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A FINITE ELEMENT METHOD FOR A MODEL OF POPULATION DYNAMICS WITH SPATIAL DIFFUSION

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1. **Introduction.** We consider the approximation of the solution $u(a,t,\mathbf{x})$ of the age distribution function in a one-sex population which moves locally in a bounded domain $\Omega \subset \mathbf{R}^2$. Here a denotes the age, t the time and $\mathbf{x} = (x, y)$ the spatial variables. We use the following degenerate parabolic equation to describe the dynamics of the population:

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial a} - \underset{\sim}{\nabla} \cdot \Big(k(p)u\underset{\sim}{\nabla} p\Big) = -\mu u, \qquad a \geq 0, \quad 0 < t \leq T, \quad \mathbf{x} \in \Omega, \tag{1.1}$$

where we have omitted the explicit writing of the time and spatial variables as arguments as we do throughout the paper when there is no ambiguity; the nonnegative function $\mu = \mu(a,t,\mathbf{x})$ is the age-specific mortality rate, $\mathbf{q}(a,t,\mathbf{x}) = k(p)u\underset{\sim}{\nabla} p$ is the flux of individuals aged a at time t at the point $\mathbf{x}$,

$$p(t,\mathbf{x}) = \int_0^\infty u(a,t,\mathbf{x})da \tag{1.2}$$

is the total population density, and $\underset{\sim}{\nabla} = (\frac{\partial}{\partial x}, \frac{\partial}{\partial y})$. Boldface characters, as well as a tilde under a character, are used to indicate vectors. The flux we consider here is one that results in directed dispersal in the direction of least crowding. Other fluxes have been considered (e.g. to describe a random dispersal in [2,8], among others), and our present analysis readily extends to those cases. The initial age distribution is given by

$$u(a,0,\mathbf{x}) = u^0(a,\mathbf{x}), \tag{1.3}$$

where u^0 is a smooth, nonnegative, compactly supported function in the age variable. Finally, the birth rate $u(0,t,\mathbf{x})$ satisfies

$$u(0,t,\mathbf{x}) = B(t,\mathbf{x}) = \int_0^\infty \beta(a,t,\mathbf{x})u(a,t,\mathbf{x})da, \tag{1.4}$$

where the nonnegative function β is the age specific birth rate of the population, and the homogeneous Neumann boundary condition

$$\mathbf{q} \cdot \underset{\sim}{\nu} = 0, \qquad \mathbf{x} \in \partial\Omega, \tag{1.5}$$

is imposed to describe a population without immigration or emigration. The vector $\underset{\sim}{\nu}$ denotes the outward unit normal to $\partial\Omega$. We assume that β and μ are continuously differentiable functions with bounded derivatives.

Several authors have treated various theoretical aspects of (1.1)–(1.5); we refer the reader to [2,5-8], just to mention a few. The numerical solution of this problem in the case of a linear flux $\mathbf{q} = k(x)\underset{\sim}{\nabla} u$ was carried out in [9]. A finite difference method of characteristics was employed in [4] for the case without diffusion. The numerical analysis of a similar two-sex model without spatial diffusion was done in [1,10], respectively, by the finite difference method of characteristics and by a finite element method in the age variables with Crank-Nicolson time discretization.

It can be shown that the initial-boundary value problem (1.1)-(1.5), with the initial and boundary conditions satisfying some compatibility and regularity constraints has a unique solution u which is compactly supported in the age variable for any time t (see, for example, [2,8]). Note that this implies that all the integrals which appear in this paper are really over a finite interval.

