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NUMERICAL ASPECTS OF COMPUTATION OF PERIODIC AND QUASIPERIODIC SOLUTIONS IN SYSTEMS OF PARTIAL DIFFERENTIAL EQUATIONS

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1. Introduction

In this paper we shall deal with some numerical aspects of computation of periodic and quasiperiodic solutions of the following system of two parabolic equations

$$
\begin{aligned}
\frac{\partial x}{\partial t} &= \frac{D_x}{L^2}\frac{\partial^2 x}{\partial z^2} + x^2 y - (B+1)x + A \\
\frac{\partial y}{\partial t} &= \frac{D_y}{L^2}\frac{\partial^2 y}{\partial z^2} + Bx - x^2 y \quad .
\end{aligned}
\qquad (1)
$$

These equations describe behaviour of a reaction - diffusion system (in spatially one-dimensional medium) with Brusselator reaction scheme [e.g.1]. Here L is a characteristic dimension of the system, $z \in [0,1]$ is dimensionless spatial coordinate, x,y concentrations and t is time. For simplicity choose boundary conditions of the Dirichlet type

$$x(t,0)=x(t,1)=\bar{x} \qquad y(t,0)=y(t,1)=\bar{y}, \qquad (2)$$

where $\bar{x}$ and $\bar{y}$ satisfy $\bar{x}^2\bar{y}-(B+1)\bar{x}+A=0$ and $B\bar{x}-\bar{x}^2\bar{y}=0$, cf. Eq. (1). We consider L as a bifurcation parameter, the values of remaining parameters are chosen: $D_x=0.008$, $D_y=0.004$, $A=2$, $B=5.45$. In our case are thus $\bar{x}=A=2$ and $\bar{y}=B/A=2.725$.

In the preceding paper [2] we presented the method of continuation of periodic solutions of Eq. (1), i.e. the way how to construct so called solution diagram of periodic solutions. The point of formation of a quasiperiodic solution was determined on the basis of an analysis of periodic solutions in dependence on L. The development of this quasiperiodic solution and its desintegration into a chaotic attractor via a cascade of the torus doubling was presented in the paper [3].

2. Computation of periodic solutions

The computation of periodic solutions for the system (1) is based on the transformation of the system of PDE's into a large set of ODE's by means of the method of lines (semidiscretization). We shall consider two discretizations of spatial derivatives here. Let us denote the approximations ($z_i = ih$, $h = 1/N$)

$$x_i(t) \sim x(t,z_i),\ y_i(t) \sim y(t,z_i),\ i = 0,1,\dots N. \tag{3}$$

From the boundary conditions (2) it follows

$$x_0(t) = \bar{x},\ x_N(t) = \bar{x},\ y_0(t) = \bar{y},\ y_N(t) = \bar{y} \quad . \tag{4}$$

The most simple discretization consists in a replacement of spatial derivatives by the folloving three-point finite difference formula (u is x or y):

$$\left.\frac{\partial^2 u}{\partial z^2}\right|_{(t,z_i)} \sim \frac{u_{i-1}-2u_i+u_{i+1}}{h^2}\ , \quad i=1,2\dots,N-1\ . \tag{5}$$

After inserting (5) into Eqs. (1) we obtain a system of ODE's ($i = 1,2,\dots,N-1$):

$$\begin{aligned}\frac{dx_i}{dt} &= \frac{D_x}{L^2h^2}(x_{i-1} - 2x_i + x_{i+1}) + x_i^2y_i - (B+1)x_i + A \\ \frac{dy_i}{dt} &= \frac{D_y}{L^2h^2}(y_{i-1} - 2y_i + y_{i+1}) + Bx_i - x_i^2y_i\end{aligned} \tag{6}$$

Second discretization uses five-point finite-difference formulas (u is x or y):

$$\left.\frac{\partial^2 u}{\partial z^2}\right|_{(t,z_i)} \sim \frac{-2u_{i-2}+32u_{i-1}-60u_i+32u_{i+1}-2u_{i+2}}{24h^2}\ , \quad i=2,3,\dots,N-2$$

$$\left.\frac{\partial^2 u}{\partial z^2}\right|_{(t,z_1)} \sim \frac{22u_0-40u_1+12u_2+8u_3-2u_4}{24h^2} \tag{7}$$

$$\left.\frac{\partial^2 u}{\partial z^2}\right|_{(t,z_{N-1})} \sim \frac{-2u_{N-4}+8u_{N-3}+12u_{N-2}-40u_{N-1}+22u_N}{24h^2}$$

After inserting (7) into Eqs (1) we obtain again a system of 2(N-1) ODE's analogously as in the preceding case, cf. Eq. (6).

