \ \textit{where}\ \ z_o := A(Ku_o - p_o)\ . \end{cases}$$

We are looking for processes u, y, e, p, z, q, r *such that*

$$(2.14)\quad \begin{cases} (u,\ y,\ e,\ p,\ z,\ q,\ r) \in H^1(S;U \times Y^3 \times Z^3), \\ y = Ku,\ K^* z = f \quad \textit{compatibility and equilibrium condition,} \\ y=e+p,\ z=q+r,\ z=Ae,\ r=Bp,\ p' \in \partial\phi(q) \ \textit{constitutive relations,} \\ (u,\ p,\ q)(0) = (u_o,\ p_o,\ q_o) \quad \textit{initial conditions.} \end{cases}$$

Problem 2 (Dynamic processes). We are given

$$(2.15)\quad \begin{cases} f \in W^{1,1}(S;H),\ u_o, v_o \in U,\ p_o \in Y,\ q_o \in D(\partial\phi) \ \textit{such that} \\ K^* z_o \in H \ \ \textit{and}\ \ q_o = z_o - Bp_o \ \ \textit{where}\ \ z_o := A(Ku_o - p_o)\ . \end{cases}$$

We are looking for processes u, y, e, p, z, q, r *such that*

$$(2.16)\quad \begin{cases} (u,\ u',\ y,\ e,\ p,\ z,\ q,\ r) \in W^{1,\infty}(S;U \times H \times Y^3 \times Z^3), \\ y=Ku,\ \rho u''+K^* z=f \quad \textit{compatibility and equilibrium condition,} \\ y=e+p,\ z=q+r,\ z=Ae,\ r=Bp,\ p' \in \partial\phi(q) \quad \textit{constitutive relations,} \\ (u,\ u',\ p,\ q)(0) = (u_o,\ v_o,\ p_o,\ q_o) \quad \textit{initial conditions.} \end{cases}$$

We shall denote by D_1 the set of all data (f, u_o, p_o, q_o) satisfying (2.13) and by D_2 the set of all data (f, u_o, v_o, p_o, q_o) satisfying (2.15). Eliminating some of the unknowns by elementary operations we can write the above problems as follows:

Problem 1'. *We are given data* $(f, u_o, p_o, q_o) \in D_1$ *and we are looking for processes* u, p, q *such that*

$$(2.17)\ \begin{cases}(u,p,q) \in H^1(S;U\times Y\times Z), & (u,p,q)(0) = (u_o,p_o,q_o),\\ q + (A + B)p = AKu, \ K^*(q + Bp) = f, & p' \in \partial\phi(q)\ .\end{cases}$$

Problem 2'. We are given data $(f, u_o, v_o, p_o, q_o) \in D_2$ *and we are looking for processes* u, p, q *such that*

$$(2.18)\ \begin{cases}(u,u',p,q) \in W^{1,\infty}(S;U\times H\times Y\times Z), \ (u,u',p,q)(0) = (u_o,v_o,p_o,q_o),\\ q + (A + B)p = AKu, \ \rho u'' + K^*(q + Bp) = f, \ p' \in \partial\phi(q)\ .\end{cases}$$

If (u, p, q) is a solution to Problem 1' (Problem 2') then we obtain the corresponding solution to Problem 1 (Problem 2) by setting

$$(2.19)\qquad y := Ku, \quad e := y - p, \quad z := Ae, \quad r := Bp\ .$$

3. Examples

In the following examples we suppose that we are given a body occupying a bounded Lipschitzian domain $G \subset \mathbb{R}^3$. By Γ we denote a closed subset of the boundary ∂G of G the surface measure of which is positive. By $\mathcal{S}$ we denote the six-dimensional Hilbert space of all symmetric 3×3 matrices.

E x a m p l e 1 . Let $U := \{u \in H^1(G;\mathbb{R}^3) | u_\Gamma = 0\}$, $Y := Z := = L^2(G; \mathcal{S})$ and let

$$(3.1)\qquad \forall u \in U: \quad (Ku)_{ij} := \tfrac{1}{2}(u_{i,j} + u_{j,i}).$$

Condition (2.10) is satisfied;it follows from Korn's inequality (see HLAVÁČEK-NEČAS [11]). The dual space U' contains body forces as well as surface forces acting on the part $\partial G \setminus \Gamma$ of the boundary. Let $\mathcal{A}, \mathcal{B} \in \mathcal{L}(\mathcal{S};\mathcal{S})$ be positive and symmetric and let

$$(3.2)\quad \forall \varepsilon \in Y: \ (A\varepsilon)(x) := \mathcal{A}\varepsilon(x), \ (B\varepsilon)(x) := \mathcal{B}\varepsilon(x) \quad \text{for a. e. } x \in G.$$

Moreover, let $F : \mathcal{S} \to \mathbb{R}$ be convex and continuous and let

$$(3.3)\quad \phi := I_C \quad \text{where} \quad C := \{\sigma \in Z | F(\sigma(x)) \leqq 0 \quad \text{for a. e.} \quad x \in G\}.$$

In this case the constitutive relations (2.7) describe a homogeneous elastoplastic body with a linear kinematic hardening rule. The function F characterizes the yield condition of the material.

<u>Example 2.</u> Let $U := \{u \in H^1(G;\mathbb{R}^3) | u_\Gamma = 0\}$, $Y := Z :=$ $= L^2(G;\mathcal{S}) \times L^2(G)$ and

(3.4) $\quad \forall u \in U: Ku := (K_1 u, 0)$ where $(K_1 u)_{ij} := \frac{1}{2}(u_{i,j} + u_{j,i})$.

Let $F : \mathcal{S} \to \mathbb{R}$ be convex and differentiable with bounded F' . We set

(3.5) $\quad \phi := I_C$ where $C := \{(\sigma,\alpha) \in Z | F(\sigma(x)) \leq \alpha(x)$ for a. e. $x \in G\}$.

It is easy to check that in this case $(\varepsilon,\lambda) \in \partial I_C(\sigma,\alpha)$ if and only if

$$(3.6) \begin{cases} \varepsilon, \sigma \in L^2(G;\mathcal{S}),\ \lambda, \alpha \in L^2(G),\ F(\sigma(x)) \leq \alpha(x) \text{ for a. e. } x \in G, \\ \varepsilon(x) = -\lambda(x) F'(\sigma(x)) \text{ where } \lambda(x) \leq 0 \text{ for a. e. } x \in G \text{ and} \\ \lambda(x) = 0 \text{ for a. e. } x \in G \text{ such that } F(\sigma(x)) < \alpha(x). \end{cases}$$

Let $A, B \in \mathcal{L}(Y;Z)$ be defined by

(3.7) $\quad \forall (\varepsilon,\lambda) \in Y: \ A(\varepsilon,\lambda) := (A_1\varepsilon, k_A\lambda),\ B(\varepsilon,\lambda) := (B_1\varepsilon, k_B\lambda)$

where A_1, B_1 are defined as A, B in Example 1 and $k_A \geq 0$, $k_B \geq$ ≥ 0 . The operators A, B represent not only Hooke's law but also a relation between values which can be regarded as internal state variables and internal stresses. The function α characterizes the actual position of the yield surface. Let $(u, p, q) =$ $= (u, (\varepsilon_p, \beta), (\tau, \alpha))$ be a solution to Problem 1'. Then (cf. (2.17), (3.6))

$\alpha + (k_A + k_B)\beta = 0$, $\varepsilon_p'(t,x) = -\beta'(t,x) F'(\tau(t,x))$, $F(\tau(t,x)) \leq \alpha(t,x)$, $\beta'(t,x) \leq 0$, $\beta'(t,x)(F(\tau(t,x)) - \alpha(t,x)) = 0$ for a. e. $(t,x) \in S \times G$.

