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STABILITY AND ASYMPTOTIC PROPERTIES OF DYNAMICAL SYSTEMS IN THE PLANE

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Consider the real dynamical system in the plane

(1) $$x' = A(t)x + h(t,x),$$

where $A(t) = (a_{jk}(t))$, $j,k = 1,2$; $t \in [t_0,\infty) =: J$ is a matrix function and $h(t,x) = (h_1(t,x), h_2(t,x))$, $x = (x_1,x_2)$ is a vector function, defined on the region $J \times \{(x_1,x_2) \in \mathbf{R}^2 : x_1^2 + x_2^2 < r \le \infty\}$.

The transformation

$$y = Bx\,, \quad B = \begin{bmatrix} 1, & i \\ 1, & -i \end{bmatrix}, \quad y = (z,\bar{z}), \quad z = x + iy$$

from real to conjugate coordinates $z,\bar{z}$ converts (1) to the equation

(2) $$y' = BA(t)B^{-1}y + Bh(t,B^{-1}y),$$

where

$$BAB^{-1} = \begin{bmatrix} a\,, & b \\ \bar{b}\,, & \bar{a} \end{bmatrix},$$

$$a = \frac{1}{2}(a_{11} + a_{22}) + \frac{i}{2}(a_{21} - a_{12}), \quad b = \frac{1}{2}(a_{11} - a_{22}) + \frac{i}{2}(a_{21} + a_{12})\,.$$

From (2) we obtain

(3) $$z' = a(t)z + b(t)\bar{z} + g(t,z,\bar{z})$$

with

$$g(t,z,\bar{z}) = h_1(t,\tfrac{1}{2}(z + \bar{z}), \tfrac{1}{2i}(z - \bar{z})) + ih_2(t,\tfrac{1}{2}(z+\bar{z}), \tfrac{1}{2i}(z - \bar{z})).$$

The second equation $\bar{z}' = \bar{a}(t)\bar{z} + \bar{b}(t)z + \bar{g}(t,z,\bar{z})$ is redundant.

Conversely, putting $a_{11} = \operatorname{Re}(a + b)$, $a_{12} = \operatorname{Im}(b - a)$, $a_{21} = \operatorname{Im}(a + b)$, $a_{22} = \operatorname{Re}(a - b)$, $h_1(t,x,y) = \operatorname{Re} g(t,x + iy, x - iy)$, $h_2(t,x,y) = \operatorname{Im} g(t,x + iy, x - iy)$, equation (3) can be written in the real form (1). Thus, it is sufficient to investigate equation (3) instead of system (1) since the asymptotic properties of (3) can be easily modified to (1).

Theorem 1. *Let the following assumptions be fulfilled.*

(A) *The functions $a,b : J \to \mathbf{C}$ have continuous first derivatives.* *The function $g : J \times \{z \in \mathbf{C},\ |z| < r \le \infty\} \to \mathbf{C}$ is continuous and any initial problem to (3) has unique solution.*

(B) $\liminf\limits_{t\to\infty} (|a(t)| - |b(t)|) > 0.$

(C) *There exists continuous function $\kappa : J \to \mathbf{R}$ such that*

$$|\gamma(t)g(t,z,\bar{z}) + c(t)\bar{g}(t,z,\bar{z})| \leq \kappa(t)|\gamma(t)z + c(t)\bar{z}|$$

for $t \geq T \geq t_0$, $|z| < r \leq \infty$, *where*

$$\gamma = |a| + \sqrt{|a|^2 - |b|^2}, \qquad c = \frac{\bar{a}b}{|a|}.$$

If

$$\limsup_{t\to\infty} \int^t \Theta(s)ds < \infty,$$

where

$$\Theta = \operatorname{Re} a + \left|\frac{b}{a}\operatorname{Re} a\right| + \frac{\operatorname{Re}(\gamma\gamma' - \bar{c}c') + |\gamma c' - \gamma' c|}{|\gamma|^2 - |c|^2} + \kappa,$$

then the trivial solution of (3) is stable, if

$$\limsup_{t\to\infty} \int^t \Theta(s)ds = -\infty$$

it is asymptotically stable.

