ON AN INTERNAL APPROXIMATION OF A CLASS OF ELLIPTIC EIGENVALUE PROBLEMS

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1 Introduction.

Let V and H be two real infinite dimensional Hilbert spaces with V compactly and densely embedded in H. Let $a : V \times V \to \mathbf{R}$ be a bilinear form which is symmetric, bounded and strongly coercive. Let $(\cdot,\cdot)$ be the inner product in H, with norm $|\cdot|$. Let V_h be a finite dimensional subspace of V. Finally, let $(\cdot,\cdot)_h$, as an approximation of $(\cdot,\cdot)$, be an inner product in V_h.
With these data we introduce the 'solution operators'

$$\begin{aligned} &T : H \to V, \quad \forall f \in H, \forall v \in V : a(Tf, v) = (f, v) \\ &\tilde{T}^h : V_h \to V_h, \quad \forall f \in V_h, \forall v_h \in V_h : a(\tilde{T}^h f, v_h) = (f, v_h)_h \end{aligned} \tag{1.1}$$

and we consider the corresponding 'exact' and 'approximate' eigenvalue problems (EVP) :

$$\begin{aligned} &\text{Find } \mu \in \mathbf{R} \text{ and } u \in V : Tu = \mu.u \\ &\text{Find } \tilde{\mu}^h \in \mathbf{R} \text{ and } \tilde{u}^h \in V_h : \tilde{T}^h\tilde{u}^h = \tilde{\mu}^h\tilde{u}^h. \end{aligned}$$

The former is the operator version of the EVP for $a(\cdot,\cdot)$ in $V \times V$, relative to $(\cdot,\cdot)$, while the latter is equivalent to the EVP for $a(\cdot,\cdot)$ in $V_h \times V_h$ relative to $(\cdot,\cdot)_h$.

This paper mainly deals with the convergence for $h \to 0$ of an approximate eigenpair, allowing for a multiple exact eigenvalue, under the following hypotheses, met in practice, ($||\cdot||$ is the norm in V),

$$\begin{aligned} &(H_1) \quad \forall v \in V : \inf\{||v - v_h||; v_h \in V_h\} \to 0 \quad \text{if } h \to 0 \\ &(H_2) \quad \forall v_h, w_h \in V_h : \ |(w_h, v_h) - (w_h, v_h)_h| \equiv |E(w_h, v_h)| \le e(h).||w_h||.||v_h||, \\ &\qquad\qquad e(h) \to 0 \text{ if } h \to 0. \end{aligned}$$

(H_1) is the standard approximation property of the finite element subspaces of the Sobolev spaces, used in weak variational EVP's for PDE's. In that context, (H_2) holds for $(\cdot,\cdot)_h$ corresponding to a suitable numerical quadrature for $(\cdot,\cdot)$.

In the case of a simple exact eigenvalue the results are incorporated in those of [3], the latter however being obtained in a less transparent manner. Moreover, the present approach can readily be extended to the case that also $a(\cdot,\cdot)$ is approximated suitably on $V_h \times V_h$.
We rely on [4] (Section V.4.3). First we recall a classical result : $T_r = T|_V$, (1.1), is a compact, self-adjoint, positive definite operator in V. Hence $\mathrm{sp}(T_r)$ consists of an infinite sequence of eigenvalues, all being strictly positive and having finite multiplicity, with zero as accumulation

point, $\mu_1 \geq \mu_2 \geq \ldots \geq \mu_n \geq \ldots \rightarrow 0$ (here **every eigenvalue** occurs as many times as given **by its** multiplicity).
In what follows, C is a generic constant, only depending on V, H and $a(\cdot,\cdot)$.

2 Uniform convergence of $\tilde{T}_e^h$ to T_r.

Let $V_h^\perp$ denote the orthogonal complement of V_h in V relative to $a(\cdot,\cdot)$. One easily proves

Proposition 2.1

$$\tilde{T}_e^h : V \rightarrow V, \quad \tilde{T}_e^h v = \tilde{T}^h v \text{ if } v \in V_h, \quad \tilde{T}_e^h v = 0 \text{ if } v \in V_h^\perp \tag{2.1}$$

defines a compact, self-adjoint, positive operator in V (equipped with $a(\cdot,\cdot)$), having the same eigenpairs as $\tilde{T}^h$, apart from the trivial eigenvalue zero.