2. **A Reformulation of the Model.** Following [2], we modify the initial-boundary value problem (1.1)-(1.5) as follows. We integrate (1.1) in a and use (1.2)–(1.5) to see that the population

function p satisfies

$$\frac{\partial p}{\partial t} - \underset{\sim}{\nabla} \cdot \left(k(p) p \underset{\sim}{\nabla} p \right) = p \int_0^\infty (\beta - \mu) v \; da, \tag{2.1}$$

with the initial and boundary conditions

$$p(0, \mathbf{x}) = \int_0^\infty u^0(a, \mathbf{x}) \; da, \qquad k(p) p \frac{\partial p}{\partial \nu} = 0. \tag{2.2}$$

Next, we introduce the *age profile function* $v = \frac{u}{p}$ which satisfies the equation

$$\frac{\partial v}{\partial t} + \frac{\partial v}{\partial a} - k(p) \underset{\sim}{\nabla} p \cdot \underset{\sim}{\nabla} v = -v \left[\mu + \int_0^\infty (\beta - \mu) \; v \; da \right], \quad a \geq 0, \quad 0 < t \leq T, \quad \mathbf{x} \in \Omega, \tag{2.3}$$

and the initial and boundary conditions

$$v(a, 0, \mathbf{x}) = \frac{u^0(a,\mathbf{x})}{\int_0^\infty u^0(a,\mathbf{x}) da}, \qquad v(0, t, \mathbf{x}) = \int_0^\infty \beta v \; da, \qquad \tfrac{1}{p} \mathbf{q} \cdot \underset{\sim}{\nu} = k(p) v \frac{\partial p}{\partial \nu} = 0. \tag{2.4}$$

Finally, by introducing the "backward characteristics" $\underset{\sim}{\phi}^\tau = ((\phi^\tau)_1, (\phi^\tau)_2)$ as solutions of

$$\frac{\partial \underset{\sim}{\phi}^\tau}{\partial t}(t, \mathbf{x}) = -k(p(t, \underset{\sim}{\phi}^\tau(t, \mathbf{x}))) \; p(t, \underset{\sim}{\phi}^\tau(t, \mathbf{x})) \left[\left(\underset{\sim}{\nabla} p \right) (t, \underset{\sim}{\phi}^\tau(t, \mathbf{x})) \right], \quad \underset{\sim}{\phi}^\tau(\tau, \mathbf{x}) = \mathbf{x}, \tag{2.5}$$

and letting

$$w(a, t, \mathbf{x}) = v(a, t, \underset{\sim}{\phi}^0(t, \mathbf{x})),$$

we see that the system (2.1)–(2.4) can be replaced by the modified system, with unknowns $\underset{\sim}{\phi}^t, w$, and p given, respectively, by (2.5),

$$\left\{ \begin{array}{l} \dfrac{\partial w}{\partial t} + \dfrac{\partial w}{\partial a} = -w \left[\mu + \displaystyle\int_0^\infty (\beta - \mu) \; w \; da \right], \qquad a \geq 0, \quad 0 < t \leq T, \quad \mathbf{x} \in \Omega, \\ w(a, 0, \mathbf{x}) = w^0(a, \mathbf{x}) = \frac{u^0(a,\mathbf{x})}{\int_0^\infty u^0(a,\mathbf{x}) da}, \quad a \geq 0, \quad \mathbf{x} \in \Omega, \\ w(0, t, \mathbf{x}) = \displaystyle\int_0^\infty \beta w \; da, \qquad 0 < t \leq T, \quad \mathbf{x} \in \Omega, \\ w(a, t, \mathbf{x}) \geq 0, \qquad a \geq 0, \quad 0 < t \leq T, \quad \mathbf{x} \in \Omega, \\ \displaystyle\int_0^\infty w \; da = 1, \qquad 0 < t \leq T, \quad \mathbf{x} \in \Omega, \\ w(a, t, \mathbf{x}) \to 0 \text{ as } a \to \infty, \qquad 0 < t \leq T, \quad \mathbf{x} \in \Omega, \end{array} \right. \tag{2.6}$$

and

$$\left\{ \begin{array}{l} \frac{\partial p}{\partial t} - \underset{\sim}{\nabla} \cdot \left(k(p) p \underset{\sim}{\nabla} p \right) = p \displaystyle\int_0^\infty (\beta - \mu) w(a, t, \underset{\sim}{\phi}^t(0, \mathbf{x})) \; da, \qquad 0 < t \leq T, \quad \mathbf{x} \in \Omega, \\ p(0, \mathbf{x}) = \displaystyle\int_0^\infty u^0(a, \mathbf{x}) \; da, \qquad \mathbf{x} \in \Omega, \\ k(p) p \dfrac{\partial p}{\partial \nu} = 0, \qquad 0 < t \leq T, \quad \mathbf{x} \in \partial\Omega, \\ p(t, \mathbf{x}) \geq 0, \qquad 0 < t \leq T, \quad \mathbf{x} \in \Omega. \end{array} \right. \tag{2.7}$$