Hence $\alpha'(t,x) \geq 0$ for almost every $(t,x) \in S \times G$, i. e. the elastic range of the body is increasing. If $F(\sigma)$ depends only on the invariants of $\sigma \in \mathcal{S}$ we are given a material subject to a superposition of isotropic and kinematic hardening.

We shall show that there is a simple connection between α

and the plastic work of the body if

(3.8) $\forall \sigma \in \mathcal{S}, \quad \forall t \in \mathbb{R} : F(t\sigma) = |t| F(\sigma).$

The relation (3.8) implies $F'(\sigma).\sigma = F(\sigma)$. Therefore, denoting by $W_p(t,x)$ the plastic work at $(t,x) \in S\times G$, we obtain for almost every $(t, x) \in S\times G$:

$$W_p(t,x)-W_p(0,x) := \int_0^t \varepsilon_p'(s,x).\tau(s,x)ds = -\int_0^t \beta'(s,x)F'(\tau(s,x)).\tau(s,x)ds$$

$$= -\int_0^t \beta'(s,x)F(\tau(s,x))ds = -\int_0^t \beta'(s,x)\alpha(s,x)ds$$

$$= \frac{1}{2(k_A + k_B)}\{(\alpha(t,x))^2 - (\alpha(0,x))^2\}.$$

Let us emphasize the fact that we defined the plastic work by means of the plastic strain ε_p and the stress τ acting on the plastic element (cf. Figure 1, note that $q = (\tau,\alpha)$).

The last considerations can be generalized to the case that $F(\sigma) = \psi(F_o(\sigma))$ where F_o satisfies the condition (3.8) and ψ is a suitable real valued function.

<u>Example 3.</u> Let $U := H^1(G;R^3)$, $Y := Z := L^2(G;\mathcal{S}) \times L^2(\Gamma;\mathbb{R}^3) \times L^2(G)$ and let

(3.9) $\forall u \in U$: $Ku := (K_1 u, u_\Gamma, 0)$ where $(K_1 u)_{ij} := \frac{1}{2}(u_{i,j}+u_{j,i})$.

Using a function $F : \mathcal{S} \to \mathbb{R}$ as in Example 2 we define

(3.10) $\phi := I_C$ where $C := \{(\sigma,\tau,\alpha) \in Z \mid F(\sigma(x)) \leqq \alpha(x)$ for a. e. $x \in G\}$.

Let $\mathcal{A}_2 \in \mathcal{S}$ be positive definite, $k_A \geqq 0$, $k_B \geqq 0$, and let

(3.11) $\forall(\varepsilon,\delta,\lambda)\in Y$: $A(\varepsilon,\delta,\lambda) := (A_1\varepsilon, A_2\delta, k_A\lambda)$, $B(\varepsilon,\delta,\lambda) := (B_1\varepsilon, 0, k_B\lambda)$

where $(A_2\delta)(x) := \mathcal{A}_2\delta(x)$ for ∂G - a. e. $x \in \Gamma$ and A_1, B_1 are defined as A, B in Example 1. We have $(\varepsilon,\delta,\lambda) \in \partial\phi(\sigma,\tau,\alpha)$ if and only if $\delta = 0$, $\tau \in L^2(\Gamma;\mathbb{R}^3)$ and (3.6) holds. The constitutive relations (2.7) include an elastic support condition for the part Γ of ∂G the elasticity law of which is given by $\mathcal{A}_2$.

The next example deals with the torsion of a cylindrical bar the cross section of which is a bounded domain $G \subset \mathbb{R}^2$.

E x a m p l e 4 . Let $U := \mathbb{R} \times U_o$ where $U_o := \{w \in H^1(G) \mid \int_G w dx = 0\}$ and let $Y := Z := L^2(G;\mathbb{R}^2)$. Furthermore, let

$$\forall(\omega,w)\in U: (Ku)(x_1,x_2) := \frac{1}{2}(-\omega x_2 + \frac{\partial w}{\partial x_1}, \omega x_1 + \frac{\partial w}{\partial x_2}) \text{ for a. e. } (x_1,x_2) \in G .$$

The elements of U' are of the form $f = (\mu, f_o)$, $\mu \in \mathbb{R}$, $f_o \in U_o'$; the meaning of μ is that of a momentum. If $z = (\sigma,\tau) \in Z$ then the equilibrium condition $K^* z = (\mu, f_o)$ includes the equation

$$\mu = \frac{1}{2} \int_G (x_1\tau(x_1, x_2) - x_2\sigma(x_1, x_2))dx .$$

The operators A, B and the functional ϕ may be defined similarly as in Example 1 ;one has only to replace $\mathcal{S}$ by $\mathbb{R}^2$.

Let us mention that MOREAU [23] has given examples of elasto-plastic systems consisting of finite elements. In that case the spaces U, Y and Z are finite dimensional. Another example dealing with simultaneous torsion and axial strain of a cylindrical bar is presented in full detail by HÜNLICH [13]. Examples of models with internal state variables are given also by NGUYEN [28].

4. Quasi-static processes with definite kinematic hardening

In this section as well as in all following sections we suppose that we are given Hilbert spaces U, H, U', Y, Z, a functional ϕ and operators K, A, B as described in Section 2. In particular, we assume (2.6), (2.8) and (2.10) to be valid without mentioning this in the formulation of the results.

Throughout this section we suppose in addition to (2.8) that

(4.1) $B \in \mathcal{L}(Y;Z)$ and $K^*AK \in \mathcal{L}(U;U')$ are positive definite.

We introduce $P \in \mathcal{L}(Z;Z)$, $Q \in \mathcal{L}(Z;Z)$ and $R \in \mathcal{L}(U';Z)$ by

(4.2) $R := AK(K^*AK)^{-1}$; $Q := RK^*$, $P := id_Z - Q$.

The operator R is defined in such a way that Rf is the stress which would correspond to the force f if the system were purely elastic with Hooke's tensor A. It is easy to check that $P^2 = P$, $Q^2 = Q$ and

(4.3) $$\mathrm{Im}\, P = \mathrm{Ker}\, K^* , \quad \mathrm{Im}\, Q = \mathrm{Im}\, AK .$$

Moreover, it is easy to see that $PA \in \mathscr{L}(Y;Z)$ is symmetric and positive. Hence $PA + B \in \mathscr{L}(Y;Z)$ is symmetric and positive definite.

4.1. Existence and uniqueness

<u>*THEOREM 4.1.*</u> *Suppose that (2.13) and (4.1) are valid. Then Problem 1' has a unique solution* (u,p,q) *and the mapping* $(f,u_o,p_o,q_o) \to (u,p,q)$ *is Lipschitzian from* D_1 *(equipped with the metric of* $W^{1,1}(S;U') \times U \times Y \times Z)$ *into* $C(S;U \times Y \times Z)$.

P r o o f . Let (u,p,q) be a solution to Problem 1'. Then (cf. (2.17), (4.3)):

(4.4) $$Pq + P(A + B)p = 0 , \quad Q(q + Bp - Rf) = 0 ,$$

(4.5) $$q + (PA + B)p = Rf ,$$

(4.6) $$q' + (PA + B)\partial\phi(q) \ni Rf' , \quad q(0) = q_o, \quad q \in H^1(S;Z) .$$

Conversely, let q be a solution to (4.6) and let $p \in H^1(S;Y)$ be determined by (4.5). Since $P(q + (A + B)p) = PRf = 0$, a function $u \in H^1(S;U)$ can be defined by

(4.7) $$AKu = q + (A + B)p \quad (\text{i. e. } u = (K^*AK)^{-1}K^*(q + (A+B)p)).$$

It is easy to check that (u,p,q) is a solution to Problem 1'. Thus Problem 1' is reduced to the problem (4.6), which is a standard type of the initial value problem. In fact, defining on Z the scalar product $(.,.)_Z$ by

(4.8) $$\forall z_1,z_2 \in Z: \ (z_1,z_2)_Z := \langle z_1, (PA+B)^{-1}z_2\rangle$$

and denoting by $\partial_Z\phi$ the subdifferential of ϕ with respect to this scalar product we can write (4.6) as

(4.9) $$q' + \partial_Z\phi(q) \ni Rf' , \quad q(0) = q_o , \quad q \in H^1(S;Z) .$$

The assertions of Theorem 4.1 now follow from standard results on evolution equations and from (4.5), (4.7) (cf. BREZIS [2], Th. 3.6, Lemme 3.1).

