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THEORY OF COMPLETELY INTEGRABLE EQUATIONS

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Let us consider a completely integrable equation

$$dx = P(x)\,dt, \tag{1}$$

where $P: G \to \mathbb{R}^{n\times m}$, G is an open domain in $\mathbb{R}^n$, $P \in C^1$ and satisfies the condition of complete integrability:

$$DP_j(x)P_i(x) = DP_i(x)P_j(x).$$

If we regard the solution of the equation (1) as a function

$$\psi(\cdot,x) : Q \to G, \quad \psi(0,x) = x, \quad Q \subset \mathbb{R}^m$$

then we can prove the local existence theorem, the local uniqueness theorem, and the continuity with respect to the initial values, i. e. the continuity of the function $\psi(t,\cdot)$ at $t \in Q$, but not the existence of maximal domain of definition [1]. Therefore alredy in [2], it was suggested to regard the solution of the equation (1) as a function

$$\varphi(\cdot,x) : Q(x) \to G,$$

where $\varphi(0,x) = x$, $Q(x) \in \mathcal{F}_\alpha(\mathbb{R}^m)$, $\mathcal{F}(\mathbb{R}^m)$ is many-sheeted space multiply covering the Euclidean space $\mathbb{R}^m$, α is the multiplicity of covering $p: \mathcal{F}_\alpha(\mathbb{R}^m) \to \mathbb{R}^m$, $1 \leq \alpha \leq +\infty$.

Thanks to such definition it is very simple to prove the existence and the uniqueness of maximal domain of definition $Q(x)$ of a solution $\varphi(\cdot,x)$.

In the standard way one can prove that every orbit $\varphi(Q(x),x)$ of the equation (1) is a smooth manifold of a dimension k, $0 \leq k \leq m$. One can prove several theorems, analogous to those in the general theory of ordinary differential equations, for instance:

1. If the rank $r(P(x_0))$ of the matrix $P(x_0)$ is equal k_0, $r(P(x)) = k < k_0$, and $x \in \partial\varphi(Q(x_0),x_0)$, then $\varphi(t,x_0) = x$ implies $\lim |t| = +\infty$.

2. If there is a compact set $B \subset G \times \mathbb{R}^m$, $x_0 \in B$, then there is a compact set $A \subset Q(x_0)$ such that $t \notin A$ implies $(\varphi(t,x_0),t) \notin B$.

In the considered theory there arises naturally the question of branching sets of definition domains of solutions. They can be defined analogously as in the theory of functions of complex variables.

The solutions of the system of equations

$$\begin{cases} dx_1 = dt_1 \\ dx_2 = dt_2 \\ dx_3 = \begin{cases} -x_3^3(x_1 dt_1 + x_2 dt_2) & \text{for } (x_1^2+x_2^2)x_3^2 \leq 1 \\ -(x_1^2+x_2^2)^{-2}(((x_1+x_2x_3)(x_1^2+x_2^2)x_3+x_2)\,dt_1 + \\ +((x_2+x_1x_3)(x_1^2+x_2^2)x_3+x_1)\,dt_2) & \text{for } (x_1^2+x_2^2)x_3^2) \geq 1 \end{cases} \end{cases}$$

with orbits in the domain $(x_1^2+x_2^2)x_3^2 > 1$ have the definition domain with one branching point. If the orbits are in the domain $(x_1^2+x_2^2)x_3^2 \leq 1$, the corresponding solutions are defined on and the branching set is empty. Indeed, in the domain $(x_1^2+x_2^2)x_3^2 \leq 1$ the orbits are

$$x_3^2 = (x_1^2 + x_2^2 + x_{30}^{-2} - x_{10} - x_{20})^{-1},$$

but in the domain $(x_1^2+x_2^2)x_3^2 > 1$, they are

$$x_3^2 = (x_1^2 + x_2^2 + (x_{30}^{-2} - x_{10} - x_{20})\exp(\varphi_1 - \varphi_{10}))^{-1}$$

where $\varphi_1 = \operatorname{Arc} tg(x_2/x_1)$, x_{10}, x_{20}, x_{30} and φ_0 are initial values.

The system

$$\begin{cases} dx_1 = (x_1^2+x_2^2)^{1-n}(Re(x_1-ix_2)^{n-1}dt_1 + Im(x_1-ix_2)^{n-1}dt_2) \\ dx_2 = (x_1^2+x_2^2)^{1-n}(-Im(x_1-ix_2)^{n-1}dt_1 + Re(x_1-ix_2)^{n-1}dt_2) \\ dx_3 = 0 \end{cases}$$

has solutions with n -sheeted domains of definition with one branching point. Here, the orbits are $x_3 = x_{30}$, $x_1^2 + x_2^2 > 0$, the solutions are

$$\varphi_1(t,x_0) = Re\sqrt[n]{t_1 - t_{10} + i(t_2 - t_{20})} + x_{10}, \quad \varphi_2(t,x_0) = Im\sqrt[n]{t_1 - t_{10} + i(t_2 - t_{20})} + x_{20}.$$

