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UNIQUENESS OF THE ROTHE METHOD FOR THE BURGERS EQUATION WITH PIECEWISE CONTINUOUS INITIAL DATA

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The ROTHE method for a longer period has been preferably applied to parabolic initial-boundary value problems. Only in the last decade it has turned out also as a suitable tool for hyperbolic problems. Especially with regard to the CAUCHY problem for the BURGERS equation and related nonlinear conservation laws, results concering to the existence, the uniqueness, and the convergence of the ROTHE approximations have been obtained [1,2,3,4,5].

The uniqueness statements in [2], however, mainly refer to continuous ROTHE solutions or to pieces where such solutions are continuous; therefore the entropy condition has not been involved. In the present paper the question of uniqueness is treated without any restriction if the initial data are piecewise continuous and compactificable at the infinity. The important role of the entropy condition in this context is pointed out in the representation.

The CAUCHY problem here considered is as follows. We are looking for a solution $u(x,t)$ in $\mathbb{R} \times [0,\infty)$ which satisfies the BURGERS equation and the corresponding entropy condition

$$u_t + \frac{1}{2}(u^2)_x = 0 \ , \tag{1}$$

$$\frac{1}{2}(u^2)_t + \frac{1}{3}(u^3)_x \leq 0 \ , \tag{2}$$

in $\mathbb{R} \times (0,\infty)$, respectively. The initial values,

$$u(x,0) = u_0(x) \ , \quad x \in \mathbb{R} \ , \tag{3}$$

are supposed to be piecewise continuous having at most a finite number of discontinuities and limits as $x \to \mp\infty$, respectively. Furthermore, the limits of $u(x,t)$ as $x \to \mp\infty$, respectively, are required to exist uniformly for every compact time interval contained in $[0,\infty)$.

The condition mentioned at last may be replaced by an equivalent one. For this purpose, let $0 < \alpha < \beta < \infty$ be chosen arbitrarily. Then because of the assured uniform convergence, the limits as $x \to \mp\infty$ of the function

$$\Phi(x) := \frac{1}{2}\int_\alpha^\beta [u(x,t)]^2 \, dt \ , \quad x \in \mathbb{R} \ ,$$

exist and may be obtained by interchange with the integration. Via application of the second L'HOSPITAL rule and by using (1) it follows that

$$0 = \lim_{x\to\mp\infty} \frac{\Phi(x)}{x} = \lim_{x\to\mp\infty} \Phi'(x) = \lim_{x\to\mp\infty} \int_\alpha^\beta \{-u_t(x,t)\} dt = \lim_{x\to\mp\infty} u(x,\alpha) - \lim_{x\to\mp\infty} u(x,\beta) \ ,$$

so the limit functions $\lim_{x\to\mp\infty} u(x,t)$ respectively must be constant for $t \in (0,\infty)$. Since for the limit functions the continuity at $t = 0$ is also ensured by the uniform convergence, and due to the initial condition (3), we find the limits

$$\lim_{x\to\mp\infty} u(x,t) = \lim_{x\to\mp\infty} u_0(x) \ , \quad t \in [0,\infty) \ , \tag{4}$$

respectively, holding uniformly for every compact time interval. This condition is obviously equivalent to the one mentioned above.

If, in view of our purpose, we assume that the time discretizised CAUCHY problem, after each time step, leads to the same situation as before, so only the first time step needs to be discussed in detail. Let the time step length be $h > 0$, then for the desired time discretizised solution $u(x),\ x \in \mathbb{R}$, the conditions

$$\frac{u-u_0}{h}+\frac{1}{2}(u^2)' = 0 \ , \tag{5}$$

$$\frac{u^2-u_0^2}{2h}+\frac{1}{3}(u^3)' \leq 0 \ , \tag{6}$$

are deduced from (1) and(2), respectively. For points of differentiability it is easily verified that *a solution of the* ROTHE *equation* (5) *also satisfies the time discretizised entropy condition* (6). With regard to (5) and (6), the transmission condition and the entropy condition for the right and left sided limits of a ROTHE solution are obtained as follows:

$$u^2|_+ - u^2|_- = 0 \ , \tag{7}$$

$$u^3|_+ - u^3|_- \leq 0 \ . \tag{8}$$

Let us for a ROTHE solution $u(x),\ x \in \mathbb{R}$, require the following properties:

(i) *$u(x)$ is piecewise continuous with at most a finite number of discontinuities where,* in accordance with (7) and (8), *the one-sided limits are subject to*

$$u|_- > 0 \ , \ u|_+ < 0 \ , \ u|_+ = -u|_- \ ; \tag{9}$$

(ii) *in every open interval of continuity, with the exception of at most a finite number of points, $u(x)$ is a continuously differentiable solution of the differential equation* (5)*;*

(iii) *for $u(x)$ the limits exist as $x \to \mp\infty$, respectively.*

As it has been shown in [2], the condition (iii) may be replaced by

$$\lim_{x\to\mp\infty} u(x) = \lim_{x\to\mp\infty} u_0(x) \ , \tag{10}$$

thus forming the time discrete counterpart of (4).

In order to prove the uniquenes of the ROTHE method, we start with two solutions $u(x)$ and $v(x),\ x \in \mathbb{R}$. For these as a consequence of (10), it can be stated immediately that

$$\lim_{x\to\mp\infty} u(x) = \lim_{x\to\mp\infty} v(x) \ . \tag{11}$$

First of all we formulate two preparatory Lemmas holding for $u(x)$ and $v(x)$.

