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THE DIRICHLET PROBLEM FOR SUBLAPLACIANS ON NILPOTENT LIE GROUPS - GEOMETRIC CRITERIA FOR REGULARITY

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In 1969 J. M. Bony [**1**] studied partial differential operators of second order

$$L = X_1^2 + \cdots + X_n^2$$

on $\mathbf{R}^m$ where $X_1, \ldots, X_n$ are $\mathcal{C}^\infty$–vector fields. Let us note three simple examples:

1. $n = m, X = \frac{\partial}{\partial x_j}$: $L = \Delta$ (Laplace operator)
2. $n = m = 2,\ X_1 = \frac{\partial}{\partial x},\ X_2 = x\frac{\partial}{\partial y}$: $L = \frac{\partial^2}{\partial x^2} + x^2\frac{\partial^2}{\partial y^2}$ (Grushin operator)
3. $n = 2, m = 3,\ X_1 = \frac{\partial}{\partial x} + 2y\frac{\partial}{\partial z},\ X_2 = \frac{\partial}{\partial y} - 2x\frac{\partial}{\partial z}$:
$$L = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + 4y\frac{\partial^2}{\partial x \partial z} - 4x\frac{\partial^2}{\partial y \partial z} + 4(x^2+y^2)\frac{\partial^2}{\partial z^2} = \Delta_K \text{ (Laplace–Kohn operator)}$$

Bony has shown the following: If the rank of the Lie algebra $\mathcal{L}(X_1, \ldots, X_n)$ is equal to m at each $x \in \mathbf{R}^m$ then $\mathcal{H}_L(U) := \{h \in \mathcal{C}^\infty(U) : Lh = 0\}, U$ open $\subset \mathbf{R}^m$, yields a Brelot space $(X, \mathcal{H}_L)$.

Clearly Bony's hypothesis is satisfied in each of the preceding simple examples ($[X_1, X_2] = \frac{\partial}{\partial y}$ in (2), $[X_1, X_2] = -4\frac{\partial}{\partial z}$ in (3)). In proving his result Bony had to find a base of regular sets. We recall that a relatively compact open set U is *regular* (with respect to L) if for every $f \in \mathcal{C}^+(\partial U)$ there exists a unique function $h \in \mathcal{C}^+(\overline{U})$ such that $h|_{\partial U} = f$ and $h|_U \in \mathcal{H}_L(U)$. Bony has shown that certain very flat lenticular sets are regular. But how about general criteria for regularity? Of course, having the harmonic space $(\mathbf{R}^m, \mathcal{H}_L)$ it is well known that an open set $U \subset \mathbf{R}^m$ (for which $\overline{U}$ is contained in a $\mathcal{P}$–set) is regular if and only if the complement of U is not thin at any $z \in \partial U$. Now in many cases geometric properties of a set permit a decision on the thinness of the set at a point. This leads to geometric criteria for the regularity of open sets.

Given $0 < \alpha < \infty$, we shall say that U satisfies a *pointwise exterior α–Hölder condition* if for every $z \in \partial U$ there exists an isometry T of $\mathbf{R}^m$ and $\varrho > 0$ such that $T(z) = 0$ and the α–Hölder cone

$$\{x \in \mathbf{R}^m : 0 < x_m < \varrho, (x_1^2 + \cdots + x_{m-1}^2)^{\alpha/2} < \varrho x_m\}$$

is contained in the complement of $T(U)$. Looking at certain model cases we shall see that an exterior $(r - \varepsilon)$–Hölder condition is in general not sufficient for regularity of U.

We fix a real finite dimensional Lie algebra $\mathcal{N} = V^1 \oplus \cdots \oplus V^r$ such that $[V^j, V^k] = V^{j+k} \neq \{0\}$ if $k + j \leqq r$, and $[V^j, V^k] = 0$ if $k + j > r$. Then $\mathcal{N}$ is nilpotent of step r and V^1 generates $\mathcal{N}$.

Example: Given a natural $k > 2$, let $\mathcal{N}$ be the set of all upper triangular $(k \times k)$–matrices (a_{pq}) ($a_{pq} = 0$ for all $p > q$) and define a product on $\mathcal{N}$ by $[A, B] = AB - BA$ (where AB denotes the usual matrix product of A and B). Then $\mathcal{N} = V^1 \oplus \cdots \oplus V^{k-1}$ where $V^i = \{(a_{pq}) \in \mathcal{N} : a_{pq} = 0 \text{ if } p + i \neq q\}$, $\dim V^i = k - i$, $\dim \mathcal{N} = \frac{k(k-1)}{2}$.