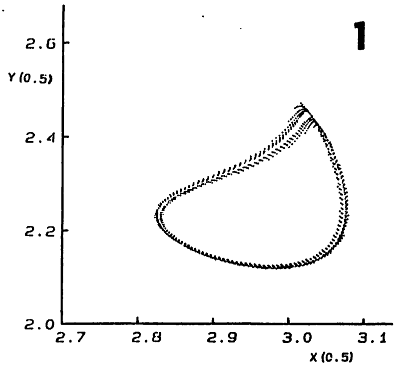
Comparison of accuracy of resulting periodic solutions obtained for different N indicates Table 1. There the values of the period T and absolute values of two leading characteristic multipliers λ_2 and λ_3 (eigenvalues of the monodromy matrix, $\lambda_1 = 1$ always for an autonomous system) are presented. Let us note that the periodic solution has in the chosen semidiscretization alltogether 2(N-1) characteristic multipliers. We can conclude from the Table that the results for (6) and N=40 are comparable with results for (7) and N=20 and that the approximation (6) gives also for N=20 satisfactory results.

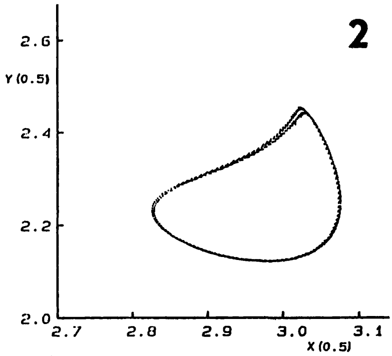
Table 1: Period T and $|\lambda_{2,3}|$ in dependence on N, L = 1.338533 (cf. the branch with tori bifurcation point T_1 in [3]). Periodic solution is stable, λ_2 and λ_3 are mutually complex conjugate.

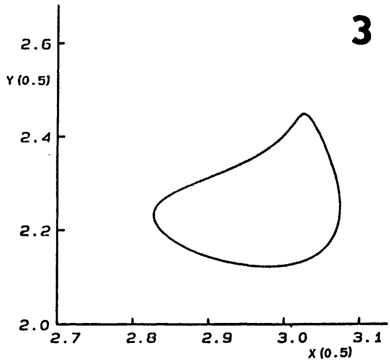
	approximation (6)		approximation (7)	
N	T	$\|\lambda_{2,3}\|$	T	$\|\lambda_{2,3}\|$
5	divergence		divergence	
10	3.412599	0.931	3.417790	0.849
20	3.412613	0.901	3.412665	0.890
40	3.412666	0.892		

3. Computation of quasiperiodic solutions

The point of invariant torus bifurcation (denoted as T_1 in [3]) has been detected on the branch considered for L $\doteq$ 1.37. At this point a branch of stable quasiperiodic solutions bifurcates. We studied behaviour of trajectories in the neighbourhood of the invariant torus by using dynamic simulation of the system (6) for N = 20. The visualization of the trajectory will be made by using a trajectory of







Figs 1–3: Poincarè maps for
L = 1.405

the Poincarè map, i.e. in our case the map from a hyperplane $x_6=2$ **into** the same hyperplane. A projection of this trajectory into the **plane** x_{10} - y_{10} (i.e. x(t, 0.5) - y (t, 0.5)) **is depicted** in Figs 1,2 and 3. In all three cases the same initial **condition** was chosen as the corresponding periodic solution perturbed on third decimal place. The points of the trajectory of Poincarè map No. 500 - 1500, 1500 - 2500 and 3000 - 4000 are depicted in Fig. 1,2 and 3, respectively. It can be concluded from the figures that the attractivity of the invariant torus is low. Fig. 3 presents already the "intersection" of the torus with the hyperplane $x_6 = 2$. The low attractivity of the torus is probably caused by a weakly repelling periodic solution in the neighbourhood. This periodic solution has only two multipliers outside the unit circle and in absolute value near to unity, cf. Table 2.

Table 2: Leading 14 multipliers for unstable periodic solution for L = 1.405, T = 3.405269. Approximation (6), N = 20. Remaining multipliers $\lambda_{15}, \dots, \lambda_{38}$ are close to zero, $\lambda_1 = 1$.

$\lvert\lambda_{2,3}\rvert$	1.1017	$\lvert\lambda_{8,9}\rvert$	0.1361
$\lvert\lambda_4\rvert$	0.7797	$\lvert\lambda_{10}\rvert$	0.0569
$\lvert\lambda_{5,6}\rvert$	0.3919	$\lvert\lambda_{11,12}\rvert$	0.0401
$\lvert\lambda_7\rvert$	0.1869	$\lvert\lambda_{13,14}\rvert$	0.0127

Literature

[1] Nicolis, G., Prigogine, I.: Self - organization in Nonequilibrium Systems. J. Wiley, New York 1977.

[2] Kubíček, M., Holodniok, M.: Periodic phenomena in reaction diffusion systems - numerical analysis. In Numerical Treatment of Differential Equations, Ed. K. Strehmel, Teubner-Texte zur Mathematik, Band 104, Teubner Leipzig 1988, pp. 299 - 305.

[3] Holodniok, M., Kubíček, M., Marek, M.: Desintegration of an invariant torus in a reaction - diffusion system. Teubner-Texte zur Mathematik, Teubner Leipzig 1990, in press.