LINEAR PERTURBATIONS OF DIFFERENTIAL SYSTEMS WITH CONSTANT MATRICES

ŠIMŠA J., BRNO, Czechoslovakia

W. F. Trench ([4], [5], [6]) and the author ([2], [3]) established recently mild sufficient conditions for a scalar linear differential equation

$$v^{(n)} = \sum_{k=1}^{n}[a_k + p_k(t)]v^{(n-k)} \qquad (t \in [0,\infty)) \tag{1}$$

to have solutions which behave for large t like solutions of the constant coefficient equation

$$v^{(n)} = \sum_{k=1}^{n} a_k v^{(n-k)}.$$

These results are considerable extensions of the classical theorems (cf. [1, Chapter X]), because they involve no smallness conditions on $p_2, p_3, \ldots, p_n$ (except p_1) which require the absolute convergence of improper integrals that occur (see Theorem 1 below). Since (1) is convertible into a linear differential system

$$x' = (A + P(t))x, \tag{2}$$

with $n \times n$ matrices A and P given by

$$A = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ a_n & a_{n-1} & a_{n-2} & \cdots & a_1 \end{pmatrix} \quad \text{and} \quad P = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 \\ p_n & p_{n-1} & \cdots & p_1 \end{pmatrix}, \tag{3}$$

the following question arises: Is it possible to extend the above mentioned results concerning (1) to a wider class of systems (2)? Our goal is to show (see Theorem 2) that such an extension is possible when some k rows of the perturbation matrix P in (2) vanish as $t \to \infty$ (notice that P in (3) has the first $(n-1)$ rows zero). We will give sufficient conditions for (2) with a constant matrix A to have a solution x_0 satisfying

$$x_0(t) = ce^{\lambda_0 t} + O(e^{\mu t}\phi(t)) \qquad \text{as } t \to \infty, \tag{4}$$

with a given function ϕ, $c \in \mathbf{C}^n$, $\lambda_0 \in \mathbf{C}$ and $\mu \in \mathbf{R}$ such that

$$Ac = \lambda_0 c \quad \text{and} \quad \rho = \mathrm{Re}\lambda_0 - \mu \geq 0. \tag{5}$$

It is convenient to collect some technical conditions on A, μ and ϕ in the following

ASSUMPTION A. Let

$$\det(\lambda I - A) = (\lambda - \lambda_0)^{n_0}(\lambda - \lambda_1)^{n_1} \cdots (\lambda - \lambda_d)^{n_d},$$

where λ_j are distinct $(0 \le j \le d)$ and $n_0 + n_1 + \cdots + n_d = n$. For a given $\mu \in \mathbf{R}$ we will write $j \in J_+$ or $j \in J_0$ if $\mu > \mathrm{Re}\lambda_j$ or $\mu = \mathrm{Re}\lambda_j$, respectively. If $J_0 \ne \emptyset$, suppose that

$$\operatorname{rank}(\lambda_j I - A) = n - n_j \qquad \text{for any } j \in J_0. \tag{6}$$

(If A is in Frobenius form (3), then (6) means that $n_j = 1$.) Let ϕ be continuous, positive and nonincreasing on $[a, \infty)$ for some $a > 0$. If $J_+ \ne \emptyset$, suppose that $e^{\delta t}\phi(t)$ is nondecreasing on $[a, \infty)$ for some δ, $0 < \delta < \min\{\mu - \mathrm{Re}\lambda_j : j \in J_+\}$.

It is to be understood below that improper integrals appearing in hypotheses are assumed to converge, and that the convergence may be conditional, unless the integrand is nonnegative. The symbols "O" and "o" refer to behavior as $t \to \infty$. The result of [5] is the following

THEOREM 1. *Suppose that A, c, λ_0, ρ and μ are as in (3) and (5) and that Assumption A holds. Given continuous scalar functions $p_1, p_2, \ldots, p_n$, define*

$$f(t) = \sum_{k=1}^{n} \lambda_0^{n-k} p_k(t)$$

and suppose that

$$\int_t^\infty f(s)e^{\rho s} ds = O(\phi(t)), \tag{7}$$

$$\int_t^\infty f(s)e^{(\lambda_0 - \lambda_j)s} ds = O(\phi(t)) \qquad (j \in J_0), \tag{8}$$

$$\int_t^\infty |p_1(s)|\phi(s) ds = o(\phi(t)) \tag{9}$$

and

$$\int_t^\infty \left| \int_s^\infty p_k(\tau) d\tau \right| \phi(s) ds = o(\phi(t)) \qquad (2 \le k \le n). \tag{10}$$