REMARK 4.1. Using Theorem 4.1 one can define the notion of a weak solution to Problem 1' for data belonging to the closure of D_1 with respect to $W^{1,1}(S;U')\times U\times Y\times Z$.

4.2 Dependence of the solution on the functional ϕ

THEOREM 4.2. Let the assumptions (2.13) and (4.1) be satisfied. Moreover, let $\phi_n : Z \to]-\infty, +\infty]$, $(f_n, u_{on}, p_{on}, q_{on}) \in D_1$, $n = 1, 2, \ldots$, *be given such that* ϕ_n *is proper convex and lower semicontinuous and that*

$$(4.10)\qquad f_n \to f \text{ in } H^1(S;U'),\quad q_{on}\to q_o \text{ in } Z,\quad \phi_n(q_{on})\to\phi(q_o),$$

$$(4.11)\qquad \begin{cases}\forall z\in Z:\phi_n(z)\to\phi(z),\ -c(1+||z||_Z)\leqq\phi_n(z)\leqq a(||z||_Z)(\phi(z)+1)\\ \text{where } a:R^+\to R^+ \text{ is increasing and } c=\text{const.},\end{cases}$$

$$(4.12)\qquad z_n \rightharpoonup z \text{ in } Z \Longrightarrow \varliminf_{n\to\infty}\phi_n(z_n)\geqq\phi(z)\ .$$

Let (u_n,p_n,q_n) *denote the solution to the problem*

$$(4.13)\qquad \begin{cases}(u_n,p_n,q_n)\in H^1(S;U\times Y\times Z),\ (u_n,p_n,q_n)(0)=(u_{on},p_{on},q_{on}),\\ q_n+(A+B)p_n=AKu_n,\ K^*(q_n+Bp_n)=f_n,\ p_n'\in\partial\phi_n(q_n).\end{cases}$$

Then if (u,p,q) *denotes the solution to Problem 1' we have*

$$(4.14)\qquad (u_n,p_n,q_n)\to(u,p,q) \text{ in } H^1(S;U\times Y\times Z)\ .$$

P r o o f . 1) In the same manner as (4.9) one can prove that

$$(4.15)\qquad q_n'+\partial_Z\phi_n(q_n)\ni Rf_n',\quad q_n(0)=q_{on},\ q_n\in H^1(S;Z)\ .$$

From (4.15) it follows (cf. BREZIS [2], Lemme 3.3):

$$\int_0^t ||q_n'(s)||_Z^2ds=\phi_n(q_{on})-\phi_n(q_n(t))+\int_0^t(Rf_n'(s),q_n'(s))_Zds$$

$$\leqq c_1\left\{1+||q_n(t)||_Z+||f_n'||_{L^2(S;U')}\left(\int_0^t||q_n'(s)||_Z^2ds\right)^{1/2}\right\}\leqq$$

$$\leqq c_2+\frac{1}{2}\int_0^t||q_n'(s)||_Z^2ds\ .$$

By (4.10) this implies that

$$(4.16) \qquad \sup_n \|q_n\|_{H^1(S;Z)} + \sup_{n,t} \phi_n(q_n(t)) < \infty .$$

2) By a standard argument it is seen that the limit of every subsequence of (q_n) converging weakly in $H^1(S;Z)$ is q. Thus

$$(4.17) \qquad q_n \rightharpoonup q \quad \text{in} \quad H^1(S;Z) .$$

3) We have

$$\overline{\lim_{n\to\infty}} \|q_n'\|^2_{L^2(S;Z)} = \overline{\lim_{n\to\infty}} \left(\phi_n(q_{on}) - \phi_n(q_n(T)) + (Rf_n', q_n')_{L^2(S;Z)}\right)$$

$$\leqq \phi(q_o) - \phi(q(T)) + (Rf',q')_{L^2(S;Z)} = \|q'\|^2_{L^2(S;Z)} .$$

This result along with (4.17) and (4.10) leads to $q_n \to q$ in $H^1(S;Z)$. Taking into account (4.5), (4.7) and the corresponding relations for (u_n,p_n,q_n) one can prove the remaining assertions of Theorem 4.2.

<u>REMARK 4.2.</u> The conditions imposed on the functionals ϕ_n are satisfied if $\phi_n = \phi_{\lambda_n}$ where $\lambda_n \downarrow 0$ and ϕ_{λ_n} denotes Yosida's approximation of ϕ corresponding to λ_n. In particular, if $\phi = I_C$, $C \subset Z$ convex closed, then we can choose $\phi_n(z) := \frac{1}{n}(\text{dist}(z,C))^2$.

<u>REMARK 4.3.</u> Yosida's approximation leads to Lipschitzian operators $\partial\phi_n$. The existence of a solution to (4.15) then follows from the theory of ordinary differential equations in Hilbert spaces and the preceding proof can be regarded as a proof of the existence of a solution to (4.9). This is in fact the usual proof of the solvability of initial value problems to evolution equations with maximal monotone operators. For problems closely related to Problems 1 and 2 this type of proof was used by NEČAS-TRÁVNÍČEK [26], DUVAUT-LIONS [4] and MOREAU [23].

<u>REMARK 4.4.</u> It was possible to replace A and B in the formulation of Theorem 4.2 by suitably chosen operators A_n and B_n. We did not do this only for the sake of simplicity.

4.3 Discretization of time

For every natural number n we set $h_n := \frac{T}{n}$ and $S_n^k :=](k-1)h_n, kh_n]$, $k = 1, \ldots, n$. If E is a Hilbert space we denote by $L_n^2(S;E)$ the subspace of $L^2(S;E)$ consisting of all functions which are constant on each interval S_n^k. For $w \in L_n^2(S;E)$ we denote by w^k the value of w on S_n^k. We define $P_n \in \mathscr{L}(L^2(S;E); L_n^2(S;E))$ and $\Delta_n^x : L_n^2(S;E) \to L_n^2(S;E)$ by

$$(4.18) \qquad \forall w \in L^2(S;E): (P_n w)^k := \frac{1}{h_n}\int_{S_n^k} w(t)dt, \quad k = 1, \ldots, n;$$

$$(4.19) \qquad \forall w \in L_n^2(S;E): (\Delta_n^x w)^k := \frac{1}{h_n}(w^k - w^{k-1}), \quad k = 1, \ldots, n, \; w^o := x.$$

Elementary estimations show that

$$(4.20) \quad \forall w \in L^2(S;E): ||P_n w||_{L^2(S;E)} \leqq ||w||_{L^2(S;E)}, \quad P_n w \to w \text{ in } L^2(S;E).$$

We shall approximate (2.17) by the following discrete-time problems:

$$(4.21) \quad \begin{cases} (u_n, p_n, q_n) \in L_n^2(S; U \times Y \times Z), & q_n + (A+B)p_n = AKu_n, \\ K^*(p_n + Bp_n) = f_n, & \Delta_n^{p_o} p_n \in \partial\phi(q_n) \end{cases}$$

where $f_n \in L_n^2(S;U')$ is defined by $f_n^k := f(kh_n)$, $k = 1, \ldots, n$ (f and p_o are the data appearing in Problem 1').

