SOLUTION OF NONLINEAR DEGENERATE ELLIPTIC-PARABOLIC SYSTEMS IN ORLICZ - SOBOLEV SPACES

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We consider the system

(1.1) $\partial_t b^j(u) - \nabla a^j(t,x, b(u), \nabla u) = f^j(t,x, b(u))$, $j = 1, \dots, m$;

$(x,t) \in Q_T = \Omega \times (0,T)$, $\Omega \subset R^N$ is bounded with the initial and boundary conditions

(1.2) $b(u) = b(u_0)$ $\quad \Omega \times \{0\}$

$u = u^D$ on $\Gamma_1 \times (0,T)$

(1.3) $a(t,x, b(u)) \cdot \nu = \varphi(t,x,u)$ on $\Gamma_2 \times (0,T)$

where $u = (u^1, \dots, u^m)$, $\Gamma_1 , \Gamma_2 \subset \partial\Omega$, $\Gamma_1 \cap \Gamma_2 = \theta$

$mes_{N-1} \Gamma_1 + mes_{N-1} \Gamma_2 = mes_{N-1} \partial\Omega$, $mes_{N-1} \Gamma_1 > 0$.

We assume

(1.4) $b(u) = \nabla\Phi(u)$ where $\Phi : R^m \to R^1$ is convex, C^1 and $b(0) = 0$.

On subsets where b is constant (1.1) is elliptic. System (1.1) includes porous medium type equations. This system has been studied by H.W. Alt and S. Luckhaus in [1] under the assumptions

(1.5) $(a(b(\eta), \xi_1) - a(b(\eta), \xi_2), \xi_1 - \xi_2) \geq c_0 |\xi_1 - \xi_2|^p$ $\quad (p \geq 2)$,

$\varphi \equiv 0$ in (1.3) and under the corresponding polynomial growth conditions concerning **a**,**f** .

Our contribution is to prove the existence of the variational solution of (1.1) only for monotone ease $c_0 = 0$ and under the very weak restrictions on the growth of a in ξ. Formally we can write it in the form

(1.6) $a(b(\eta), \xi)$ is monotone in ξ (i.e. (1.5) holds with $c_0 = 0$) and it is continuous in their variables;

(1.7) $|a(b(\eta), \xi)| \leq c(1 + h(\eta) + |g(\xi)|)$

(1.8) $a(b(\eta), \xi) \cdot \xi \geq c\, \xi \cdot g(\xi)$

(1.9) $g \in C(R)$, $g(\xi) \to \infty$ for $\xi \to \infty$, $\xi \cdot g(\xi)$ is even and convex for $|\xi| \geq \xi_0 > 0$. (In the fact we understand that to each component of ξ belongs a corresponding component of the vector function g).

The growth conditions in a more general and more precise form are considered in [3].

Denote by G, $G(\xi) := \xi\, g(\xi)$ (for $|\xi| \geq \xi_0 > 0$) the N-function (see [5]) and by $\overline{G}$ the conjugate (N-function) with respect to $G(\overline{G}(x) = \max_{y \in R} (x \cdot y - G(y))$. By the same way we construct

(1.10) $\Psi(x) := \overline{\phi(x) - \phi(0))}$ and $B(\eta) := \Psi(b(\eta))$

Then $h(\eta)$ in (1.7) has to satisfy

(1.11) $h(\eta) \leq \overline{G}^{-1}(B(\eta))$

Moreover, we assume

(1.12) $|f(b(\eta))| \leq c(1 + h(\eta))$

(1.13) $u^D \in W^{1,1}_{\infty}(Q_T)$, $u_0 := u^D(0)$.

Our variational solution is an element of the Orlicz-Sobolev space V defined as follows. Let $L_G(Q_T) \equiv L_G$ be the Orlicz space

$$L_G \equiv \{u \in L_1(Q_T) : \exists\, k > 0 \quad \text{such that} \quad \int_{Q_T} G(ku) < \infty\}$$

with the norm $\|v\|_G = \inf\{r > 0 : \int_{Q_T} G(v/r) \le 1\}$.

L_G is B-space and $L_G \equiv (E_{\bar{G}})^*$ where $E_{\bar{G}}$ is the closure of bounded functions in the norm of the space $L_{\bar{G}}$. When $g(\xi) = |\xi|^{p-2}\xi$ then $L_G \equiv L_p$ and $L_{\bar{G}} = L_q$ (with $p^{-1} + q^{-1} = 1$). Then our Orlicz-Sobolev space V is defined as follows

$$V \equiv W_G^{1,0}(Q_T) = \{u : u^j \in L_{G_0^j} \quad \text{for} \quad j = 1, \dots, m,\ D^i u^j \in L_{G_i^j}$$

$$\text{for } i = 1, \dots, N \quad \text{and} \quad u/_{\Gamma_1 \times (0,T)} = 0\}$$

with the norm $\|u\|_V = \sum_{j=1}^{m} \sum_{i=0}^{N} \|D^i u^j\|_{G_i^j}$ where we take $G_0^j := \min_{1 \le i \le N} G_i^j$. Evidently $W_G^{1,0} \subset W_1^{1,0}(Q_T)$. With respect to φ in (1.3) we assume

(1.14) $\varphi(t,x,\eta)$ is continuous in their variables and is monotone in η;

(1.15) $$|\varphi(\eta) \cdot \xi| \le c_1 + c_2(\eta \cdot \varphi(\eta) + \xi \cdot \varphi(\xi))$$

(1.16) $$|\varphi^j(\eta)) \le c(1 + \sum_{i=1}^{m} (\bar{G}_0^j)^{-1}[G_0^i(\eta^i)]) \qquad \forall j = 1, \dots, m.$$

The more general conditions are considered in [4].