It is not necessary that all sheets have the same branching set. For instance, the system

$$\begin{cases} dx_1 = dt_1 \\ dx_2 = dt_2 \\ dx_3 = \begin{cases} h_{01}(x)\,dt_1 + h_{02}(x)\,dt_2 & \text{for } \psi(x) \leq 0 \\ h_{11}(x)\,dt_1 + h_{12}(x)\,dt_2 & \text{for } 0 \leq \psi(x) \leq 1 \\ h_{21}(x)\,dt_1 + h_{22}(x)\,dt_2 & \text{for } \psi(x) \geq 1 \end{cases} \end{cases}$$

where

$$dx_3 = h_{01}(x)\,dt_1 + h_{02}(x)\,dt_2 \Leftrightarrow D\psi(x)(\psi(x)-1)dx = 0,$$
$$\psi(x) = -x_3^{-2} + (x_1^2-1)^2 + x_2^2,$$

$$dx_3 = h_{11}(x)dt_1 + h_{12}(x)dt_2 \Leftrightarrow ((2\psi(x)-1)D\psi(x) + \psi(x)(\psi(x)-1)D\varphi_1(x))dx = 0,$$

$$\varphi_1(x) = \mathrm{Arc\,tg}(x_2/(x_1+1)),$$

$$dx_3 = h_{21}(x)dt_1 + h_{22}(x)dt_2 \Leftrightarrow ((2\psi(x)-1)D\psi(x) + \psi(x)(\psi(x)-1)(D\varphi_1(x) + D\varphi_2(x)))dx = 0,$$

$$\varphi_2(x) = \mathrm{Arc\,tg}(x_2/(x_1-1))$$

on the sheets for which the values of the solution are in the set $\{x : 0 < \psi(x) \leq 1\}$ has one branching point, but on the sheets for which the values of the solution are in the set $\{x : \psi(x) > 1\}$ have two branching points. In the domain $\psi(x) \leq 0$, orbits are

$$\psi(x)(\psi(x)-1) = \psi(x_0)(\psi(x_0)-1)$$

in the domain $0 < \psi(x) \leq 1$, they are

$$\psi(x)(\psi(x)-1)\exp(\varphi_1(x) - \varphi_1(x_0)) = \psi(x_0)(\psi(x_0)-1)$$

and in the domain $\psi(x) > 1$, they are

$$\psi(x)(\psi(x)-1)\exp(\varphi_1(x) + \varphi_2(x)) = \psi(x_0)(\psi(x_0)-1)\exp(\varphi_1(x_0) + \varphi_2(x_0))$$

For $n = 3$, $m = 2$, $\alpha = \infty$ there are orbits, which are dense in some domain. For instance, orbits of the system

$$\begin{cases} dx_1 = dt_1 \\ dx_2 = dt_2 \\ dx_3 = -(2(x^2-1)x_1 + f(x)f_1(x))x_3^3\,dt_1 - (x_2 + f(x)f_2(x))x_3^3\,dt_2 \end{cases}$$

where $f(x) = -(((x_1^2-1)^2 + x_2^2)x_3^2 - 1) + |((x_1^2-1)^2 + x_2^2)x_3^2 - 1|)/4,$

$$f_1(x) = (((x_1-1)^2 + x_2^2)^{-1} + \alpha((x_1+1)^2 + x_2^2)^{-1})x_2 x_3^{-2},$$

$$f_2(x) = (((x_1^2-1)^2 + x_2^2)^{-1}(x_1-1) + \alpha((x_1+1)^2 + x_2^2)^{-1}(x_1+1))x_3^{-2}$$

in the domain $((x_1^2-1)^2 + x_2^2)x_3^2 - 1 > 0$ are everywhere dense if α is irrational. In the domain $((x_1^2-1)^2 + x_2^2)x_3^2 > 1$ orbits are

$$x_3^2 = ((x_1^2-1)^2 + x_2^2 + (x_{30}^{-2} - (x_{10}^2-1)^2 - x_{20}^2)\exp(\varphi_1(x) - \varphi_1(x_0) + \alpha(\varphi_2(x) - \varphi_2(x_0))))^{-1}$$

The theorem of Kronecker involves that at constant x_1 and x_2 the exponent has arbitrary value from an everywhere dense set.

The existence of branching set of a solution shows that for completely integrable equations there is no theorem of reparametrization and therefore one can not associate a dynamical system with the group $\mathbb{R}^m$ to such equation. Nevertheless, thanks to the theorem of the existence and the uniqueness of the maximal domain of definition one can define several concepts of dynamical systems.

The limit set of the solution $\varphi(\cdot, x_0)$ is the set

$$L(x_0) = \bigcap_i \overline{\varphi(Q(x_0) \setminus K_i(x_0), x_0)} \setminus \partial G$$

where $K_i(x_0)$ is a compact, $K_i(x_0) \subset K_{i+1}(x_0)$, $\bigcup_i K_i(x_0) = Q(x_0)$.

The prolongation of the orbit $\varphi(Q(x_0), x_0)$ is the set

$$D(x_0) = \bigcap_{\delta > 0} \overline{\bigcup \{\varphi(Q(x), x) : x \in \mathcal{U}(x_0, \delta)\}} \setminus \partial G .$$

Analogously one can define the prolongational limit set:

$$J(x_0) = \bigcap_{\delta > 0} \bigcap_i \overline{\bigcup \{\varphi(Q(x) \setminus K_i(x), x) : x \in \mathcal{U}(x_0, \delta)\}} \setminus \partial G.$$

If one considers, instead of the equation (1), the equation

$$dx = P(x, t)\, dt \tag{2}$$

ha can reduce it to the system

$$dx = P(x, t)\, d\tau, \quad dt = d\tau \tag{3}$$

which has the same form (1). Orbits of the system (3) are naturally to call simple integral manifolds of the equation (2).

References

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