<u>LEMMA 4.1.</u> *The operator* $\Delta_n^x : L_n^2(S;Z) \to L_n^2(S;Z)$, $x \in Z$, *is monotone. More precisely, if* $z_1, z_2 \in L_n^2(S;Z)$ *and* $z = z_1 - z_2$ *then*

$$(4.22) \quad \int_0^{kh_n} ((\Delta_n^x z_1 - \Delta_n^x z_2)(t), z(t))_Z dt \geqq \frac{1}{2}||z^k||_Z^2, \quad k = 1, \ldots, n.$$

P r o o f . We have $\Delta_n^x z_1 - \Delta_n^x z_2 = \Delta_n^o z$. Therefore

$$\int_0^{kh_n} ((\Delta_n^x z_1 - \Delta_n^x z_2)(t), z(t))_Z dt = \int_0^{kh_n} ((\Delta_n^o z)(t), z(t))_Z dt =$$

$$= \sum_{j=1}^{k} (z^j - z^{j-1}, z^j)_Z \geqq \sum_{j=1}^{k} \{||z^j||_Z^2 - ||z^{j-1}||_Z ||z^j||_Z\} \geqq$$

$$\geqq \frac{1}{2}\sum_{j=1}^{k} \{||z^j||_Z^2 - ||z^{j-1}||_Z^2\} = \frac{1}{2}||z^k||_Z^2 \quad \text{where } z^o = 0.$$

THEOREM 4.3. Suppose that (2.13) and (4.1) are valid. Then for every $n \in \mathbb{N}$ *the problem (4.21) has a unique solution* (u_n, p_n, q_n). *Moreover, if* (u,p,q) *denotes the solution to Problem 1' then*

$$(4.23)\qquad (u_n, p_n, q_n) \to (u,p,q) \quad \text{in} \quad L^\infty(S; U \times Y \times Z),$$

$$(4.24)\qquad (\Delta_n^{u_o} u_n, \Delta_n^{p_o} p_n, \Delta_n^{q_o} q_n) \to (u', p', q') \qquad \text{in } L^2(S; U \times Y \times Z).$$

P r o o f . 1) From (4.21) it follows (cf. (4.5), (4.6), (4.9)):

$$(4.25)\qquad q_n + (PA + B)p_n = Rf_n \;,$$

$$(4.26)\qquad \Delta_n^{q_o} q_n + \partial_Z \phi(q_n) \ni R\Delta_n^{f(0)} f_n = RP_n f' \;,\; q_n \in L_n^2(S;Z) \;.$$

The operator $\Delta_n^{q_o}$ is monotone and Lipschitzian (cf. Lemma 4.1). Therefore $\Delta_n^{q_o} + \partial_Z \phi : L_n^2(S;Z) \to L_n^2(S;Z)$ is maximal monotone.

2) From (4.26) it follows that

$$(4.27)\qquad ||\Delta_n^{q_o} q_n||^2_{L^2(S;Z)} + \phi(q_n^n) - \phi(q_o) \leqq \leqq ||RP_n f'||_{L^2(S;Z)} ||\Delta_n^{q_o} q_n||_{L^2(S;Z)} \;.$$

Since

$$- \phi(q_n^n) \leqq c(||q_n^n||_Z + 1) = c\left(\left|\left| \int_S (\Delta_n^{q_o} q_n)(t)\,dt + q_o \right|\right|_Z + 1\right)$$

by (4.27) it is immediate that $(\Delta_n^{q_o} + \partial_Z \phi)^{-1} : L_n^2(S;Z) \to L_n^2(S;Z)$ is bounded. In view of the maximal monotonicity of this operator this implies the solvability of (4.26) (cf. BREZIS [2], Th. 2.3). The unicity of the solution to (4.26) can be proved by a standard argument (using (4.22)).

3) If q_n is a solution to (4.26) we obtain a solution to (4.21) if and only if we define p_n by (4.25) and u_n by $AKu_n = q_n + (A+B)p_n$. The last step is possible since $P(q_n + (A+B)p_n) = PRf_n = 0$. Thus we have established the unique solvability of (4.21) for each $n \in \mathbb{N}$.

4) Let $w_n \in L_n^2(S;Z)$ be defined by $w_n^k := q(kh_n)$, $k = 1, \ldots, n$. Then $\Delta_n^{q_o} w_n = P_n q'$ and (cf. (4.20)):

$$(4.28) \qquad w_n \to q \quad \text{in} \quad L^\infty(S;Z), \qquad \Delta_n^{q_o} w_n \to q' \quad \text{in} \quad L^2(S;Z) .$$

Using (4.22) we obtain for $k = 1, \ldots, n$:

$$\frac{1}{2}\|q_n^k - w_n^k\|_Z^2 \leqq \int_0^{kh_n} ((\Delta_n^{q_o} q_n - \Delta_n^{q_o} w_n)(t),\ (q_n - w_n)(t))_Z dt =$$

$$= \int_0^{kh_n} ((\Delta_n^{q_o} q_n - RP_n f' + Rf' - q')(t),\ (q_n - w_n)(t))_Z dt \leqq$$

$$\leqq \int_0^{kh_n} ((\Delta_n^{q_o} q_n - q')(t), (q - w_n)(t))_Z dt \leqq$$

$$\leqq c\|q - w_n\|_{L^2(S;Z)} \to 0 \quad \text{as} \quad n \to \infty .$$

Thus $q_n \to q$ in $L^\infty(S;Z)$.

5) By the previous results it is easily seen that the limit of every subsequence of $(\Delta_n^{q_o} q_n)$ converging weakly in $L^2(S;Z)$ is necessarily q'. Therefore $\Delta_n^{q_o} q_n \rightharpoonup q'$ in $L^2(S;Z)$.

6) By arguments similar to those of the last step of the proof of Theorem 4.2 one can prove first the assertion $\Delta_n^{q_o} q_n \to q'$ in $L^2(S;Z)$ and then the remaining assertions of Theorem 4.3.

<u>REMARK 4.5.</u> The discrete-time problem (4.21) can be written more explicitly as follows:

$$(4.29) \begin{cases} q_n^k + (A+B)p_n^k = AKu_n^k,\ K^*(q_n^k + Bp_n^k) = f_n^k,\ \frac{1}{h_n}(p_n^k - p_n^{k-1}) \in \partial\phi(q_n^k), \\ u_n^k \in U,\ p_n^k \in Y,\ q_n^k \in Z,\ k = 1, \ldots, n,\ p_n^o = p_o . \end{cases}$$

Eliminating p_n we obtain

$$(4.30) \begin{cases} K^*(B(A+B)^{-1}AKu_n^k + A(A+B)^{-1}q_n^k) = f_n^k, \\ (A+B)^{-1}(q_n^k - q_n^{k-1} - AKu_n^k + AKu_n^{k-1}) + h_n\partial\phi(q_n^k) \ni 0, \\ u_n^k \in U,\ q_n^k \in Z,\ k = 1, \ldots, n,\ q_n^o = q_o,\ u_n^o = u_o . \end{cases}$$

Since $\partial\phi \subset Z \times Y$ is maximal monotone, the operator $(A+B)^{-1} + h_n\partial\phi$ has a Lipschitzian inverse operator. From $(4.30)_2$ it follows that

$$q_n^k = ((A+B)^{-1} + h_n\partial\phi)^{-1}((A+B)^{-1}AKu_n^k + r_n^{k-1})$$

where $r_n^{k-1} := (A+B)^{-1}(q_n^{k-1} - AKu_n^{k-1})$. Obviously, the operator

$M_n^k : Y \to Z$ defined by

$$\forall y \in Y : M_n^k y := A(A+B)^{-1}((A+B)^{-1} + h_n \partial\phi)^{-1}((A+B)^{-1}Ay + r_n^{k-1})$$

is monotone and Lipschitzian. Using the notation M_n^k we can write $(4.30)_1$ as follows:

$$(4.31) \qquad K^*(B(A+B)^{-1}A + M_n^k)Ku_n^k = f_n^k \ .$$

We shall show in the following lemma that $K^*B(A+B)^{-1}AK \in \mathcal{L}(U;U')$ is symmetric and positive definite. Therefore the equations (4.31) can be solved successively for $k = 1, \ldots, n$ by means of standard iteration or projection-iteration methods if the operator M_n^k appearing in (4.31) can be handled numerically. This is the case for some nontrivial examples (cf. HÜNLICH [13]).