To 'compare' this extension $\tilde{T}_e^h$ with T, we use the 'intermediate' operator

$$T^h : V \rightarrow V_h, \quad \forall f \in V, \forall v_h \in V_h : a(T^h f, v_h) = (f, v_h).$$

Note that $T_h = \pi_h T_r$, where $\pi_h : V \rightarrow V_h$ is the projection operator relative to $a(\cdot,\cdot)$. Invoking (H_1) and the compactness of T_r, one has, using [1] (Theorem 3.2 p. 124)

Proposition 2.2

$$||T_r - T^h|| \equiv \sup\{||(T - T^h)v||; v \in V, ||v|| \leq 1\} \rightarrow 0 \text{ if } h \rightarrow 0. \tag{2.2}$$

Theorem 2.1

$$||T_r - \tilde{T}_e^h|| \equiv \sup\{||(T - \tilde{T}_e^h)v||; v \in V, ||v|| \leq 1\} \rightarrow 0 \text{ if } h \rightarrow 0.$$

Proof. By (2.2) it is sufficient to consider $(T^h - \tilde{T}_e^h)$. Denoting by α the coercivity constant of a, one has

$$\forall v \in V, \quad \alpha||(T^h - \tilde{T}_e^h)v||^2 \leq (v - \pi_h v, (T^h - \tilde{T}_e^h)v) + E(\pi_h v, (T^h - \tilde{T}_e^h)v).$$

Using (H_1), the continuity of $i : V \rightarrow H$ and the coercivity and boundedness of a, one gets

$$||(T^h - \tilde{T}_e^h)v|| \leq C.[|v - \pi_h v| + \epsilon(h).||v||].$$

From a variant of the Aubin-Nitsche lemma, cfr. [1] (Lemma 4.26 p. 215), one finds (with $\hat{H}$ the unit ball in H)

$$|v - \pi_h v| \leq C.||v - \pi_h v||.\sup\{||w - \pi_h w||; w \in T(\hat{H})\} \leq \epsilon(h).||v||.$$

Invoking the compactness of i and the spectral decomposition theorem of a compact operator, T, (1.1), may be shown to be compact. (H_1) then implies that $\epsilon(h) \rightarrow 0$ if $h \rightarrow 0$. ∎

3 Convergence of the eigenvalues.

Relying on [4] (Section V.4.3) one readily obtains

Lemma 3.1 *Let μ be an eigenvalue of T_r with multiplicity m and isolation distance d. If h is sufficiently small, then the open interval $(\mu - d/2, \mu + d/2)$ contains exactly m eigenvalues of $\tilde{T}^h$, counting with their multiplicity.*

Lemma 3.2

$$\sup_{\tilde{\nu}^h} \inf_{\nu} |\tilde{\nu}^h - \nu| \text{ and } \sup_{\nu} \inf_{\tilde{\nu}^h} |\tilde{\nu}^h - \nu| \leq ||T_r - \tilde{T}_e^h|| ,$$

where ν and $\tilde{\nu}^h$ run over $\mathrm{sp}(T_r)$ and $\mathrm{sp}(\tilde{T}_e^h)$ respectively.

We number the nonzero eigenvalues $\tilde{\mu}_l^h$, $1 \leq l \leq \dim V_h$ of $\tilde{T}_e^h$, similarly to those of T_r. Then, combining the two lemmas, we arrive at

Theorem 3.1

$$|\tilde{\mu}_l^h - \mu_l| \leq ||T_r - \tilde{T}_e^h||, \quad 1 \leq l \leq \dim V_h, \quad h \text{ sufficiently small.}$$

Consequently, from Theorem 2.1, $\tilde{\mu}_l^h \to \mu_l$, $l \geq 1$, if $h \to 0$.

