HOMOGENIZATION AND CORRECTORS FOR NONLINEAR ELLIPTIC EQUATIONS

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INTRODUCTION

We deal with a homogenization problem for a nonlinear equation

(1) $$Au + g(.,u) = f .$$

The nonlinear elliptic operator $A : H^1(G) \longrightarrow H^{-1}(G)$ is of type $Au = -\operatorname{div} a(.,Du)$, where the coefficient $a(x,t) : G \times R^n \longrightarrow R^n$ is strongly monotone and Lipschitz continuous in variable t .

The homogezation consists in considering a sequence of equations (1) denoted by superscript $\varepsilon \in E = \{\varepsilon_i > 0 \ , \ \varepsilon_i \longrightarrow 0\}$ with a limit equation of the same type denoted by superscript o , and in investigation of convergence of the corresponding solutions u^ε to the solution u^o of the limit equation. In periodic case the sequence of operators A^ε has periodic coefficients with diminishing period ε defined by $a^\varepsilon(x,t) = a(x/\varepsilon,t)$, where $a(y,t)$ is a function periodic in variable y . In the generalized non-periodic case the diminishing period of the coefficients is replaced by H-convergence (Tartar [2]) of the operators. This "external" characterization of operator convergence by means of their solutions can be replaced by N-condition requiring existence of auxiliary functions N^ε satisfying some convergences. These functions appear in a formula for coefficient a^o of the limit operator A^o . In case of linear operators this "internal" characterization of operator convergence was introduced by Zhikov-Kozlov-Oleinik-Ngoan [4].

The homogenization result (Giachetti [5]) yields weak convergence $u^\varepsilon - u^o \rightharpoonup 0$ in $H^1(G)$. For a fixed small $\varepsilon > 0$ the function u^o represents an approximation of solution u^ε to the problem (1). This approximation can be improved by adding a corrector. Correctors for linear problems were introduced by Bensoussan-Lions-Papanicolaou [3]. Using the auxiliary functions N^ε for $u^o \in C^2(G)$ we obtain the corrector $N^\varepsilon(x,Du^o)$ such that the corrected solution $U^\varepsilon = u^o + N^\varepsilon$ converges strongly : $u^\varepsilon - U^\varepsilon \longrightarrow 0$. Thus for a small fixed $\varepsilon > 0$ the function U^ε approximates u^ε together with its gradient Du^ε. This is important for applications, since e.g. in elasticity $a^\varepsilon(.,Du^\varepsilon)$ describes stresses.

In the first section we shall deal with operator convergences and their characterizations, the second contains homogenization results with correctors. For proofs, more details and comments see [6].

1. OPERATOR CONVERGENCES

We shall deal with a special class of nonlinear operators of type

(2) $$A : u \longmapsto - \operatorname{div} a(.,Du)$$

in a bounded domain G in R^n with Lipschitz boundary. The Carathéodory functions

(3) $$a(x,t) : G \times R^n \longrightarrow R^n$$

are supposed to satisfy the following assumptions

(4) $$\begin{aligned} &(a(x,t) - a(x,s), t - s) \geq \alpha \, |t - s|^2 && \alpha > 0 , \\ &|a(x,t) - a(x,s)| \leq M \, |t - s| && M > 0 , \\ &a(x,0) = 0 . \end{aligned}$$

Definition. The class of operators $A : H^1(G) \longrightarrow H^{-1}(G)$ of type (2) with coefficients (3) satisfying (4) will be denoted by $\mathcal{E}(\alpha,M,G)$.

We introduce operator convergences on the class $\mathcal{E}(\alpha,M,G)$. The sequences will be denoted by superscript $\varepsilon \in E = \{\varepsilon_i > 0 \, , \, \varepsilon_i \rightarrow 0\}$. For homogenization the following notion of H-convergence (Tartar [2]) seems to be the most convenient operator convergence :

Definition. We say that a sequence $\{A^\varepsilon\}$ H-converges to an operator A^0, iff for each u^ε , $u^0 \in H^1(G)$, $f \in H^{-1}(G)$ the following implication holds :

(5) $$\begin{aligned} &u^\varepsilon \rightharpoonup u^0 \quad \text{in } H^1(G) \quad \text{and} \quad A^\varepsilon u^\varepsilon \equiv - \operatorname{div} a^\varepsilon(.,Du^\varepsilon) = f \\ &\text{imply} \quad a^\varepsilon(.,Du^\varepsilon) \rightharpoonup a^0(.,Du^0) \quad \text{in } [L_2(G)]^n . \end{aligned}$$

Remarks.