<u>LEMMA 4.2.</u> *If* $A + B$ *is positive definite then* $B(A+B)^{-1}A \in \mathcal{L}(Y;Z)$ *is symmetric and positive. If (4.1) is satisfied then* $K^*B(A+B)^{-1}AK \in \mathcal{L}(U;U')$ *is positive definite. (As before we assume (2.8) to be valid.)*

P r o o f . 1) Since $B(A+B)^{-1}A = B(A+B)^{-1}(A+B-B) = B-B(A+B)^{-1}B = B - (A+B-A)(A+B)^{-1}B = A(A+B)^{-1}B$ the operator $B(A+B)^{-1}A$ is symmetric. The positivity of this operator follows from

$$(4.32) \begin{cases} \langle B(A+B)^{-1}Ay, \ y\rangle = \langle B(A+B)^{-1}Ay, \ (A+B)^{-1}Ay + (A+B)^{-1}By\rangle \\ = \langle B(A+B)^{-1}Ay, \ (A+B)^{-1}Ay\rangle + \langle A(A+B)^{-1}By, \ (A+B)^{-1}By\rangle . \end{cases}$$

2) If (4.1) is satisfied then by (4.32) we obtain for $u \in U$

$$\langle K^*B(A+B)^{-1}AKu, \ u\rangle = \langle B(A+B)^{-1}AKu, \ Ku\rangle \geqq c_1||(A+B)^{-1}AKu||_Y^2 \geqq$$
$$\geqq c_2||AKu||_Z^2 \geqq c_3||K^*AKu||_{U'}^2 \geqq c_4||u||_U^2 \quad \text{where} \quad c_i > 0 \ , \ i=1,2,3,4.$$

<u>REMARK 4.6.</u> The discretization of time considered in this section may be combined with Galerkin's method and also with a regularization of the functional ϕ of the type mentioned in the second part of this section.

5. Dynamic processes with definite kinematic hardening

THEOREM 5.1. Suppose that (2.15) and (4.1) are valid and that $\rho > 0$ *is fixed. Then Problem 2' has a unique solution* (u,p,q) *and the mapping* $(f,u_o,v_o,p_o,q_o) \to (u,u',p,q)$ *is Lipschitzian from* D_2 *(equipped with the metric of* $(W^{1,1}(S;U')+L^1(S;H))\times U \times H \times Y \times Z$*) into* $C(S;U \times H \times Y \times Z)$.

The proof of this theorem is based on two lemmas. First we introduce some notation. Let

(5.1) $\quad U_1 := U \times Z \,, \quad H_1 := H \times Z \,, \quad U_1' := U' \times Z \,.$

In this section we define a scalar product on Z by

(5.2) $\quad \forall z_1, z_2 \in Z : (z_1,z_2)_Z := \frac{1}{\rho}\langle z_1, (A+B)^{-1}z_2\rangle .$

Using this scalar product we regard U_1' as dual to U_1 and identify H_1 with its dual.

LEMMA 5.1. Let $L \in \mathcal{L}(U_1;U_1')$ *be defined by*

(5.3) $\quad \forall (u,z) \in U_1 : L(u,z) := \frac{1}{\rho}(K^*B(A+B)^{-1}AKu, 0) \,.$

Then L *is symmetric and*

(5.4) $\forall (u,z) \in U_1 : \langle L(u,z),(u,z)\rangle \geq c_o(||(u,z)||^2_{U_1} - ||(u,z)||^2_{H_1}), c_o > 0.$

P r o o f . This lemma is an immediate consequence of Lemma 4.2.

LEMMA 5.2. Let $M := M_1 + M_2$ *where*

(5.5) $$\forall (u,z) \in U_1 : \begin{cases} M_1(u,z) := (\frac{1}{\rho}K^*A(A+B)^{-1}z, -AKu) \,, \\ M_2(u,z) := \{(0,(A+B)\eta) \,|\, \eta \in \partial\phi(z)\}. \end{cases}$$

Then $M \subset U_1 \times U_1'$ *is maximal monotone.*

P r o o f . By the choice of the scalar product on Z (cf. (5.2)) it is easy to see that M_1 and M_2 are monotone. Since M_1 is continuous it is maximal monotone. M_2 is maximal monotone because subdifferential mappings are maximal monotone. The maximal monotonicity of M now follows from a basic result on the maximality of monotone operators (see BREZIS [2], Cor. 2.7).

P r o o f   o f   T h e o r e m 5.1. 1) Eliminating p from (2.18) and introducing a new unknown function w by

$$(5.6) \qquad \forall t \in S : \quad w(t) := \int_0^t q(s)ds$$

we can formulate Problem 2' in the following way:

$$(5.7) \quad u \in W^{1,\infty}(S;U), \ (u,w) \in W^{2,\infty}(S;H_1), \ (u,u',w,w')(0)=(u_o,v_o,0,q_o),$$

$$(5.8) \begin{cases} \rho u'' + K^*B(A+B)^{-1}AKu + K^*A(A+B)^{-1}w' = f , \\ w'' + (A+B)\partial\phi(w') \ni AKu' . \end{cases}$$

Using the operators L and M introduced before we can rewrite (5.8) as follows:

$$(5.9) \qquad (u'', w'') + M(u', w') + L(u,w) \ni (\tfrac{1}{\rho}f, 0).$$

The existence-uniqueness result of Theorem 5.1 now follows from a result on second order evolution equations formulated by BARBU [1] (Th. 1.1, Ch. V). We are allowed to apply this result to the problem (5.7), (5.9) because of Lemma 5.1, Lemma 5.2 and the following relations (cf. (2.15)):

$$(f,0) \in W^{1,1}(S;H_1), \qquad (v_o,q_o) \in D(M),$$

$$L(u_o,0) + M(v_o,q_o) =$$
$$= \{(\tfrac{1}{\rho}K^*(B(A+B)^{-1}AKu_o+A(A+B)^{-1}q_o \ , \ -AKv_o+(A+B)\eta) \,|\, \eta\in \partial\phi(q_o)\}$$
$$= \{(\tfrac{1}{\rho}K^*z_o, \ -AKv_o+(A+B)\eta) \,|\, \eta \in \partial\phi(q_o)\} \subset H_1 .$$

2) Let $(f_i, u_{oi}, v_{oi}, p_{oi}, q_{oi}) \in D_2$, $i = 1, 2$, and let (u_i,p_i,q_i) denote the solution to Problem 2' corresponding to the data $(f_i, u_{oi}, v_{oi}, p_{oi}, q_{oi})$. Furthermore, let $\bar{u} := u_1 - u_2$, $\bar{p} := = p_1 - p_2$ etc. Finally, let $\bar{f} = g + h$ where $g \in W^{1,1}(S;U')$, $h \in L^1(S;H)$. Then (cf. (5.7), (5.8) and Lemma 4.2)

$$0 \geqq \int_0^t \{<(\rho\bar{u}''+K^*B(A+B)^{-1}AK\bar{u}-g-h)(s),\bar{u}'(s)>+<\bar{q}'(s),(A+B)^{-1}\bar{q}(s)>\}ds \geqq$$
$$\geqq c_o(||\bar{u}'(t)||_H^2+||\bar{u}(t)||_U^2+||\bar{q}(t)||_Z^2) - c_1(||\bar{v}_o||_H^2+||\bar{u}_o||_U^2+||\bar{q}_o||_Z^2)$$
$$-<g(t),\bar{u}(t)>+<g(0),\bar{u}_o>-\int_0^t \{||g'(s)||_{U'}||\bar{u}(s)||_U+||h(s)|_H||\bar{u}'(s)|_H\}ds, \quad c_o > 0 .$$

