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THE OPTIMAL CONTROL PROBLEM IN COEFFICIENTS FOR THE PSEUDOPARABOLIC VARIATIONAL INEQUALITY

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We shall deal with an optimal control problem for a pseudoparabolic variational inequality with controls appearing in operator coefficients, right hand sides as well as in convex sets of states. In addition to [3] the control problem is approximated by a penalized problem enabling us to deduce generalized optimality conditions due to V. Barbu. For simplicity, we consider the time independent operators in the left-hand side of the inequality. A similar problem for the elliptic case was solved in [4], [5].

Let U be a Hilbert space, U_{ad} a set of admissible controls compact in U, V a real Hilbert space with an inner product $(.,.)$, a norm $\|.\|$, V^* its dual space with a norm $\|.\|_*$ and the duality pairing $\langle .,. \rangle$.

Now, we recall the convergence of set and functional sequences in V via Mosco ([6]) :

<u>Definition 1</u>. A sequence $\{K_n\}$ of subsets of V converges to the set $K \subset V$, if

i) K contains all weak limits of sequences $\{u_k\}$, $u_k \in K_{n_k}$, where $\{K_{n_k}\}$ is an arbitrary subsequence of $\{K_n\}$,

ii) every $v \in K$ is the strong limit of some sequence $\{v_n\}$, $v_n \in K_n$.

Notation : $K = \operatorname{Lim}_{n\to\infty} K_n$.

<u>Definition 2</u>. A sequence $\{j_n\}$ of functionals from V into $(-\infty,\infty]$ converges to $j : V \to (-\infty,\infty]$ in V, if $\operatorname{epi} j = \operatorname{Lim}_{n\to\infty} \operatorname{epi} j_n$ with $\operatorname{epi} j : \{(v,\beta) \in V \times R : j(v) \le \beta\}$.

Notation : $j = \operatorname{Lim}_{n\to\infty} j_n$.

Let us introduce the systems $\{K(e)\}, \{u_0(e)\}, \{B(e)\}, \{A_0(e)\}, \{A_1(e)\}$, $e \in U_{ad}$, of convex closed subsets $K(e) \subset V$, the elements $u_0(e) \in K(e)$, $B(e) \in V^*$ and linear bounded operators $A_i(e) \in L(V,V^*)$, $i = 0, 1$, satisfying the assumptions :

(1) $$\bigcap_{e \in U_{ad}} K(e) \neq \emptyset$$

(2) $e_n \to e$ in $U \Rightarrow K(e) = \mathrm{Lim}_{n\to\infty} K(e_n)$

(3) $\langle A_i(e)u,v\rangle = \langle A_i(e)v,u\rangle$ for all $u,v \in V$

(4) $\langle A_i(e)u,u\rangle \geq \alpha_i \|u\|^2$, $\alpha_i > 0$; for all $u \in V$

$$(5)\qquad e_n \to e \text{ in } U \Rightarrow \begin{cases} \text{i) } A_i(e_n) \to A_i(e) \text{ in } L(V,V^*) \\ \text{ii) } u_o(e_n) \to u_o(e) \text{ in } V \\ \text{iii) } B(e_n) \to B(e) \text{ in } V^* . \end{cases}$$

Further, let $f \in C^1([0,T],V^*)$. Using the method of penalization (see [3]) the following theorem can be verified.

<u>Theorem 1</u>. There exists for every $e \in U_{ad}$ the unique solution $u(e) := u(.,e) \in W_2^1([0,T],V)$ of the initial value problem

(6) $u(t,e) \in K(e)$ for all $t \in [0,T]$,

(7) $\langle A_1(e)u_t'(t,e) + A_o(e)u(t,e),\ v - u(t,e)\rangle \geq$
$\langle f(t) + B(e), v - u(t,e)\rangle$ for all $v \in K(e)$, $t \in [0,T]$,

(8) $u(0,e) = u_o(e)$.

Now, we link with (6), (7), (8) a minimum problem

(9) $J(u(\bar{e}),\bar{e}) = \min_{e \in U_{ad}} J(u(e),e)$,

where a functional $J : W_2^1([0,T],V) \times U \to R$ fulfils the assumption

(10) $u_n \rightharpoonup u,\ e_n \to e \Rightarrow J(u,e) \leq \liminf_{n\to\infty} J(u_n,e_n)$

<u>Theorem 2</u>. There exists at least one solution $\bar{e} \in U_{ad}$ of the Optimal control problem (6) - (9).

The state inequality (7) can be rewritten in a form

(7') $\langle A_1(e)u_t'(t,e) + A_o(e)u(t,e), v - u(t,e)\rangle +$
$+ \phi(e,v) - \phi(e,u(t,e)) \geq \langle f(t) + B(e),\ v - u(t,e)\rangle$
for all $e \in U_{ad}$, $v \in V$, $t \in [0,T]$,

where

$$\Phi(e,v) = \begin{cases} 0\ , & \text{if } v \in K(e) \\ +\infty\ , & \text{if } v \notin K(e)\ . \end{cases} \tag{11}$$

We regularize the functional Φ by the system of convex Frechet differentiable functionals $\Phi^{\varepsilon}(e,.) : V \to R$ fulfilling the conditions :

(12) $\Phi^{\varepsilon}(e,v) \geq -c\ (\|v\| + 1)$ for all $\varepsilon > 0$, $e \in U_{ad}$, $v \in V$,

(13) $\lim\limits_{\varepsilon \to 0} \Phi^{\varepsilon}(e,v) = \Phi(e,v)$ for all $e \in U_{ad}$, $v \in V$,