In the general case let $n_i := \dim V^i$, $1 \leq i \leq r$. Then $m := \dim \mathcal{N} = n_1 + \cdots + n_r$. Fixing a basis $\{Y_{ij} : 1 \leq j \leq n_i\}$ for each V^i we identify $(x_{ij}) \in \mathbf{R}^m$ with $\Sigma x_{ij} Y_{ij} \in \mathcal{N}$. Using the map exp from $\mathcal{N}$ on the corresponding simply connected Lie group we obtain a product $x \cdot y \in \mathbf{R}^m$ for $x, y \in \mathbf{R}^m$ by $\exp(x \cdot y) = \exp(x)\exp(y)$. Then $(\mathbf{R}^m, \cdot)$ is a group such that $x \cdot (-x) = 0$ for every $x \in \mathbf{R}^m$. By the Campbell–Hausdorff formula

$$(x \cdot y)_{ij} = x_{ij} + y_{ij} + p_{ij}(x, y)$$

such that $p_{ij}(x,y)$ is a linear combination of monomials $z_{k_1l_1}\dots z_{k_sl_s}$ where each z is x or y and $\sum_{\nu=1}^{s} k_\nu = i$. In particular, the Lebesgue measure λ^m on $\mathbf{R}^m$ is invariant under left translations $l_u : x \mapsto u\cdot x$, $u \in \mathbf{R}^m$.

We now define left invariant vector fields $X_1,\dots,X_{n_1}$ on $\mathbf{R}^m$ by

$$X_j f(0) = \frac{\partial f}{\partial x_{1j}}(0)\ ,\quad X_j f(u) = X_j(f\circ l_u)(0)\ .$$

Then

$$L = X_1^2 + \dots + X_{n_1}^2$$

is the corresponding *sublaplacian* on $\mathbf{R}^m$. It is the unique left invariant differential operator satisfying

$$Lf(0) = \sum_{j=1}^{n_1} \frac{\partial^2 f}{\partial x_{1j}}(0)\ .$$

Since $\mathcal{N}$ is generated by $Y_{11},\dots,Y_{1n_1}$, we have $\mathcal{L}(X_1,\dots,X_{n_1})(0) = \mathcal{N}$ and hence by left translation $\mathcal{L}(X_1,\dots,X_n)(x) = \mathcal{N}$ for every $x \in \mathbf{R}^m$. So L satisfies Bony's hypothesis.

Note that n_1 may be much smaller than m : In our example of triangular matrices, $\frac{n_1}{m} = \frac{2}{k}$! In the case $k = 3$ we have $m = 3$,

$$(x,y,z)\cdot(x',y',z') = (x+x', y+y', z+z' + \tfrac{1}{2}(xy' - x'y))\ ,$$

and hence

$$X_1 = \frac{\partial}{\partial x} - \frac{1}{2}y\frac{\partial}{\partial z}\ ,\quad X_2 = \frac{\partial}{\partial y} + \frac{1}{2}x\frac{\partial}{\partial z}\ .$$

Replacing z by $-4z$ we obtain the Heisenberg group $\mathbf{H}_1$ and Δ_K .

We have natural dilations δ_R , $R > 0$:

$$\delta_R((x_{ij})) = (R^i x_{ij})\ .$$

Clearly, $\delta_R(x+y) = \delta_R(x) + \delta_R(y)$ and by the Campbell–Hausdorff formula $\delta_R(x\cdot y) = \delta_R(x)\cdot\delta_R(y)$. As a consequence L is homogeneous of order 2, i.e.,

$$L(f\circ\delta_R) = R^2(Lf)\circ\delta_R\ .$$

In the following let us assume that the *homogeneous dimension*

$$Q = n_1 + 2n_2 + \dots + rn_r$$

is at least 3. Then there is a symmetric Green function satisfying

$$G(\delta_R(x),0) = R^{2-Q}G(x,0)\ .$$

Defining

$$\|x\| := \max_{i,j} |x_{ij}|^{1/i} \qquad (x \in \mathbf{R}^m)$$

we have $\|\delta_R(x)\| = R\|x\|$. Therefore there exists a constant $C > 0$ such that

$$C^{-1}\|x\|^{2-Q} \le G(x,0) \le C\|x\|^{2-Q}\ .$$

Indeed, having such inequalities on the compact set $\{x \in \mathbf{R}^m : \|x\| = 1\}$ we obtain these inequalities on $\mathbf{R}^m$ since $G(\delta_R(x),0) = R^{2-Q}G(x,0)$ and $\|\delta_R(x)\| = R\|x\|$. Since L is invariant under left translations, $G(ux,uy) = G(x,y)$ for all $u,x,y \in \mathbf{R}^m$. Defining

$$d(x,y) := \|x^{-1}\cdot y\| \qquad (x,y \in \mathbf{R}^m)$$

we conclude that

$$C^{-1}d^{2-Q} \le G \le Cd^{2-Q}\ .$$

The Campbell–Hausdorff formula yields that d is almost a distance on $\mathbf{R}^m$. We have $d(x,y) > 0$ if $x \neq y$, $d(x,y) = d(y,x)$ and $d(x,y) \le D(d(x,z) + d(z,y))$ for all $x,y,z \in \mathbf{R}^m$ with a constant $D = D(\mathcal{N})$. H. Hueber [3] has shown that such a property is sufficient to obtain a Wiener criterion for regularity:

Theorem. *Let E be a Borel subset of $\mathbf{R}^m$, $0 < \eta < 1$, and define*

$$E_{(s)} = \{x \in E : \|x\| < \eta^s\} \ , \ E_s = E_{(s)} \setminus E_{(s+1)} \ .$$

Then the following statements are equivalent:

(1) *E is thin at 0 .*
(2) $\sum_{s \in \mathbf{N}} \hat{R}_1^{E_s}(0) < \infty$.
(3) $\sum_{s \in \mathbf{N}} \eta^{(2-Q)s}$ *cap*$(E_s) < \infty$.
(4) $\sum_{s \in \mathbf{N}} \eta^{(2-Q)s}$ *cap*$(E_{(s)}) < \infty$.

Of course, the capacity of a Borel set $A \subset \mathbf{R}^m$ is given by

$$\begin{aligned} \operatorname{cap}(A) &= \sup\{\mu(\mathbf{R}^m) : G\mu \le 1, \mu(\complement A) = 0\} \ , \\ &= \inf\{\nu(\mathbf{R}^m) : G\nu \ge 1 \text{ on } A\} \ , \end{aligned}$$

and we have

$$\operatorname{cap}(y \cdot A) = \operatorname{cap}(A) \ , \qquad \operatorname{cap}(\delta_R(A)) = R^{Q-2} \operatorname{cap}(A) \ .$$

Corollary. *If $0 < \eta < 1$ and $\delta_\eta(E) \subset E$ then E is thin at 0 if and only if cap$(E) = 0$.*

Let

$$I = \{(x_{ij}) \in \mathbf{R}^m : |x_{ij}| \le 1 \text{ for all } i,j \} \ .$$

For every $0 < \gamma \le 1$, let

$$V_\gamma = \{x \in I : x_{11} = 0 \ , \ |x_{12}| \le \gamma\} \ .$$

Using $(m-1)$–dimensional Lebesgue measure on V_γ it can be shown that

$$\operatorname{cap} V_\gamma \approx \frac{1}{1 - \ln \gamma}$$

(where $a_\gamma \approx b_\gamma$ means that there is $C > 0$ such that $C^{-1} b_\gamma \le a_\gamma \le C b_\gamma$ for all γ). For every $0 < \gamma \le 1$, $1 \le k \le r$ and $1 \le l \le n_k$ let

$$W_\gamma^{kl} = \{x \in I : |x_{ij}| \le \gamma \text{ for all } 1 \le i \le k \ , \ 1 \le j \le n_k \ , \ j \le l \text{ if } i = k\} \ .$$

Using Lebesgue measure λ^m it can be shown that if $k \ge 2$ or $l \ge 3$ then

$$\operatorname{cap} W_\gamma^{kl} \approx \begin{cases} \dfrac{\lambda^m(W_\gamma^{kl})}{\gamma^2} \ , & n_1 \ge 3 \ , \\ \dfrac{\lambda^m(W_\gamma^{kl})}{\gamma^2(1 - \ln \gamma)} \ , & n_1 = 2 \ . \end{cases}$$

These estimates allow us to prove the following geometric criteria for regularity:

Proposition. *Suppose that $Q \ge 4$, $n_1 \ge 3$, $0 < \varepsilon < 1$ and $\beta > 0$. Then the set*

$$E = \{x \in \mathbf{R}^m : \beta |x_{ij}|^{r-\varepsilon} \le x_{rn_r} \text{ for } j = 1,2,3\}$$

is thin at 0 .

Proposition. *For all $\beta, p > 0$ the set*

$$E = \{x \in \mathbf{R}^m : \beta |x_{ij}|^{\frac{r}{i}} \le x_{rn_r} \le 1/\beta \text{ for all } (i,j) \ne (r, n_r), \ x_{11} = 0, \ \beta |x_{12}|^p \le x_{rn_r}\}$$

is not thin at 0 .

Fix $\alpha, \beta > 0$ and let

$$A = \{x \in \mathbf{R}^m : \Big(\sum_{(i,j) \neq (r,n_r)} x_{ij}^2 \Big)^{\alpha/2} \leq \beta x_{rn_r} \} .$$

Theorem. *Suppose that $n_1 \geq 3$, $Q \geq 4$. Then A is thin at 0 if and only if $\alpha < r$.*

Theorem. *Suppose that $n_1 = 2$ and $r \geq 3$. Then A is thin at 0 if and only if $\alpha < \frac{r}{2}$.*

Theorem. *An outer ball condition is sufficient for regularity if and only if $r \leq 2$ or $n_1 = 2$ and $r \leq 4$.*

Details can be found in [2].

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