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THE ACCURACY OF NUMERICALLY COMPUTED ORBITS OF DYNAMICAL SYSTEMS

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In their papers [2,3] Hammel, Yorke and Grebogi have given a procedure which determines the accuracy of numerically computed orbits of dynamical systems. They apply their procedure to maps which exhibit a large amount of *hyperbolicity*. However their procedure does not use the hyperbolicity explicitly. In this paper we give a procedure for one-dimensional maps which does use the hyperbolicity explicitly. Unlike the procedure of Hammel et al., our procedure works forward. After N iterates we can decide whether our theorem applies and, if it does, we can estimate how far the computed orbit is from a true orbit.

Now we state the main theorem. Let $f : [0,1] \to [0,1]$ be a C^2 function and let $\{y_n\}_{n=0}^{N+1}$ be a *pseudo-orbit* of this map, i.e., $| y_{n+1} - f(y_n) |$ is small for $n = 0, 1, ..., N$. We define the quantities

$$\sigma = \sup_{n=0}^{N} \sum_{m=n}^{N} | Df(y_n)^{-1} Df(y_{n+1})^{-1} ... Df(y_m)^{-1} |,$$

which measures the *expansiveness* of the map, and

$$\tau = \sup_{n=0}^{N} | \sum_{m=n}^{N} Df(y_n)^{-1} Df(y_{n+1})^{-1} ... Df(y_m)^{-1} [y_{m+1} - f(y_m)] | .$$

It turns out that τ gives a good measure of how close the pseudo-orbit (of course, our numerically computed orbits will be pseudo-orbits) is to a true orbit.

THEOREM. *Let $f : [0,1] \to [0,1]$ be a C^2 function with*

$$M = \sup\{| D^2 f(x) | : 0 \le x \le 1\}.$$

Let $\{y_n\}_{n=0}^{N+1}$ be a pseudo-orbit of f such that

$$2M\sigma\tau \le 1.$$

Then there is an exact orbit $\{x_n\}_{n=0}^{N}$ with

$$(1 + 1/2(1 + \sqrt{1 - 2M\sigma\tau}))^{-1}\tau \le \sup_{n=0}^{N} | x_n - y_n | \le 2(1 + \sqrt{1 - 2M\sigma\tau})^{-1}\tau.$$

Outline of proof. Denote by S the set of sequences $\mathbf{x} = \{x_n\}_{n=0}^{N}$ with $| x_n - y_n | \le \varepsilon$ for $n = 0, 1, ..., N$, where

$$\varepsilon = 2\tau/(1 + \sqrt{1 - 2M\sigma\tau}).$$

S is a compact convex subset of $\mathbf{R}^{N+1}$. We define a mapping T on S. If $\mathbf{x} \in S$ we define

$$(T\mathbf{x})_n = y_n - \sum_{m=n}^{N} Df(y_n)^{-1} Df(y_{n+1})^{-1} .... Df(y_m)^{-1} h_m \qquad (n = 0, ..., N),$$

where

$$h_n = f(x_n) - y_{n+1} - Df(y_n)(x_n - y_n).$$

It turns out that T is a continuous mapping of S into itself and so, by Brouwer's fixed point theorem, has a fixed point $\mathbf{x} = \{x_n\}_{n=0}^{N}$. This is the exact orbit that we wanted.

Note that the idea of this proof was suggested by the proofs of the *shadowing lemma* given in Palmer [4] and Chow, Lin and Palmer [1].

The Method of Computation

Let $f : [0,1] \to [0,1]$ be a C^2 mapping. Suppose our computer starts with a number y_0 in $[0,1]$ and computes an orbit $\{y_n\}_{n=0}^{N+1}$ of f in single precision. $\{y_n\}$ will be, in fact, a pseudo-orbit. To use the theorem we have to find the quantities σ and τ. For large N it would not be practical to compute the sums $\sum_{m=n}^{N}$. Instead we calculate the quantities

$$\sigma_p = \sup_{n=0}^{N} \sum_{m=n}^{\min(n+p,N)} | Df(y_n)^{-1} ... Df(y_m)^{-1} |,$$
$$\tau_p = \sup_{n=0}^{N} | \sum_{m=n}^{\min(n+p,N)} Df(y_n)^{-1} ... Df(y_m)^{-1} [y_{m+1} - f(y_m)] | ,$$

where p is an integer, $0 \le p \le N$, such that

$$\mu_p = \sup_{n=0}^{N-p} | Df(y_n)^{-1} ... Df(y_{n+p})^{-1} | < 1.$$

It turns out that

$$\sigma \le (1-\mu_p)^{-1}\sigma_p, \ \tau \le (1-\mu_p)^{-1}\tau_p. \tag{1}$$

The computation of μ_p, σ_p, τ_p is done in double precision. We have fully analyzed the effect of round-off error on these computations. Unless the hyperbolicity if very weak (i.e. σ is large and $\mu_p < 1$ only for large p), it turns out that the effect of round-off error is very slight.

Example. We consider the quadratic map $f(x) = ax(1-x)$ with $a = 3.8$. Then $M = 2a = 7.6$. The computations were done on an IBM compatible computer using Microsoft Quickbasic. For $N = 426{,}000, p = 30$ and $y_0 = .3$, we find that

$$\mu_p = 2.297433184600331 * 10^{-3},$$
$$\sigma_p = 375.6005726956602,$$
$$\tau_p = 9.60282364278178 * 10^{-6}.$$

Using the inequalities (1) and taking into account the round-off error, we find that

$$\sigma \le 376.4658, \ \tau \le 9.624939 * 10^{-6}.$$

Then $2M\sigma\tau \le .05507661$ and

$$2\tau/(1+\sqrt{1-2M\sigma\tau}) \le 9.76125 * 10^{-6}.$$

Our theorem enables us to conclude that during 426,000 iterates our computed orbit differs by at most $1/10^5$ from a true orbit. Note that the orbit was computed only in single precision, that is to an approximate accuracy of 7 decimal digits. So over 426,000 iterates we have only lost two digits of accuracy.

References

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