LEMMA 1. *Let (a, b) be a finite or an infinite open interval of the real axis where both solutions $u(x)$ and $v(x)$ are continuous. Let further $u(a)$, $v(a)$ and $u(b)$, $v(b)$ exist in the sense of one-sided limits as $x \to a$ and $x \to b$, respectively. Then from*

(12)	$u(a) \neq v(a)$,	$u(b) \neq v(b)$,
(13)	$\dfrac{u(a)+v(a)}{2} \leq 0$,	$\dfrac{u(b)+v(b)}{2} \geq 0$,

it follows that

(14)	$u(b) \neq v(b)$,	$u(a) \neq v(a)$,
(15)	$\dfrac{u(b)+v(b)}{2} < 0$.	$\dfrac{u(a)+v(a)}{2} > 0$.

PROOF, may be restricted to the first case. We assume that there is a point in the interval (a,b) with coincident values for $u(x)$ and $v(x)$. Let c denote the infimum of all points of (a,b) where $u(x)$ and $v(x)$ have coinciding values. Then with regard to (12) we have $c \in (a,b)$, and due to continuity $u(c) = v(c)$. Since the differential equation (5) is valid for $u(x)$ as well as for $v(x)$, by (inproper) integration we obtain that

$$\frac{1}{h}\int_a^c \{u(x) - v(x)\}\,dx = -\frac{1}{2}\int_a^c \left\{\frac{d}{dx}[u(x)]^2 - \frac{d}{dx}[v(x)]^2\right\} dx = \frac{u(a)+v(a)}{2}\{u(a) - v(a)\} .$$

Observing here the continuity of the difference $u(x) - v(x)$ for $x \in (a,c]$, the integral on the left hand side must have the same sign as $u(a) - v(a) \neq 0$. Hence it follows as a contradiction to (13) that the arithmetic mean of $u(a)$ and $v(a)$ is positive. Consequently, there are no zeroes of the difference $u(x) - v(x)$ in (a,b) and, again by continuity, $u(x) - v(x)$ has a fixed sign in (a,b). With this we get from (5) that

$$\begin{aligned}\frac{1}{h}\int_a^b \{u(x) - v(x)\}\,dx &= -\frac{1}{2}\int_a^b \left\{\frac{d}{dx}[u(x)]^2 - \frac{d}{dx}[v(x)]^2\right\} dx \\ &= \frac{u(a)+v(a)}{2}\{u(a) - v(a)\} - \frac{u(b)+v(b)}{2}\{u(b) - v(b)\},\end{aligned}$$

where both the integral on the left and $u(a) - v(a) \neq 0$ have the same sign. So after division by $u(a) - v(a) \neq 0$ and because of (13) we are led to the inequality

$$\frac{u(b)+v(b)}{2}\,\frac{u(b)-v(b)}{u(a)-v(a)} < 0 \tag{16}$$

which, in particular, implies (14). Since both differences

$$\begin{aligned} u(a) - v(a) &= \lim_{x\to a}\{u(x) - v(x)\} \neq 0 \ , \\ u(b) - v(b) &= \lim_{x\to b}\{u(x) - v(x)\} \neq 0 \ , \end{aligned}$$

have the same sign as $u(x) - v(x) \neq 0$ has for all $x \in (a,b)$, so (15) is obtained from (16).

LEMMA 2. *At a point of discontinuity for at least one of the solutions, $u(x)$ or $v(x)$, from*

(17)	$\frac{u\vert_- + v\vert_-}{2} < 0$	$\frac{u\vert_+ + v\vert_+}{2} > 0$

it follows that

(18)	$u\vert_+ \neq v\vert_+$,	$u\vert_- \neq v\vert_-$,
(19)	$\frac{u\vert_+ + v\vert_+}{2} \leq 0$.	$\frac{u\vert_- + v\vert_-}{2} \geq 0$.

PROOF, only for the first case. Without a loss of generality, we may assume that $u(x)$ is discontinuous at the point under consideration. From (9) and (17) it follows that $v|_- < -u|_- < 0$; so due to (9), the solution $v(x)$ must be continuous, i.e. $v|_+ = v|_-$. Using again (9) and (17), then (18) is deduced from

$$u|_+ = -u|_- = v_+ - (u|_- + v|_-) > v|_+ \ .$$

From (9) and as seen before we have $u|_+ < 0$ and $v|_+ = v|_- < 0$; this verifies (19).

THEOREM 1. *If the* ROTHE *method is applied to the* BURGERS *equation and the corresponding entropy condition, and if the initial data are piecewise continuous, then with the exception of at most a finite number of points, a solution is uniquely determined.*

PROOF. We assume that there is a point $c \in \mathbb{R}$ where both solutions, $u(x)$ and $v(x)$, are continuous with different values

(20) $$u(c) \neq v(c) \ .$$

Then from the further assumption

(21) $$\frac{u(c) + v(c)}{2} \leq 0$$

we can conclude from Lemma 1 either that $u(\infty) \neq v(\infty)$ if there is no point of discontinuity on the right of c, or that the inequality

$$\frac{u|_- + v|_-}{2} < 0$$

holds at the next point on the right of c where at least one solution is discontinuous. In the latter case it follows from Lemma 2 that

$$u|_+ \neq v|_+ \ , \quad \frac{u|_+ + v|_+}{2} \leq 0 \ ;$$

thus with regard to (20) and (21), the same initial situation as before at c is taken on at the point of discontinuity under consideraion. So after at most a finite number of steps we are again led to $u(\infty) \neq v(\infty)$. Since this contradicts (11), so (21) cannot hold. Analogously, together with (20), the assumption

$$\frac{u(c) + v(c)}{2} \geq 0$$

leads to the contradiction $u(-\infty) \neq v(-\infty)$ if we proceed from c to the left. So we get the final contradiction that the arithmetic mean of $u(c)$ and $v(c)$ is positive as well as negative.

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