This implies (cf. BREZIS [2], Lemme A.5)

$$||(\bar{u},\bar{u}',\bar{q})||_{C(S;U\times H\times Z)} \leq c(||\bar{f}||_{W^{1,1}(S;U')+L^1(S;H)}+||\bar{u}_o||_U+||\bar{v}_o||_H+ +||\bar{q}_o||_Z).$$

Since $p_i = (A+B)^{-1}(AKu_i-q_i)$ this result establishes the desired Lipschitz continuity of the mapping $(f,u_o,v_o,p_o,q_o) \mapsto (u,u',p,q)$.

REMARK 5.1. Using Theorem 5.1 one can define the notion of a weak solution to Problem 2' for data belonging to the closure of D_2 with respect to $(W^{1,1}(S;U') + L^1(S;H))\times U \times H \times Y \times Z$.

6. Quasi-static processes without definite kinematic hardening

In this section we shall use instead of (4.1) the following assumptions:

(6.1) $A + B \in \mathcal{L}(Y;Z)$ and $K^*AK \in \mathcal{L}(U;U')$ are positive definite,

(6.2) $w + (A+B)^{-1}AKv \in \partial\phi(z) \Longrightarrow ||v||_U \leq c(||w||_Y+||z||_Z + 1)$,

$c = \text{const.}$

For some results we shall need the assumption

$$(6.3)\ \begin{cases}(v_n,w_n,z_n) \in L^2(S;U\times Y\times Z),\ w_n + (A+B)^{-1}AKv_n \in \partial\phi(z_n)\ , \\ (w_n,z_n) \to (w,z) \text{ in } L^2(S;Y\ Z) \Longrightarrow v_n \to v \text{ in } L^2(S;U)\ .\end{cases}$$

A justification of the assumptions (6.2), (6.3) is given by the following remarks.

REMARK 6.1. The relations (6.2), (6.3) hold if $\partial\phi$ is Lipschitz-continuous. This is true in particular if $\partial\phi = (\partial I_C^* + M)^{-1}$ provided $M : Y \to Z$ is strongly monotone (cf. (2.3)).

REMARK 6.2. Let the suppositions of Example 2 (Section 3) be satisfied. Then condition (6.2) is also satisfied:

$$(\varepsilon,\lambda) + (A+B)^{-1}AKv = (\varepsilon + (A_1+B_1)^{-1}A_1K_1v,\lambda) \in \partial\phi(\sigma,\alpha)$$

implies (cf. (3.6)) $\varepsilon + (A_1+B_1)^{-1}A_1K_1v = -\lambda F'(\sigma)$. Thus

$$||v||_U \leq c_1||\varepsilon+\lambda F'(\sigma)||_{L^2(G;\mathcal{S})} \leq c_2(||\varepsilon||_{L^2(G;\mathcal{S})}+||\lambda||_{L^2(G)}) \leq$$
$$\leq c_3||(\varepsilon,\lambda)||_Y\ .$$

Let us suppose in addition that F' is continuous. If the premises of (6.3) are satisfied for $w_n = (\varepsilon_n, \lambda_n)$, $z_n = (\sigma_n, \alpha_n)$ then

$$\overline{\lim_{n,m\to\infty}} ||v_n - v_m||_{L^2(S;U)} \leqq$$
$$\leqq c_1 \overline{\lim_{n,m\to\infty}} ||\varepsilon_n - \varepsilon_m + \lambda_n F'(\sigma_n) - \lambda_m F'(\sigma_m)||_{L^2(S;L^2(G;\mathcal{f}))} \leqq$$
$$\leqq c_1 \overline{\lim_{n,m\to\infty}} ||\lambda(F'(\sigma_n) - F'(\sigma_m)||_{L^2(S;L^2(G;\mathcal{f}))} = 0$$

where λ denotes the limit of (λ_n) in $L^2(S;L^2(G))$. We made use of Krasnoselski's theorem on the continuity of Nemycki operators (cf. KRASNOSELSKI [17]). Thus the assumption (6.3) is satisfied in the case mentioned here.

In what follows we denote by D_ϕ the set $D(\phi)$ equipped with the topology generated by the basis $\{V_{\delta,\zeta} | \delta > 0,\ \zeta \in D(\phi)\}$ where

$$V_{\delta\ \zeta} := \{z \in D(\phi) |\ ||z - \zeta||_Z < \delta,\ \phi(z) < \phi(\zeta) + \delta\}.$$

The topology of D_ϕ is chosen so that $\phi : D_\phi \to \mathbb{R}$ is continuous.

THEOREM 6.1. Suppose that (2.13), (6.1) and (6.2) are valid. Then Problem 1' has a solution (u,p,q) *where* q *is uniquely determined. If in addition (6.3) is satisfied then the solution* (u,p,q) *to Problem 1' is unique and the mapping* $(f,u_o,p_o,q_o) \mapsto (u,p,q)$ *is continuous from* D_1 *(equipped with the topology of* $H^1(S;U')\times U\times Y\times D_\phi$*) into* $H^1(S;U\times Y\times Z)$.

P r o o f . 1) We shall approximate Problem 1' by a problem describing a system with definite kinematic hardening. We choose a sequence (δ_n) such that $\delta_n \downarrow 0$ and we set

$$(6.4)\quad \begin{cases} B_n := B + \delta_n(A + B),\ p_{on} := (A + B_n)^{-1}(AKu_o - q_o)\ , \\ \forall t \in S : f_n(t) := f(t) - f(0) + K^*(q_o + B_n p_{on})\ . \end{cases}$$

This choice ensures that $K^* z_{on} = f_n(0)$ and $q_o = z_{on} - B_n p_{on}$ where $z_{on} := A(Ku_o - p_{on})$ (cf. (2.13)). Let (u_n,p_n,q_n) denote the unique solution to the problem

$$(6.5)\quad \begin{cases} (u_n,p_n,q_n) \in H^1(S;U\times Y\times Z),\ (u_n,p_n,q_n)(0) = (u_o,p_{on},q_o)\ , \\ q_n + (A+B_n)p_n = AKu_n,\ K^*(q_n + B_n p_n) = f_n,\ p_n' \in \partial\phi(q_n)\ . \end{cases}$$

2) A priori estimates. We have $A + B_n = (1 + \delta_n)(A+B)$. Therefore (6.5) yields

(6.6) $$(1 + \delta_n)^{-1}(A + B)^{-1}(AKu_n' - q_n') \in \partial\phi(q_n) \ .$$

By (6.5) and (6.2) this implies (cf. BREZIS [2], Lemme 3.3)

$$(1 + \delta_n)^{-1}\langle q_n', (A+B)^{-1}q_n'\rangle + \phi(q_n(T)) - \phi(q_o) =$$

$$= (1 + \delta_n)^{-1}\langle q_n', (A+B)^{-1}AKu_n'\rangle = \langle q_n' - B_n(A+B_n)^{-1}q_n', Ku_n'\rangle =$$

$$= \langle q_n' + B_n p_n' - B_n(A+B_n)^{-1}AKu_n', Ku_n'\rangle \leq \langle f', u_n'\rangle \leq$$

$$\leq c_1 ||f'||_{L^2(S;U')} (||q_n'||_{L^2(S;Z)} + ||q_n||_{L^2(S;Z)} + 1) \ .$$

Hence (q_n) is bounded in $H^1(S;Z)$. Using once more (6.6) and (6.2) we see that (u_n) is bounded in $H^1(S;U)$. Finally, the relation $p_n = (A + B_n)^{-1}(AKu_n - q_n)$ proves the boundedness of (p_n) in $H^1(S;Y)$.

