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PARTIAL DIFFERENTIAL INEQUALITIES AS EQUATIONS WITH HYSTERESIS

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The aim of this note is to emphasize the parallelism between inequalities of evolution type, free boundary problems and equations with hysteresis. Hysteresis thus arises as a natural feature of classical problems of mathematical physics. This approach can give new results if we profit from the knowledge of the structure of the hysteresis memory.

I. An ordinary differential inequality.

Problem. For a given interval $K \subset R^1$, $0 \in K = \overline{K}$ and a function $\varepsilon : [0,T] \to R^1$ find $\sigma : [0,T] \to R^1$ such that

$$\sigma(t) \in K \qquad \forall t \in [0,T] \ , \tag{1}$$

$$(\sigma' - \varepsilon')(\sigma - z) \leq 0 \qquad \forall z \in K, \text{ a.e. } t \in [0,T] \ , \tag{2}$$

$$\sigma(0) = P_K(\varepsilon(0)) \ , \tag{3}$$

where P_K is the projection of R^1 onto K.

Physical interpretation. Relations (1) - (3) represent Prandtl's constitutive law for a one-dimensional elasto-plastic material, where ε, σ are the strain and stress, respectively, Int K is the domain of elasticity with modul 1 and ∂K are the yield points.

In [6] we can find the proof of the following theorem (cf. also [3], [1]).

Theorem. *For every* $\varepsilon \in W^{2,1}(0,T)$ *there exists a unique solution* $\sigma \in W^{1,\infty}(0,T)$ *of* (1) - (3). *Moreover, the mapping* $f_K : W^{2,1}(0,T) \to W^{1,\infty}(0,T) : \varepsilon \mapsto \sigma$ *thus defined can be extended to an operator* $C([0,T]) \to C([0,T])$ *and* $W^{1,p}(0,T) \to W^{1,p}(0,T)$, $1 \leq p \leq \infty$ *such that*

(i) $f_K : C([0,T]) \to C([0,T])$ *is Lipschitz,*

(ii) $f_K : W^{1,1}(0,T) \to W^{1,1}(0,T)$ *is Lipschitz,*

(iii) $f_K: W^{1,p}(0,T) \to W^{1,p}(0,T)$ *is continuous, but not Lipschitz unless* $K = \{0\}$ *or* $K = R^1$,

(iv) $f_K: W^{1,\infty}(0,T) \to W^{1,\infty}(0,T)$ *is discontinuous unless* $K = \{0\}$ *or* $K = R^1$.

The following properties can be derived immediately from (1)-(3).

Rate independence. $f_K(\varepsilon \circ s) = f_K(\varepsilon) \circ s$
for every increasing bijection $s: [0,T] \to [0,T]$.

Monotonicity.
$(f_K(\varepsilon_1) - f_K(\varepsilon_2))(\varepsilon_1' - \varepsilon_2') \geq \frac{1}{2} \frac{d}{dt} \left(f_K(\varepsilon_1) - f_K(\varepsilon_2)\right)$ a.e.
for every $\varepsilon_1, \varepsilon_2 \in W^{1,1}(0,T)$.

Energy inequalities.

$$\frac{1}{2} f_K^2(\varepsilon)(t) - \frac{1}{2} f_K^2(\varepsilon)(s) \leq \int_s^t f_K(\varepsilon)\varepsilon' \, d\tau \qquad \forall \varepsilon \in W^{1,1}(0,T),$$

$$\frac{1}{2} (f_K(\varepsilon))'^2(t) - \frac{1}{2} (f_K(\varepsilon))'^2(s) \leq \int_s^t (f_K(\varepsilon))'\varepsilon'' \, d\tau$$

$$\forall \varepsilon \in W^{2,1}(0,T) \qquad \text{for a.e. } 0 \leq s < t \leq T.$$

Memory (cf. [4], [5]). Let the expression *memory* denote the set of those values of $\varepsilon(\tau)$, $\tau \in [0,t]$, which are necessary for determining the value of $f_K(\varepsilon)(t)$. This set is always *finite*.

Terminology. The operator f_K is called *Prandtl hysteresis operator corresponding to K*.

II. One-dimensional one-phase Stefan problem.

Problem. Find the functions $\theta(x,t)$ (temperature) and $s(x)$ (interface between ice and water) such that

$\theta(x,t) = 0$ for $t < s(x)$ (ice) , (4)

$\theta_t - \theta_{xx} = 0$ for $t > s(x)$ (water) , (5)

$\theta_x(x,s(x))\, s'(x) = -k(x)$ (conservation of energy) , (6)
where $k(x)$ is a given function (latent heat).

We prescribe e.g. the Newton boundary condition

$$\theta_x(0,t) = b(\theta(0,t) - \theta_1(t)) \quad \text{and} \quad \theta(1,t) = 0\ , \tag{7}$$

where $b > 0$ and θ_1 are given.

The weak formulation of (4) - (7) (cf. [2]) consists in transforming the problem into a parabolic inequality. In [6] it is shown that this problem is equivalent to the system

$$w_t - f_{R+}(w_x)_x = -K(x) \tag{8}$$

$$w_t(0,t) = -K(0) + b(f_{R+}(w_x)(0,t) - T_1(t)) \tag{9}$$

$$w(1,t) = w(x,0) = 0\ , \tag{10}$$

where $K(x) = -\int_x^1 k(y)\,dy,\quad T_1(t) = \int_0^t \theta_1(\tau)\,d\tau\ ,$

$$w(x,t) = \int_0^t \int_{s(x)}^{\tau} \theta_x(x,\sigma)\,d\sigma\,d\tau - tK(x)\ ,$$

and f_{R+} is the Prandtl hysteresis operator corresponding to $R^+ = [0,+\infty)$. The hysteresis formulation yields under the same assumptions slightly more regular solutions. The free boundary $t=s(x)$ appears as the interface between memory levels of the operator f_{R+} , i.e. the cardinality of the memory passes from 1 to 2.

III. Longitudinal vibrations of an elasto-plastic beam.

Problem. Find the functions $\sigma(x,t)$ (stress), $\varepsilon(x,t)$ (strain), $u(x,t)$ (displacement), $v(x,t)$ (velocity) satisfying the system

$$\varepsilon = u_x\ ,\quad v = u_t \tag{11}$$

$$\sigma = f_K(\varepsilon),\ K = [-h,h] \quad \text{(Prandtl's constitutive law)} \tag{12}$$

$$u_{tt} = \sigma_x \quad \text{(equation of motion)} \tag{13}$$

with initial conditions

$$v(x,0) = v_0(x),\ u(x,0) = u_0(x)\ . \tag{14}$$

Instead of rewriting (11) - (13) in the form of a hyperbolic inequality (cf. [2], [7]) we investigate the system

$$\begin{aligned} \varepsilon_t &= v_x \\ v_t &= f_K(\varepsilon)_x\ . \end{aligned} \tag{15}$$

Its hyperbolicity is confirmed by the finite speed of propagation. The construction of the solution relies on energy estimates which are strong enough for ensuring the global solvability of (15), (14) even with more complex (nonlinear) constitutive laws (cf. references in [6]). We can observe again a nontrivial (noncharacteristic) free boundary between the regions of plasticity ($\sigma_t \neq \varepsilon_t$) and elasticity ($\sigma_t = \varepsilon_t$).

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