EQUADIFF 7

K. Kirchgässner

Nonlinear surface waves under external forcing

In: Jaroslav Kurzweil (ed.): Equadiff 7, Proceedings of the 7th Czechoslovak Conference on Differential Equations and Their Applications held in Prague, 1989. BSB B.G. Teubner Verlagsgesellschaft, Leipzig, 1990. Teubner-Texte zur Mathematik, Bd. 118. pp. 47--58.

Persistent URL: http://dml.cz/dmlcz/702344

Terms of use:

© BSB B.G. Teubner Verlagsgesellschaft, 1990

Institute of Mathematics of the Academy of Sciences of the Czech Republic provides access to digitized documents strictly for personal use. Each copy of any part of this document must contain these *Terms of use*.

This paper has been digitized, optimized for electronic delivery and stamped with digital signature within the project *DML-CZ: The Czech Digital Mathematics Library* http://project.dml.cz

Nonlinear Surface Waves under External Forcing

KIRCHGÄSSNER K., STUTTGART, FRG

1 Reduction of nonlinear elliptic systems

In recent years quasilinear elliptic systems have been found to be reducible to ordinary differential equations when considered in an unbounded domain $\Omega = \mathbb{R} \times D$, where $D \subset \mathbb{R}^n$ is bounded and smooth ([6],[10], [12]). The reduction can be achieved when bounded solutions are sought near a known solution being independent of the unbounded variable x. The method applied is a combination of arguments from elliptic systems and dynamical systems. Although the initial value problem in x is not well posed for elliptic equations, the proofs for the existence of a center manifold still work with some (nontrivial) modifications. For a good introduction into the main ideas for the semilinear case c.f.[16]. This manifold contains all solutions of small 'amplitude', and these are determined as the flow of a finite dimensional vector field.

The method has been successful by resolving a number of long standing open problems. Let me mention just two, whose resolution I consider as particularly satisfying: the Saint-Venant problem of nonlinear elasticity [13] and the existence of solitary waves on the free surface of an inviscid fluid under the influence of gravity and surface tension [1]. The latter work required that the surface tension is not too small, i.e. the 'Bond number', measuring the relative strength of surface forces to gravity forces, should be greater than one third.

In this contribution this result is reproduced with a method modifying the one used in [1]. We add the additional difficulty of a steady external forcing and describe the response completely. This elaborates our work in [7] and is, in its discussion of the reduced system, identical with a remarkable analysis by Mielke [11] for a physically different problem. Our main motivation, however, is the wish, to present this new point of view for nonlinear elliptic systems to a wider readership by describing a real but relatively simple problem of physical interest.

2 The basic equations

The inviscid fluid is located in a domain $\mathbb{R} \times D_\xi$, where $D_\xi = \{(\xi, \eta)/0 < \eta < z(\xi), \xi \in \mathbb{R}\}$, and $z(\xi)$ describes the free boundary (see Figure 1). We use nondimensional quantities by taking h, the asymptotic height of the fluid layer, and the

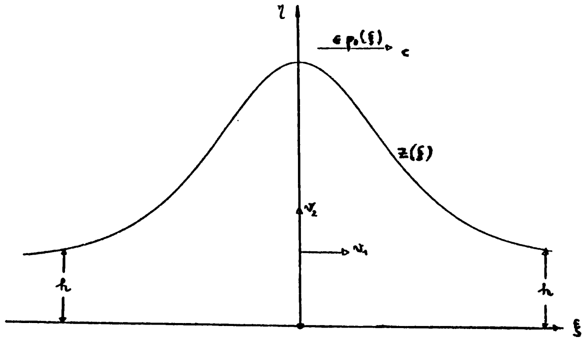

Figure 1

wave speed c as reference length and velocity. Due to Galilean invariance we may choose a moving coördinate system in which the motion is time independent. The term $\epsilon\ p_0(\xi)$ describes the external forcing. Thus $\epsilon = 0$ is the unforced situation. We shall analyze the case of p_0 having compact support and thus treat the nonlinear interaction between a solitary wave, travelling with constant speed c, and a localized pressure wave moving with the same speed.