4 Convergence of the eigenfunctions.

Let $\mu_{k-1} < \mu_k = \mu_{k+1} = \ldots = \mu_{k+m} < \mu_{k+m+1}$, i.e. let μ_k be an $(m+1)$-fold eigenvalue of T_r. Denote by u_{k+r}, $0 \leq r \leq m$, eigenfunctions of T_r, corresponding to μ_k, orthonormal in H. Let E be the space spanned by these eigenfunctions. Likewise, let $\tilde{u}_{k+r}^h$, $0 \leq r \leq m$, be eigenfunctions of $\tilde{T}^h$, corresponding to the eigenvalues $\tilde{\mu}_{k+r}^h$, $0 \leq r \leq m$, and being orthonormalized with respect to $(\cdot,\cdot)_h$. Set $\tilde{E}^h = \mathrm{span}(\tilde{u}_k^h, \ldots, \tilde{u}_{k+m}^h)$. Finally, let $\tilde{P}^h$ be the spectral projection of V onto $\tilde{E}^h$. Similarly as in [5] (Section VIII.5), one has

Proposition 4.1 *Let $w_k \in E$, then, for sufficiently small h,*

$$||w_k - \tilde{P}^h w_k|| \leq C.||(T - \tilde{T}_e^h) w_k|| .$$

Corollary 4.1

$$\delta(E, \tilde{E}^h) \equiv \sup\{d(w_k, \tilde{E}^h); w_k \in E, ||w_k|| = 1\} \leq C.||T_r - \tilde{T}_e^h||.$$

Consequently, from Theorem 2.1, the distance between the two 'eigenspaces' tends to zero with h. Moreover one has

Theorem 4.1 *There exists a set of eigenfunctions U_{k+r}^*, $0 \leq r \leq m$ of T, corresponding to μ_k and being orthonormalized with respect to $(\cdot,\cdot)$, such that, with $\tilde{u}_{k+r}^h$, $0 \leq r \leq m$ as above,*

$$||U_{k+r}^* - \tilde{u}_{k+r}^h|| \to 0 \text{ if } h \to 0, \quad 0 \leq r \leq m. \tag{4.1}$$

Proof. This adapts the two basic ideas of the proof in [2] (Theorem XII.4.5, p. 907-909), but is more involved. First one defines the non-singular square matrix $(\beta) = (\beta_{ri})$ by

$$\tilde{P}^h u_{k+r} = \sum_{i=0}^{m} \beta_{ri}.\tilde{u}^h_{k+i}, \quad 0 \leq r \leq m$$

and one introduces

$$U_{k+t} = \sum_{l=0}^{m} (\beta^{-1})_{tl}.u_{k+l}, \quad 0 \leq t \leq m.$$

Using Proposition 4.1, one may show that $||U_{k+t} - \tilde{u}^h_{k+t}|| \to 0$, $0 \leq t \leq m$, if $h \to 0$. From this convergence, (4.1) can be derived by induction, whence U^*_{k+r} is generated from U_{k+r}, $0 \leq r \leq m$, by the Gram-Schmidt orthonormalization procedure. ∎

5 Approximation of the bilinear form.

The analysis above may be adapted to the case that $a(\cdot,\cdot)$ is suitably approximated on $V_h \times V_h$ and $(\cdot,\cdot)$ is retained exactly. By superposition one arrives at the case where both $a(\cdot,\cdot)$ and $(\cdot,\cdot)$ are suitably approximated on $V_h \times V_h$. Thus define

$$\hat{T}^h : H \to V_h, \quad \forall f \in H, .\forall v \in V_h : a_h(\hat{T}^h f, v_h) = (f, v_h)\,,$$

where $a_h(\cdot,\cdot)$ is a symmetric, uniformly bounded and uniformly strongly coercive bilinear form on $V_h \times V_h$ (uniformly with respect to h), fulfilling a hypothesis similar to (H_2).
To show the convergence for $h \to 0$ of the eigenvalues and eigenfunctions of $\hat{T}^h$, which is a compact, self-adjoint, positive definite operator in H, one proves that

$$|T - \hat{T}^h| \equiv \sup\{|(T - \hat{T}^h)v|; v \in H, |v| \leq 1\} \to 0 \text{ if } h \to 0.$$

References.

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