<u>*1.17 Theorem*</u> If (1.6) - (1.16) are satisfied then there exists a variational solution u of (1.1), i.e., $u - u^D \in V$, $b(u) \in L_1(Q_T)$, $a(b(u), \nabla u) \in L_{\bar{G}}(Q_T)$, $f(b(u)) \in L_{\bar{G}}(Q_T)$ and

(1.18) $$\int_{Q_T} (b(u_0) - b(u)) \cdot \partial_t v + \int_{Q_T} a(b(u), \nabla u) \cdot \nabla v + \int_{S_T} \varphi(u) \cdot v = \int_{Q_T} f(b(u)) \cdot v,$$
$\forall v \in V \cap L_\infty(Q_T)$ with $\partial_t v \in L_\infty(Q_T)$, $v(T) = 0$.

1.19 Remark In the fact there exists $\partial_t b(u) \in V^*$ and $\int_{Q_T} (b(u_0 - b(u)) \cdot \partial_t v =$ $<\partial_t b(u), v>_T$ ($<\cdot,\cdot>_T$ is the duality between V^* and V) where v is from (1.18). Then in the place of (1.18) we have

(1.18') $$<\partial_t b(u), v>_T + \int_{Q_T} a(b(u), \nabla u) \cdot \nabla v + \int_{S_T} \varphi(u) \cdot v = \int_{Q_T} f(b(u)) \cdot v \quad \forall v \in V.$$

To prove Theorem 1.17 we discretize (1.1) in time and space (modified time discretized Galerkin method). We obtain energy type a priori estimates

$$\int_{Q_T} G(\nabla u_\alpha) \le c, \ \sup_{t \in (0,T)} \int_\Omega B(u_\alpha(t)) \le c \quad \text{and}$$
$$\int_0^{T-\tau} \int_\Omega (b(u_\alpha(t+\tau)) - b(u_\alpha(t))) \cdot (u_\alpha(t+\tau) - u_\alpha(t)) \le c\,\tau$$

uniformly with respect to the discretization index α ($\alpha = (\Delta t, \lambda^{-1})$, $\Delta t = \frac{T}{n}$, λ being the dimension of $V_\lambda = \mathrm{span}\{e_1, \dots, e_\lambda\}$). In the parabolic part of the equation we follow [1] (using compactness argument and integration by parts formula $\int_0^t \int_\Omega \partial_t b(u_\alpha) \cdot u_\alpha = \sum_\Omega B(u_\alpha(t)) - \int_\Omega B(u_0))$. In the elliptic part of the equation we follow the idea of Minty-Browder. Some special properties of the Orlicz-Sobolev spaces are used and some results from elliptic equations [2] concerning Orlicz-Sobolev spaces are applied. The detail proofs are in [3] for $\varphi \equiv 0$. The case $\varphi \neq 0$ and also nonmonotonicity of φ will be discussed in [4].

When the system (1.1) is diagonal we can prove L_∞-boundedness of the variational solution. Moreover we can remove the restrictions of a,φ with respect to the growth in $b(\eta)$, η, respectively. We consider

(2.1) $$\partial_t b^j(u^j) - \nabla a^j(t,x, b(u), \nabla u^j) = f^j(t,x, b(u)) + F^j(t,x, b(u))$$
with (1.2), (1.3). We assume

(2.2) b(s) is strictly monotone and $|b(s)| \to \infty$ for $|s| \to \infty$.

(2.3) $|a(b(\eta), \xi)| \le \mu(|\eta|)(1 + |g(\xi)|)$, $a(b(\eta), 0) \equiv 0$;

(2.4) $a(b(\eta), \xi) \cdot \xi \geq \nu(|\eta|)\, \xi \cdot g(\xi)$

where $\nu, \mu > 0$ are continuous ($\nu(s) \to 0$ for $s \to \infty$);

(2.5) $\partial_{\xi_i} a^j_k (b(\eta), \xi) = \partial_{\xi_k} a^j_i (b(\eta), \xi), \forall j, \forall i,k$;

(2.6) $\partial_{\eta_j} \varphi^j(\eta) \geq 0$ and $\varphi^j(\eta) \cdot \eta^j \geq 0 \quad \forall |\eta| \geq K > 0 \quad \forall j$;

(2.7) $|f(\eta)| \leq c_1(d + |\eta|), d \in L_\infty(Q_T)$

and

(2.8) $\partial_{\eta_j} F^j(\eta) \leq 0, \forall j$; $\sum_{j=1}^{m} \eta_j^p F^j(\eta) \leq c_2 |\eta|^{p+1} + c_3$

$\forall p = 2k+1, k \geq k_0 > 0, \forall |\eta| \leq D_T + 1$ where

(2.9) $D_T := (\|b(u^D)\|_{\infty,Q_T} + \|d\|_{\infty,Q_T} + b_K)\, e^{(c_1(m+1)+c_2)T}$

and $b_K := \max_j (\max\{b^j(K), -b^j(-K)\})$.

2.10 Theorem Let (2.2) - (2.9), (1.6), (1.9) and (1.13) are satisfied. Then there exists a bounded variational solution of (2.1), (1.2), (1.3). Moreover $\|b(u)\|_{\infty,Q_T} \leq D_T$ where D_T is from (2.9).

The assertion of Theorem 2.10 can be extended to the case when a is of the form $a(t,x,M(u), \nabla u)$ with a rather general Volterra operator $M : L_\infty(Q_T) \to L_\infty(Q_T)$. The proof can be found in [3] for the case $\varphi \equiv 0$. The case $\varphi \neq 0$ will be discussed in [4].

References

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