Considering irrotational flow we obtain

$$\begin{aligned} \operatorname{div} \underline{v} &= \operatorname{curl} \underline{v} = 0, \quad 0 < \eta < z(\xi) \\ v_2 &= 0, \qquad\qquad\quad \eta = 0 \\ v_1 \partial_\xi z - v_2 &= 0 \\ \tfrac{1}{2}|\underline{v}|^2 + p + \lambda z &= C = \text{const.}, \quad \eta = z(\xi) \ , \end{aligned} \tag{2.1}$$

where

$$\lambda = \frac{gh}{c^2}\,, \qquad b = \frac{T}{gh^2}\ .$$

T is the coefficient of surface tension, b the Bond number. We seek solutions satisfying

$$\lim_{\xi \to \pm\infty} \underline{v}(\xi, \eta) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \mathcal{O}(\epsilon) \tag{2.2a}$$

which correspond, due to the coordinate system moving in time, to solutions of the Euler equation vanishing at infinity for $\epsilon = 0$. The pressure p on the free surface $\eta = z(\xi)$, satisfies

$$p(\xi) = -b\frac{z''(\xi)}{(1+z'(\xi)^2)^{3/2}} + \epsilon p_0(\xi) \ . \tag{2.2b}$$

We shall treat $\epsilon = 0$ (unforced case) and $\epsilon \neq 0$, p_0 has compact support (forced case).

Let me mention a few contributions to this problem in the mathematical literature. The case of pure gravity cnoidal (periodic) waves, ($b = \epsilon = 0$) has been settled by Levi-Civitá [9] and Nekrasov [14], existence of solitary waves was shown by Lavrentiev [8], Friedrichs and Hyers [5], and also by Ter Krikerov [15]. Finite amplitude waves and the proof of Stokes' conjecture was given by Amick and Toland [2]. Capillary-gravity cnoidal waves ($b > 0, \epsilon = 0$) were analyzed by Beckert [3] and Zeidler [17]. The existence of solitary waves for $b > \frac{1}{3}$ was proved by Amick and the author in [1]. The forced case is en vogue in the applied literature but essentially untouched mathematically, except for [7]. The case $0 < b < \frac{1}{3}, \epsilon = 0$, seems to be of particular difficulty.

3 Transformations and Symmetries

For $\epsilon = 0$ the system (2.1) is equivariant to translations in ξ and reflections: $\xi \mapsto -\xi$, properties which may be broken for $\epsilon \neq 0$. The stream function ψ is defined by

$$\partial_\xi \psi = -v_2, \qquad \partial_\eta \psi = v_1, \qquad \psi|_{\eta=0} = 0$$

then $\psi = \eta +$ higher order terms holds. The almost identical transformations

$$\begin{array}{rclrcll} x &=& \xi, & y &=& \psi(\xi,\eta) & \\ z &=& 1+Z, & v_1 &=& 1+V_1, & v_2 = V_2 \end{array} \tag{3.1a}$$

and

$$\begin{array}{rcl} W_1 &=& \frac{1}{2}((1+V_1)^2 + V_2^2 - 1 \\ W_2 &=& V_2(1+V_1)^{-1} \end{array} \tag{3.1b}$$

are invertible whenever Z, V_j resp. W_j are sufficiently small in modulus.

Via an elementary calculation one derives from (2.1) an equivalent system (for small $|W_j|$)

$$\partial_x \underline{w} = A(\lambda, b)\underline{w} + F(\lambda, b, \epsilon, \underline{w}) \ , \tag{3.2}$$

where

$$
\underline{w} = \begin{pmatrix} \beta \\ W_1 \\ W_2 \end{pmatrix} \in C_b^1(\mathbb{R}; X) \cap C_b^0(\mathbb{R}, D(A))
\tag{3.2a}
$$

$$
\begin{aligned}
X &= \mathbb{R} \times (L_2(0,1))^2 \\
D(A) &= \mathbb{R} \times (H^1(0,1))^2 \cap \{W_2(0) = 0,\ W_2(1) = \beta\}
\end{aligned}
$$

and subscript b indicates the boundedness of elements. The linear operator A represents the F-derivature at $\underline{w} = 0$ of $A + F$ as a mapping from $\mathbb{R} \times (H^1)^2$ into $\mathbb{R} \times (L_2)^2$, observe that β is a scalar, and is given by

$$
A(\lambda, b)\underline{w} = \begin{pmatrix} \frac{1}{b}(W_1(1) - [W_1]) \\ -\partial_y W_2 \\ \partial_y W_1 \end{pmatrix}, \quad [W_1] = \int_0^1 W_1 dy . \tag{3.3}
$$

Observe that $A \in \mathcal{L}\ (D(A); X)$ has a compact resolvent in X. The symmetries given above can be described as follows: set

$$
R = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad \tau x = x + \tau \tag{3.4}
$$

then

$$
AR = -RA, \qquad F_0 R = -RF_0, \qquad (A + F_0)\tau = \tau(A + F_0)\,, \tag{3.4a}
$$

where $F_0 = F|_{\epsilon=0}$. In short, we say that system (3.2) is reversible.