(14) $e_n \to e$ in $U \Rightarrow \Phi(e,.) = \operatorname*{Lim}\limits_{n \to \infty} \Phi^{\varepsilon}(e_n,.)$,

(15) $e_n \to e$ in U, $\varepsilon_n \to 0 \Rightarrow \Phi(e,.) = \operatorname*{Lim}\limits_{n \to \infty} \Phi^{\varepsilon_n}(e_n,.)$,

(16) $\|\frac{\partial}{\partial u}\Phi^{\varepsilon}(e,u_1) - \frac{\partial}{\partial u}\Phi^{\varepsilon}(e,u_2)\|_* \leq M_1(\varepsilon)\|u_1 - u_2\|$
for all $\varepsilon > 0$, $e \in U_{ad}$; $u_1, u_2 \in V$,

(17) $\|\frac{\partial}{\partial u}\Phi^{\varepsilon}(e,v_o)\|_* \leq M_2$ for any $v_o \in V$ and all $e \in U_{ad}$, $\varepsilon > 0$.

Now, for each $\varepsilon > 0$ we consider the approximating

<u>Problem P_{ε}</u>. To find a couple $[e_{\varepsilon}, u_{\varepsilon}] \in \mathcal{U}_{\varepsilon}$ such that

$$J(e_{\varepsilon},u_{\varepsilon}) + \tfrac{1}{2}\|e - \bar{e}\|_U^2 = \min_{[e,u] \in \mathcal{U}_{\varepsilon}} [J(e,u) + \tfrac{1}{2}\|e - \bar{e}\|_U^2]\ , \tag{18}$$

where

$$\begin{aligned} \mathcal{U}_{\varepsilon} = \{[e,u] \in U_{ad} \times W_2^1([0,T],V) : u(0) = u_o(e)\ , \\ A_1(e)u_t'(t) + A_o(e)u(t) + \tfrac{\partial}{\partial u}\Phi^{\varepsilon}(e,u) = f(t) + B(e)\}\ . \end{aligned} \tag{19}$$

In a similar way as in [4] for the elliptic case the following theorem can be verified :

<u>Theorem 3</u>. There exists for every $\varepsilon > 0$ at least one optimal pair $[e_{\varepsilon}, u_{\varepsilon}] \in \mathcal{U}_{\varepsilon}$ for the Problem P_{ε}. If $\lim\limits_{n \to \infty} \varepsilon_n = 0$, then there exists a subsequence $\{\varepsilon_k\}$ of $\{\varepsilon_n\}$ such that

(20) $e_{\varepsilon_k} \to \bar{e}$ in U

(21) $u_{\varepsilon_k} \to \bar{u} = u(\bar{e})$ in $W_2^1([0,T],V)$,

where $\bar{e}$ is a solution of the Optimal control problem (6) - (9).

If we add some differentiability assumptions and if $B : U \to V^*$ is the linear bounded operator, then it is possible to derive the optimality system for the Problem P_ε.

Theorem 4. If $[e_\varepsilon, u_\varepsilon]$ is the optimal pair for the Problem P_ε, then there exists $p_\varepsilon \in W_2^1([0,T],V)$ satisfying the system

$$(22) \qquad A_1(e_\varepsilon)u_\varepsilon'(t) + A_0(e_\varepsilon)u_\varepsilon(t) + \frac{\partial}{\partial u}\phi^\varepsilon(e_\varepsilon, u_\varepsilon(t)) = f(t) + B(e_\varepsilon)$$

$$(23) \qquad u_\varepsilon(0) = u_0(e_\varepsilon)$$

$$(24) \qquad -A_1(e_\varepsilon)p_\varepsilon'(t) + A_0(e_\varepsilon)p_\varepsilon(t) + \frac{\partial^2}{\partial u^2}\phi^\varepsilon(e_\varepsilon, u_\varepsilon)p_\varepsilon(t) = \frac{\partial J}{\partial u}(e_\varepsilon, u_\varepsilon)$$

$$(25) \qquad p_\varepsilon(T) = 0$$

$$(26) \qquad \left\langle B\, p_\varepsilon(t) + \frac{\partial J}{\partial e}(e_\varepsilon, u_\varepsilon), e-e_\varepsilon \right\rangle_U + (e_\varepsilon - \bar{e}, e-e_\varepsilon)_U \geq$$
$$\geq \left\langle \left[\frac{\partial A_1}{\partial e}(e_\varepsilon)(e-e_\varepsilon)\right]u_\varepsilon'(t) + \left[\frac{\partial A_0}{\partial e}(e_\varepsilon)(e-e_\varepsilon)\right]u_\varepsilon(t) + \right.$$
$$\left. \left[\frac{\partial}{\partial e}\frac{\partial}{\partial u}\phi^\varepsilon(e_\varepsilon, u_\varepsilon(t))\right](e-e_\varepsilon)\ ,\ p_\varepsilon(t) \right\rangle \quad \text{for all} \quad e \in U_{ad}\ .$$

It can be verified that the set $\{p_\varepsilon\}$ is bounded in $W_2^1([0,T],V)$. Then there exists a sequence $\{\varepsilon_k\}$, $\varepsilon_k \to 0$, and p_0 such that

$$(27) \qquad p_{\varepsilon_k} \rightharpoonup p_0 \quad \text{in} \quad W_2^1([0,T],V)$$

A function p_0 can be considered as the generalized adjoint state to the system (6) - (9).

R e f e r e n c e s

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