Then (1) has a solution $v_0(t)$ such that

$$v_0^{(r)}(t) = \lambda_0^r e^{\lambda_0 t} + O(e^{\mu t}\phi(t)), \quad 0 \le r \le n-1. \tag{11}$$

We now announce our result. Having fixed an integer k $(1 \le k \le n)$, for any matrix $P \in C^n$ and any vector $x \in \mathbf{C}^n$ we will write

$$P = \begin{pmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{pmatrix} \quad \text{and} \quad x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix},$$

where $x_1 \in \mathbf{C}^k$, $x_2 \in \mathbf{C}^{n-k}$, and the blocks $P_{11}, P_{12}, P_{21}, P_{22}$ are matrices of sizes $k \times k$, $k \times (n-k)$, $(n-k) \times k$ and $(n-k) \times (n-k)$, respectively.

Theorem 2. *Suppose that A, c, λ_0, ρ and μ are as in (5) and that Assumption A holds. Suppose also that $P(t)$ is locally integrable on $[0, \infty)$,*

$$P_{11}(t)c_1 + P_{12}(t)c_2 = O(e^{-\rho t}\phi(t)) \quad \text{a.e.} \tag{12}$$

$$P_{1j}(t) = o(1) \quad \text{a.e.} \quad (j = 1, 2) \tag{13}$$

and that the following conditions are satisfied.

(i) If $J_0 = \emptyset$, then

$$\int_t^\infty (P_{21}(s)c_1 + P_{22}(s)c_2)ds = O(e^{-\rho t}\phi(t)), \tag{14}$$

$$\sup_{s \geq t}(1 + s - t)^{-1}\left\| \int_t^s P_{21}(r)dr \right\| = o(1) \tag{15}$$

and

$$\sup_{s \geq t}(1 + s - t)^{-1} \int_t^s \|P_{22}(r)\|\phi(r)dr = o(\phi(t)). \tag{16}$$

(ii) If $J_0 \neq \emptyset$, then

$$\int_t^\infty P(s)ce^{(\lambda_0 - \lambda_j)s}ds = O(\phi(t)) \quad \text{for any } j \in J_0, \tag{17}$$

$$\int_t^\infty \|P_{i2}(s)\|\phi(s)ds = o(\phi(t)) \quad (i = 1, 2) \tag{18_i}$$

and

$$\int_t^\infty \left\| \int_s^\infty P_{i1}(\tau)d\tau \right\| \phi(s)ds = o(\phi(t)) \quad (i = 1, 2). \tag{19_i}$$

Then (2) has a solution x_0 satisfying (4).

The proof of Theorem 2 is based on the simple Banach contraction principle. The solution x_0 can be found in the form $x_0(t) = ce^{\lambda_0 t} + y_0(t)$, where y_0 is a fixed point of the affine map

$$\mathcal{L}y = u[Py] + u[P(t)ce^{\lambda_0 t}]$$

and $u[.]$ is an operator defined in the following

Lemma 1. *Suppose that Assumption A holds and that $b = b(t)$ is a locally integrable n-vector function which satisfies the following conditions (i) and (ii) with a constant $K_b \geq 0$.*

(i) If $J_0 = \emptyset$, then

$$\sup_{s \geq t}(1 + s - t)^{-1}\left\| \int_t^s b(\tau)e^{-\omega\tau}d\tau \right\| \leq K_b\phi(t) \quad (t_0 \leq t < \infty)$$

for some $\omega \in \mathbf{C}$ such that $Re\omega = \mu$.

(ii) If $J_0 \neq \emptyset$, then

$$\|\int_t^\infty b(s)e^{-\lambda_j s}ds\| \leq K_b\phi(t) \quad (t_0 \leq t < \infty\,,\, j \in J_0)\,.$$

Then there exists a constant $K_{A,\omega}$ such that the system

$$u' = Au + b(t)$$

has a solution $u = u[b](t)$ which is linear in b and satisfies

$$\|u[b](t)\| \leq K_{A,\omega}K_b e^{\mu t}\phi(t) \quad (t_0 \leq t < \infty)\,.$$

The detailed proofs of Theorem 2 and Lemma 1 are rather complicated and so they will be published elsewhere. We finish here by remarking that Theorem 2 (with $k = 1$ and zero matrices P_{11}, P_{12}) is *an improvement* of Theorem 1. In fact, if A and P are as in (3),

$$P_{21} = (p_n, p_{n-1}, \ldots, p_2)\,,\ P_{22} = (p_1) \text{ and } c = (1, \lambda_0, \ldots, \lambda_0^{n-1})\,,$$

then (12), (13), (18_1) and (19_1) hold automatically, (7) implies (14), (10) implies (15), (9) implies (16), while the pairs of conditions (8)&(17) or (9)&(18_2) or (10)&(19_2) are equivalent.

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