(a) The introduced H-convergence represents a weak inverse operator convergence besides G-convergence (Spagnolo [1]) and strong G-convergence (in linear case Zhikov-Kozlov-Oleinik-Ngoan [4]).

(b) In the definition we can replace the equality $A^\varepsilon u^\varepsilon = f$ by

(7) $$A^\varepsilon u^\varepsilon = f^\varepsilon \longrightarrow f^0 \quad \text{in } H^{-1}(G)$$

without change of the concept.

Further we introduce another characterization of operator convergence inspired by N-condition introduced for linear operators in [4].

Definition. We say that the sequence of coefficients a^ε satisfies N-condition with respect to the coefficient a^0 iff there exists a sequence of functions $N^\varepsilon(x,t) : G \times R^n \longrightarrow R$ continuous in x and Lipschitz continuous in t such that for $\varepsilon \longrightarrow 0$ the functions N^ε satisfy the following convergences locally uniformly in t :

(N1) $$N^\varepsilon \rightharpoonup 0 \quad \text{in } H^1(G) \quad ,$$

(N2) $a^{\varepsilon}(.,t + D_x N^{\varepsilon}(.,t)) \longrightarrow a^{o}(.,t)$ in $[L_2(G)]^n$,

(N3) $- \operatorname{div}[a^{\varepsilon}(.,t + D_x N^{\varepsilon}(.,t)) - a^{o}(.,t)] \longrightarrow 0$ in $H^{-1}(G)$,

(N4) $D_t N^{\varepsilon} \longrightarrow 0$ in $[L_2(G)]^n$ for a. e. $t \in R^n$.

Theorem. Let A^{ε}, $A^{o} \in \mathcal{E}(\alpha,M,G)$ and let a^{ε}, a^{o} be the corresding coefficients. Then the following implications hold :

(i) If A^{ε} satisfy N-condition with respect to A^{o} , then A^{ε} H-converge to A^{o}.

(ii) If A^{ε} H-converge to A^{o} then a^{ε} satisfy N-condition with respect to a^{o} with functions N^{ε} satisfying (N1) - (N3).

Remark. In linear or periodic cases H-convergence implies even (N4). In the nonlinear non-periodic case the proof of (N4) is not complete.

For the proof see [6]. The functions N^{ε} are defined as the solution to the problem :

(8) Find $N^{\varepsilon}(.,t) \in H^1_0(G)$ satisfying $A^{\varepsilon}(tx + N^{\varepsilon}(.,t)) = A^{o}(tx)$.

2. HOMOGENIZATION AND CORRECTORS

We consider a sequence of boundary value problems

$$A^{\varepsilon}u^{\varepsilon} + g^{\varepsilon}(.,u^{\varepsilon}) = f^{\varepsilon} \text{ in } G ,$$

(9)

$$u^{\varepsilon} = \varphi^{\varepsilon} \text{ on } \Gamma_0 , \qquad (a^{\varepsilon}(.,Du^{\varepsilon}),n) = \psi^{\varepsilon} \text{ on } \Gamma_1 ,$$

where $\partial G = \Gamma_0 \cup \Gamma_1$ and n is the unit normal vector to ∂G . The corresponding homogenized problem has the form

$$A^{o}u^{o} + g^{o}(.,u^{o}) = f^{o} \text{ in } G ,$$

(10)

$$u^{o} = \varphi^{o} \text{ on } \Gamma_0 , \qquad (a^{o}(.,Du^{o}),n) = \psi^{o} \text{ on } \Gamma_1 .$$

We adopt the following assumptions :