Note that the variable $\mathbf{x}$ in (2.6) appears only as a parameter. It is possible to solve (2.6) for w independently of (2.5) and (2.7). Once w has been found, (2.5) and (2.7) form a coupled system for p and $\underset{\sim}{\phi}$ which, for some special cases, has been analyzed in [2]. This formulation separates the hyperbolic and the degenerate parabolic behavior of (1.1), and so, in particular, w is smoother than u (see [2]). Note that $u(a, t, \mathbf{x}) = w(a, t, \underset{\sim}{\phi}^t|_{t=0}) p(t, \mathbf{x})$. Hence, once w, $\underset{\sim}{\phi}$ and p have been approximated, an approximation to $u(a_i, t^n, \mathbf{x})$ can be immediately obtained from this expression.

3. A Numerical Method. We begin by discretizing (2.6) using the finite difference method of characteristics employed in [4]. Let $S > 0$ be a fixed integer and let $\Delta t = T/S$ and $t^n = n\Delta t$, $0 \leq n \leq S$. Let $\mathcal{G} = \{a_i = i\Delta t, \quad 0 \leq i < \infty\}$ be the grid for the age discretization. We seek a discrete function $W = W_i^n(\mathbf{x})$ which approximates the value $w_i^n = w(a_i, t^n, \mathbf{x})$. We use the notation $f_i^n = f(a_i, t^n)$ for any function f of the arguments a and/or t. The algorithm to approximate w is given by:

$$\left\{\begin{aligned} & W_i^0 = w^0(a_i), \qquad i \geq 0, \\ & \frac{W_i^n - W_{i-1}^{n-1}}{\Delta t} = -W_i^n \left[\mu_i^n + \sum_{j=1}^{\infty}(\beta_{j-1}^{n-1} - \mu_{j-1}^{n-1})W_{j-1}^{n-1}\Delta t\right], \quad n > 0, i \geq 1, \\ & W_0^n = \sum_{j=1}^{\infty} \beta_j^n W_j^n \Delta t \quad n > 0. \end{aligned}\right. \tag{3.1}$$

The approximation (3.1) converges to the solution of (2.6) at an optimal rate [11].

Theorem 3.1: *Let the solution w of (2.6) be continuously differentiable with bounded derivatives. Then, there exists a constant $C > 0$, independent of Δt, such that, for Δt sufficiently small,*

$$\max_{0 \leq n \leq S} \sup_{i \geq 0} \{|w_i^n - W_i^n|\} \leq C\Delta t.$$

We use a modification of the method of [12] to approximate p while simultaneously approximating $\underset{\sim}{\phi}$ (which is the solution of a system of ordinary differential equations) by Euler's method. Let us simplify the notation by introducing an antiderivative of $k(p)p$. Let $\sigma(p) = \int_0^p k(s)s\, ds, \quad p \geq 0$. Note that the flux which appears in (2.7), is now just $\underset{\sim}{\nabla}\sigma$, and thus (2.7) can be rewritten in the form

$$\left\{\begin{aligned} & \tfrac{\partial p}{\partial t} - \Delta\sigma(p) = R\left(t, \underset{\sim}{\phi}^t(0, \mathbf{x})\right)p, \quad 0 < t \leq T, \ \mathbf{x} \in \Omega, \\ & \frac{\partial\sigma}{\partial\nu} = 0, \qquad\qquad 0 < t \leq T, \quad \mathbf{x} \in \partial\Omega, \end{aligned}\right.$$

where we have set

$$R(t, \mathbf{y}) = \int_0^\infty [\beta(a, t, \mathbf{y}) - \mu(a, t, \mathbf{y})]\, w(a, t, \mathbf{y})\, da, \tag{3.2}$$

while (2.5) is rewritten as

$$\frac{\partial \underset{\sim}{\phi}^\tau}{\partial t} = \left(-\underset{\sim}{\nabla}\sigma(p)\right)(\underset{\sim}{\phi}^\tau), \qquad \underset{\sim}{\phi}^\tau(\tau, \mathbf{x}) = \mathbf{x}. \tag{3.3}$$