3) Existence. Extracting, if necessary, a subsequence we may assume

(6.7) $$(u_n,p_n,q_n) \rightharpoonup (u,p,q) \quad \text{in} \quad H^1(S;U \times Y \times Z) \ .$$

By (6.5) this implies that

(6.8) $$q + (A+B)p = AKu, \quad K^*(q+Bp) = f, \quad (u,p,q)(0) = (u_o,p_o,q_o) \ .$$

In view of the previous results and of Lemma 4.2

$$\overline{\lim_{n\to\infty}} \langle q_n,p_n'\rangle = \overline{\lim_{n\to\infty}} \langle q_n,(A+B_n)^{-1}(AKu_n' - q_n')\rangle =$$

$$= \overline{\lim_{n\to\infty}} \langle A(A+B_n)^{-1}q_n, Ku_n'\rangle - \langle q_n, (A+B)^{-1}q_n'\rangle \leq$$

$$\leq \langle f,u'\rangle + \tfrac{1}{2}\{\langle B(A+B)^{-1}AKu_o, Ku_o\rangle - \langle B(A+B)^{-1}AKu(T), Ku(T)\rangle +$$

$$+ \langle q_o, (A+B)^{-1} q_o\rangle - \langle q(T), (A+B)^{-1}q(T)\rangle\} =$$

$$= \langle q+Bp-B(A+B)^{-1}AKu, Ku'\rangle - \langle q, (A+B)^{-1}q'\rangle = \langle q, p'\rangle.$$

This inequality along with (6.5), (6.7) shows that $p' \in \partial\phi(q)$ (cf. BREZIS [2], Prop. 2.5). Thus (u,p,q) is a solution to Problem 1'.

4) Uniqueness. Let (u_i,p_i,q_i), $i = 1, 2$, be two solutions to Problem 1' and let $(u,p,q) := (u_1-u_2,\ p_1-p_2,\ q_1-q_2)$. Then $(u,p,q)(0) = (0,0,0)$, $q + (A+B)p = AKu$, $K^*(p + Bp) = 0$ and (since $\partial\phi$ is monotone)

$$0 \leq \int_0^t \langle q(s), p'(s)\rangle ds = -\frac{1}{2}\{\langle \cdot (A+B)^{-1}AKu(t), Ku(t)\rangle + \langle q(t), (A+B)^{-1}q(t)\rangle\}$$

(cf. the preceding calculation). Thus $q_1 = q_2$. If (6.3) is satisfied then $u_1 = u_2$ in view of $(A+B)^{-1}(AKu_i' - q_i') \in \partial\phi(q_i)$, $i = 1, 2$. Finally, $p_1 = p_2$ since $p_i = (A+B)^{-1}(AKu_i - q_i)$, $i = 1, 2$.

5) The asserted continuity of the dependence of the solution on the data can be proved by arguments quite similar to those used in the previous steps of the proof. Therefore we omit the details.

REMARK 6.3. MANDEL [22] has pointed out that u is not necessarily unique in cases of isotropic hardening governed by TRESCA's yield condition. These cases are special cases of our Example 2 with a function F such that F' is bounded but not continuous (condition (6.2) is satisfied but not necessarily condition (6.3), cf. Remark 6.2).

REMARK 6.4. Under the hypotheses of Theorem 6.1 it is possible to prove results on discrete-time problems analogous to those established in Theorem 4.3. The crucial step is again to find sufficiently strong a priori estimates. Let us mention that it would be possible to prove the existence result of Theorem 6.1 via dicretization of time.

7. Dynamic processes without kinematic hardening

In this section we shall show that it is possible to prove results on dynamic processes for systems without definite kinematic hardening. For the sake of simplicity we restrict ourselves to processes without any kinematic hardening, i. e. to $B = 0$. Simultaneously we suppose that

(7.1) $\qquad A \in \mathcal{L}(U;U')$ is positive definite.

In this case we have $q = z$ (cf. Figure 1 and (2.7)). Introducing $v := u'$ as a new unknown function we may formulate Problem 2' as follows:

We are given

(7.2) $\quad f \in W^{1,1}(S;H),\ v_o \in U\ ,\ z_o \in D(\partial\phi)$ such that $K^* z_o \in H$.

We are looking for processes $v,\ z$ such that

$$(7.3)\quad \begin{cases} v \in L^2(S;U),\ (v,z) \in W^{1,\infty}(S;H\times Z),\ (v,z)(0) = (v_o,\ z_o), \\ \rho v' + K^* z = f,\quad A^{-1}z' + \partial\phi(z) \ni Kv\ . \end{cases}$$

<u>THEOREM 7.1.</u> *If (7.1), (7.2) and (6.2) (with* $B = 0$*) are satisfied then there exists a unique solution* (v,z) *to the problem (7.3) and the mapping* $(f,v_o,z_o) \mapsto (v,z)$ *is Lipschitzian from the set of data satisfying (7.2) (equipped with the metric of* $L^1(S;H)\times H\times Z$*) into* $C(S;H\times Z)$.

The proof of this theorem is based on two lemmas. We shall use again the spaces $U_1,\ H_1,\ U_1'$ introduced in Section 5 (cf. (5.1), (5.2)). We define a multivalued mapping M_H in H_1 by

$$(7.4)\quad \forall(v,z) \in H_1:\ M_H(v,z) := \{(\tfrac{1}{\rho}K^* z, A(\eta - Kv))\,|\, v \in U, K^* z \in H, \eta \in \partial\phi(z)\}.$$

<u>LEMMA 7.1.</u> *The operator* $M_H \subset H_1 \times H_1$ *is maximal monotone.*

P r o o f . Evidently, $M_H = M \cap (H_1 \times H_1)$ where $M \subset U_1 \times U_1'$ denotes the operator introduced in Lemma 5.2 (with $B = 0$). Therefore M_H is monotone. To prove the maximality of M_H we show first that $(M + id_{U_1})^{-1}$ is bounded. Let $(M + id_{U_1})(v,z) = (\tilde{v},\tilde{z})$, i. e.

$$(7.5)\quad \tfrac{1}{\rho}K^* z + v = \tilde{v}\ ,\quad A(\eta - Kv) + z = \tilde{z}\ ,\ \eta \in \partial\phi(z)\ ,$$

where $(\tilde{v},\tilde{z}) \in U_1'$. From (7.5) it follows (if we denote by η_o an element from $\partial\phi(z_o)$, cf. (7.2))

$$0 \le \langle Kv - A^{-1}(z - \tilde{z}) - \eta_o,\ z - z_o\rangle\ .$$

Hence

$$(7.6)\quad \begin{cases} \langle A^{-1}z,\ z\rangle \le c_1(||z||_Z||\tilde{z}||_Z + ||z||_Z + 1) + \langle v, K^* z - K^* z_o\rangle \\ \le c_1(||z||_Z||\tilde{z}||_Z + ||z||_Z + 1) + \langle v, \rho(\tilde{v} - v) - K^* z_o\rangle \\ \le c_2(||z||_Z||\tilde{z}||_Z + ||z||_Z + ||v||_U||\tilde{v}||_{U'} + ||v||_U + 1) \end{cases}$$

where $c_1,\ c_2$ do not depend on $\tilde{v},\ \tilde{z}$. In view of (6.2) the relations (7.5) imply that

(7.7) $$||v||_U \leqq c(||z||_Z + ||\tilde{z}||_Z + 1) .$$

The inequalities (7.6),(7.7) show that for $||(v,z)||_{U_1}$ there exists a bound depending only on $||(\tilde{v}, \tilde{z})||_{U_1'}$. Thus $(M + id_{U_1})^{-1}$ is bounded and, consequently, the range of $M + id_{U_1}$ is the whole U_1' (cf. BREZIS [2], Th. 2.3). If $(\tilde{v},\tilde{z}) \in H_1$ then $K^*z \in H$, i. e. $(v,z) \in D(M_H)$. Therefore $M_H + id_{H_1}$ is surjective and this implies the maximality of $M_H \subset H_1 \times H_1$.