(11)

$$A^{\varepsilon} , A^{o} \in \mathcal{E}(\alpha,M,G) , \; A^{\varepsilon} \text{ H-converge to } A^{o} ,$$

g^{ε} , g^{o} : $G \times R \longrightarrow R$ are Carathéodory functions,

$$|g^{\varepsilon}(x,t)| \leq h_0(x) + c|t| , \quad c > 0 , \; h_0 \in L_2(G) ,$$

$$g^{\varepsilon}(x,t) \cdot \operatorname{sign} t \geq - h_1(x) , \qquad h_1 \in L_2(G) ,$$

$$g^{\varepsilon}(.,t) - g^{o}(.,t) \longrightarrow 0 \text{ in } L_\infty(G) \text{ uniformly in } t ,$$

$$f^{\varepsilon} , f^{o} \in H^{-1}(G) , \qquad f^{\varepsilon} \longrightarrow f^{o} \text{ in } H^{-1}(G) ,$$

$$\varphi^{\varepsilon}, \varphi^{o} \in H^1(G) , \qquad \varphi^{\varepsilon} \longrightarrow \varphi^{o} \text{ in } H^1(G) ,$$

$$\psi^{\varepsilon}, \psi^{o} \in L_2(\Gamma_1) , \qquad \psi^{\varepsilon} \longrightarrow \psi^{o} \text{ in } L_2(\Gamma_1) .$$

Under the introduced assumptions the problems have solution, which need not be unique. We can formulate the following result :

Theorem. Let $\{u^{\varepsilon}\}$ be a sequence of solutions to the problem (9). Then there exists a subsequence $\{u^{\varepsilon'}\} \subset \{u^{\varepsilon}\}$ such that

$$u^{\varepsilon'} - u^0 \longrightarrow 0 \quad \text{in} \quad H^1(G) \,,$$
(12)
$$a^{\varepsilon'}(.,Du^{\varepsilon'}) - a^0(.,Du^0) \longrightarrow 0 \quad \text{in} \quad [L_2(G)]^n \,,$$
where u^0 is a solution to the homogenized problem (10).

Let moreover $u^0 \in C^2(G)$ and let the coefficients a^ε satisfy N-condition with respect to a^0 with functions $N^\varepsilon(x,t)$. Then using correctors we can define the corrected solution

(13) $$U^\varepsilon(x) = u^0(x) + N^\varepsilon(x,Du^0(x)) \,,$$

such that the convergences (12) become strong :

$$u^{\varepsilon'} - U^{\varepsilon'} \longrightarrow 0 \quad \text{in} \quad H^1(G) \,,$$
(14)
$$a^{\varepsilon'}(.,Du^{\varepsilon'}) - a^{\varepsilon'}(.,DU^{\varepsilon'}) \longrightarrow 0 \quad \text{in} \quad [L_2(G)]^n \,.$$

If the homogenized equation admits unique solution, or $u^\varepsilon \longrightarrow u^0$, then the whole sequences (12) and (14) converge.

<u>Periodic case.</u> Let the coefficients a^ε of operators A^ε be defined

(15) $$a^\varepsilon(x,t) = a(x/\varepsilon,t) \,,$$

where $a(y,t)$ is a function periodic in variable y with period Y, $Y = [0,\overline{y}_1] \times \ldots \times [0,\overline{y}_n]$ satisfying conditions (4). Let $N(y,t)$ be the solution periodic in y to the following problem :

(16) $$-\operatorname{div} a(y,t + D_yN(y,t)) = 0 \quad , \quad \int_Y N(y,t)dy = 0 \,.$$

Then the following theorem holds:

<u>Theorem.</u> The operators A^ε H-converge to a constant coefficient operator A^0 with coefficient

(17) $$a^0(t) = \int_Y a(y,t + D_yN(y,t))dy \,/\, \text{meas}(Y) \,.$$

Further the coefficients a^ε satisfy N-condition with respect to a^0 with auxiliary functions $N^\varepsilon(x,t) = \varepsilon\, N(x/\varepsilon,t)$, where N is defined by the problem (14).

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