Note that (3.2), together with the assumptions on the data, imply that R is uniformly bounded. Let

$$V = H^1(\Omega) \cap \left\{\tfrac{\partial f}{\partial\nu} = 0 \quad \text{on} \quad \partial\Omega\right\}, \qquad W = L^2(\Omega),$$

and denote the standard L^2-inner product by $(\ \cdot\ ,\ \cdot\ )$. Let $\{\mathcal{T}_h\}_{h\to 0}$ be a regular family of triangulations of Ω of characteristic parameter h, and, associated to it, let us consider the following finite element subspaces of V and W:

$$V^h = \{\text{continuous piecewise linear polynomials on } \mathcal{T}_h\} \cap \left\{\tfrac{\partial f}{\partial\nu} = 0 \text{ on } \partial\Omega\right\},$$

$$W^h = \{\text{discontinuous piecewise constant polynomials on } \mathcal{T}_h\}.$$

Let $\Pi : L^2(\Omega) \longrightarrow W^h$ be the orthogonal L^2-projection onto W^h. We need an approximation of R, ρ, naturally given by replacing w in (3.2) by the continuous age-time piecewise bilinear interpolant W of its nodal approximation W_i^n, $0 \leq n \leq S$, $0 \leq i$, given by (3.1) and prolonged by zero outside Ω:

$$\rho(t, \mathbf{y}) = \int_0^\infty [\beta(a,t,\mathbf{y}) - \mu(a,t,\mathbf{y})] W(a,t,\mathbf{y})\, da.$$

The finite element method we propose is the following: Find functions $\underset{\sim}{F}^n(\mathbf{x}) \in W^h \times W^h$, $P^n \in W^h$ and $Z^n \in V^h$, such that

$$\begin{cases} \underset{\sim}{F}^0(\mathbf{x}) = [\Pi \times \Pi](\mathbf{x}), \\ (P^0 - p^0, \psi) = 0, \qquad \psi \in W^h, \\ (\underset{\sim}{\nabla}[Z^0 - \sigma^0], \underset{\sim}{\nabla}\chi) = (\sigma^0 - Z^0, \chi), \qquad \chi \in V^h, \\ \underset{\sim}{F}^n(\mathbf{x}) = \underset{\sim}{F}^0(\mathbf{x}) + \Delta t \sum_{i=1}^{n} \underset{\sim}{\nabla} Z^{i-1}, \qquad n > 0, \\ (Z^n, \chi) + \frac{\Delta t}{\lambda}(\underset{\sim}{\nabla} Z^n, \underset{\sim}{\nabla}\chi) = \left(\sigma(P^{n-1}) + \frac{\Delta t}{\lambda} P^{n-1} \rho(t^n, \underset{\sim}{F}^n), \chi\right), \ \chi \in V^h, \ n > 0, \\ P^n = P^{n-1} + \lambda\left[\Pi Z^n - \sigma(P^{n-1})\right], \ n > 0, \end{cases} \tag{3.5}$$

where we have chosen some positive constant λ smaller than the reciprocal of the Lipschitz constant for σ, L_σ.

The next to last equation in (3.5) results in a linear system of algebraic equations with a symmetric positive definite matrix and, thus, is uniquely solvable.

The unconditional stability of this algorithm was established in [11].

Theorem 3.2: *Let Ω be a convex polygon and let $\sigma(p)$ be Lipschitz continuous growing at least linearly at infinity (that is, $|p| \leq C_1 + C_2|\sigma(p)|$). Then, there is $C > 0$, independent of h and Δt, such that*

$$\|P\|_{l^\infty(L^2)} + \|\sigma(P)\|_{l^\infty(L^2)} + \|\underset{\sim}{\nabla} Z\|_{l^2(L^2)} + \sum_{j=1}^{S} \|P^j - P^{j-1}\|_0 \leq C.$$

The convergence of the algorithm for the approximation of p was established in [11] only for small t. The author is presently extending that result to be global in time.

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