LEMMA 7.2. *Let* M_{HS} *and* $\tilde{M}_{HS}$ *be multivalued mappings from* $L^2(S;H_1)$ *into itself defined by*

$$M_{HS}(v,z):=\{(\bar{v},\bar{z}) \in L^2(S;H_1) \mid (\bar{v},\bar{z})(t) \in M_H(v,z)(t) \quad \textit{for a. e.} \quad t \in S\},$$

$$\tilde{M}_{HS}(v,z):=\{(\tfrac{1}{\rho}K^*v, A(\eta-Kv)) \in L^2(S;H_1) \mid v \in L^2(S;U),\ \eta(t) \in \partial\phi(z(t)) \quad \textit{for a. e.} \quad t \in S\}.$$

Then $M_{HS} = \tilde{M}_{HS}$.

P r o o f . The operator M_{HS} is maximal monotone (see BREZIS [2], Example 2.3.3). The operator $\tilde{M}_{HS}$ is also maximal monotone. This can be proved in the same way as the maximal monotonicity of M_H. Since M_{HS} is an extension of $\tilde{M}_{HS}$ both operators must be equal.

REMARK 7.1. In what follows we shall write M_H instead of M_{HS}.

P r o o f o f T h e o r e m 7.1. The problem (7.3) may be written as

(7.8) $$(v',z')+M_H(v,z) \ni (\tfrac{1}{\rho}f,0), \quad (v,z)(0)=(v_o,z_o), \quad (v,z) \in W^{1,\infty}(S;H_1).$$

Note that $v \in L^2(S;U)$ if (v,z) is a solution to (7.8) by Lemma 7.2. Since $(v_o,z_o) \in D(M_H)$ the assertions of Theorem 7.1 follow from well known results on evolution equations with maximal monotone operators (see BREZIS [2], Prop. 3.3 and Lemme 3.1).

In what follows we shall show that there exists a "weak formulation" of the problem (7.3) which leads to an existence-uniqueness result even without the assumption (6.2). We need the following notations

(7.9) $\bar{Z}:=\{z\in Z\,|\,K^*z\in H\},\quad ||z||_{\bar{Z}}:=(||z||_Z^2+||K^*z||_H^2)^{\frac{1}{2}},\quad \bar{Y}:=\bar{Z}',\quad \bar{\phi}:=\phi|\bar{Z}$.

Since $\bar{Z}$ is dense in Z we can regard Y as a subset of $\bar{Y}$. Obviously

$$\forall u\in U:||Ku||_{\bar{Y}}=\sup_{\|z\|_{\bar{Z}}=1}\langle z,Ku\rangle=\sup_{\|z\|_{\bar{Z}}=1}\langle K^*z,u\rangle\leqq||u||_H .$$

Thus K can be extended by continuity to a mapping $\bar{K}\in\mathcal{L}(H;\bar{Y})$. The adjoint operator $\bar{K}^*$ of $\bar{K}$ is the restriction of K^* to $\bar{Z}$. Later on we shall make use of the trivial implication

(7.10) $$\eta\in\partial\phi(z),\ z\in\bar{Z}\Longrightarrow\eta\in\partial\bar{\phi}(z) .$$

If (7.2) holds and (v,z) is a solution to (7.3) then

(7.11) $$\begin{cases}(v,z)\in W^{1,\infty}(S;H\times Z),\ (v,z)(0)=(v_o,z_o)\ ,\\ \rho v'+\bar{K}^*z=f\ ,\quad A^{-1}z'+\partial\bar{\phi}(z)\ni\bar{K}v\ .\end{cases}$$

Therefore we may regard (7.11) as a weak formulation of (7.3). Let us mention that in the case of Prandtl-Reuss' equations, DUVAUT-LIONS [4] used a formulation equivalent to (7.11).

<u>*THEOREM 7.2.*</u> *If (7.1), (7.2) are satisfied then there exists a unique solution* (v,z) *to the problem (7.11) and the mapping* $(f,v_o,z_o)\mapsto(v,z)$ *is Lipschitzian from the set of data satisfying (7.2) (equipped with the metric of* $L^1(S;H)\times H\times Z)$ *into* $C(S;H\times Z)$.

P r o o f . We define $\bar{M}_H\subset H_1\times H_1$ by

(7.12) $$\forall(v,z)\in H_1:\ \bar{M}_H(v,z):=\{(\tfrac{1}{\rho}K^*z,A(\eta-\bar{K}v))\,|\,\eta\in\partial\bar{\phi}(z),\eta-\bar{K}v\in Y\}.$$

Then (7.11) is valid if and only if

(7.13) $$(v,z)\in W^{1,\infty}(S;H_1),\ (v,z)(0)=(v_o,z_o),\ (v',z')+\bar{M}_H(v,z)\ni(\tfrac{1}{\rho}f,0).$$

By (7.2) we have $(v_o,z_o)\in D(\bar{M}_H)$. Therefore Theorem 7.2 follows from the theory of evolution equations if we can prove $\bar{M}_H\subset H_1\times H_1$ to be maximal monotone. This will be done in the following lemma.

<u>*LEMMA 7.3.*</u> *The operator* $\bar{M}_H$ *defined by (7.12) is maximal monotone.*

P r o o f . It is easy to see that $\bar{M}_H$ is monotone. To prove its maximality we show that $\bar{M}_H\dot{+}id_{H_1}$ is surjective. To this end we consider the problem

(7.14) $\frac{1}{\rho}\overline{K}^* z + v = \tilde{v}, \ A(\eta - \overline{K}v) + z = \tilde{z}, \eta \in \partial\overline{\phi}(z), \ \eta - \overline{K}v \in Y$,

where $(\tilde{v},\tilde{z}) \in H_1$ is given arbitrarily. From (7.14) it follows

$$\partial\overline{\phi}(z) + (\tfrac{1}{\rho}\overline{KK}^* + A^{-1})z \ni A^{-1}\tilde{z} + \overline{K}\tilde{v} \ . \tag{7.15}$$

Evidently, $\frac{1}{\rho}\overline{KK}^* + A^{-1} \in \mathscr{L}(\overline{Z};\overline{Y})$ is positive definite. This fact along with the maximal monotonicity of $\partial\overline{\phi} \subset \overline{Z} \times \overline{Y}$ proves the existence of a solution $z \in \overline{Z}$ to (7.15). Defining $v \in H$ by $v := \tilde{v} - \frac{1}{\rho}\overline{K}^* z$ we obtain a solution to (7.14). This completes the proof of Lemma 7.3.

REMARK 7.2. If we are given a solution (v,z) to problem (7.11) and an initial value u_o for the displacement we can define $u \in W^{2,\infty}(S;H)$, $y \in W^{1,\infty}(S;\overline{Y})$, $e \in W^{1,\infty}(S;Y)$ and $p \in W^{1,\infty}(S;\overline{Y})$ by

$$\forall t \in S: \ u(t) := u_o + \int_0^t v(s)ds, \ y := \overline{K}u \ , \ e := A^{-1}z \ , \ p := y - e$$

(cf. (2.19)).

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