Outline of the proof. Let $z = z(t)$ be any nontrivial solution of (3). Then

(4) $$V(t) = |\gamma(t)z(t) + c(t)\bar{z}(t)|$$

is a Lyapunov function for (3) satisfying the inequality

$$V' \leq V\left(\operatorname{Re} a + \left|\frac{b}{a}\operatorname{Re} a\right| + \operatorname{Re}\frac{\gamma' z + c'\bar{z}}{\gamma z + c\bar{z}} + \kappa\right) \leq V\Theta.$$

This inequality implies the statement.

Corollary. *Let* $a,b \in \mathbf{C}$, $|a| > |b|$ *and let* $p,q : J \to \mathbf{C}$ *be continuous functions. If*

$$\limsup_{t\to\infty} \int^t \left[\frac{(|a|-|b|)^2}{\sqrt{|a|^2 - |b|^2}} \frac{\operatorname{Re} a}{|a|} + |p(s)| + |q(s)|\right] ds < \infty,$$

then the trivial solution of the equation

(5) $$z' = (a + p(t))z + (b + q(t))\bar{z}$$

is stable. If

$$\lim_{t\to\infty} \int^t \left[\frac{(|a|-|b|)^2}{\sqrt{|a|^2 - |b|^2}} \frac{\operatorname{Re} a}{|a|} + |p(s)| + |q(s)|\right] ds = -\infty$$

then the trivial solution is asymptotically stable.

Remark. If we apply Theorem 1 to the case $a(t) \equiv a$, $b(t) \equiv b$, $q(t,z,\bar{z}) \equiv 0$, that means to the equation

(6) $$z' = az + b\bar{z}, \quad a,b \in \mathbf{C},$$

the assumption (B) implies $|a| > |b|$ and the origin is a single singular point which

is a

focus, if $|b| < |\text{Im } a|$ and $\text{Re } a \neq 0$,
centre, if $|b| < |\text{Im } a|$ and $\text{Re } a = 0$,
node, if $|b| \geq \text{Im } a$.

The equation $|\gamma z + c\bar{z}| = \lambda$ where λ is a parameter (compare with (4)) represents a pencil of ellipses which intersect the trajectories of (6) at a constant angle $\omega = \arg(i\bar{a})$. If

$$|b| > |a|, \tag{7}$$

the origin is a saddle point; note that the condition (7) implies $|b| > |\text{Im } a|$ (see Theorem 2). In this case there exists one parameter family of solutions of (6) which converge to the origin and two parameter family of solutions which tend to infinity as $t \to \infty$.

To receive analogous result in the nonconstant case the equation (3) is transformed by means of the mapping

$$w = h(t)z + k(t)\bar{z},$$

$$h(t) = i\,\text{Im } a(t), \quad k(t) = \sqrt{|b^2(t)| - \text{Im}^2 a(t)} + b(t)$$

to an equation suitable for the use of Ważewski topological method with the following result (it is formulated for simplicity for the linear case only).

Theorem 2. *Let $a,b \in \mathbf{C}$, $|b| > |\text{Im } a|$ and let $p,q : J \to \mathbf{C}$ be continuous functions. Put* $\xi = 2(|h|+|k|)/||h|-|k||$, $\alpha = \text{Re } a$, $\beta = \sqrt{|b|^2 - \text{Im}^2 a}$. *Let*

$$\limsup_{t\to\infty} (|p(t)| + |q(t)|) < \frac{\beta}{\xi}.$$

***Then** there exists a solution $z_0 \neq 0$ of (5) such that*

$$\lim_{t\to\infty} z_0(t) \exp\{-\varepsilon t - \xi \int_{t_0}^{t} (|p(s)| + |q(s)|)ds\} = 0$$

for any $\varepsilon > \alpha - \beta$. Every solution z linearly independent to z_0 has the property

$$z(t) \exp\{-\eta t + \xi \int_{t_0}^{t} (|p(s)| + |q(s)|)ds\} \to \infty$$

for any $\eta < \alpha + \beta$.