4 The unforced motion

Observe that the spectrum ΣA consists of eigenvalues σ of finite multiplicities, which appear in pairs $(\sigma, -\sigma)$. Moreover, due to the ellipticity of A, only finitely many can live on the imaginary axis $i\mathbb{R}$. This implies that $\Sigma A = \Sigma_0 \cup \Sigma_1$ where Σ_0 is finite and Σ_1 is confined to a double cone $|\mathrm{Re}\ \sigma| < q|\mathrm{Im}\ \sigma|$. If Σ_0 is empty, it is easily seen that $\underline{w} = 0$ is an isolated solution. Therefore, the eigenvalues on $i\mathbb{R}$ are 'critical', as they may cause bifurcation. The eigenvalues are given by the equation (c.f. [7])

$$
(\lambda - b\sigma^2)\sin\sigma = \sigma\cos\sigma, \qquad \sigma \in \mathbb{C}\ .
$$

The critical part Σ_0, i.e. all $\sigma \in \Sigma A$ with $|\mathrm{Re}\,\sigma|$ small, is shown in Figure 2.

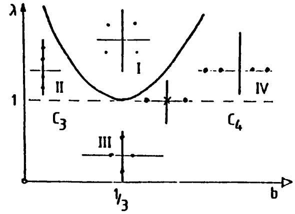

Figure 2

Observe that ΣA is infinite dimensional in the positive and negative halfplane. The case we consider is $\lambda \approx 1, b > \frac{1}{3}$ fixed, then

$$A_0 = A_0(1,b) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \tag{4.1}$$

where A_0 denotes the central part of A.

Let us decompose X according to the splitting $\Sigma = \Sigma_0 \cup \Sigma_1$, of ΣA

$$X = X_0 \oplus X_1, \qquad A(X_j \cap D(A)) \subset X_j$$

for $j = 1, 2$. Set $\underline{w} = w_0 + w_1$, then (3.2) reads

$$\begin{aligned} \partial_x w_0 &= A_0 w_0 + \tilde{F}_0(\underline{\lambda}, w_0 + w_1) \\ \partial_x w_1 &= A_1 w_1 + \tilde{F}_1(\underline{\lambda}, w_0 + w_1), \end{aligned} \tag{4.2}$$

where $\underline{\lambda} := (\lambda, b, \epsilon), \tilde{F} := F + (A(\lambda, b) - A(1,b))\underline{w}$ and $A(1,b) = A_0 \oplus A_1$. For (4.2) the initial value problem is not solvable, in general. Nevertheless , working in spaces of bounded functions as given in (3.2a), one may still use concepts of the theory of dynamical systems. In particular, a local center manifold exists. It contains all sufficiently small bounded solutions. For a precise version and the proofs see [10] and [12]. The only fact we shall exploit here, is the reducibility of (4.2) to a system in w_0. Under the assumptions stated, w_1 is a pointwise smooth function of w_0

$$w_1 = h(\underline{\lambda}, x, w_0,) = h^0(\lambda, b, w_0) + h^1(\lambda, b, \epsilon, x, w_0), \tag{4.3}$$

where $h^0 = \mathcal{O}((\lambda - 1)w_0 + |w_0|^2)$, $h^1 = \mathcal{O}(\epsilon)$. h^0 describes the reduction in the unforced case, i.e. $\epsilon = 0$. It inherits the reversibility from the original equations and therefore satisfies

$$h^0(\lambda, b, R_0 w_0) = R_1 h^0(\lambda, b, w_0), \qquad R = R_0 \oplus R_1 .$$

Therefore, we can reduce (4.2) by (4.3) to the system

$$\partial_x w_0 = A_0 w_0 + \tilde{F}_0(\lambda, w_0 + h(\lambda, \cdot, w_0)) . \tag{4.4}$$

The case of interest is $\mu := \lambda - 1 > 0$. As Figure 2 shows, the eigenvalues of A_0 are real then. Therefore, (4.4) is a first order system in $\mathbb{R}^2$. For $\epsilon = 0$ we have

$$\partial_x w_0 = A_0 w_0 + \tilde{F}_0^0(\lambda, b, w_0 + h^0(\lambda, b, w_0)) \tag{4.5}$$

and this system is reversible with respect to

$$R_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} .$$

First we solve (4.5), the unforced case. We suppress b and write $\mu = \lambda - 1$. According to [4], (4.5) can be transformed into (c.f.[7])

$$\partial_x s = A_0 s + N_k(\mu, s) + R_k , \tag{4.6}$$

where

$$A_0^* N_k(\mu, s) = D_s N_k(\mu, s) A_0^* s \tag{4.7}$$

for all $s \in \mathbb{R}^2$. Here A_0^* denotes the adjoint of A_0, k any integer $k \geq 2, N_k$ a polynominal in s of order k, and $N_k = \mathcal{O}(|s|^2 + \mu|s|), R_k = \mathcal{O}(|s|^{k+1})$. Write $N_k = (N_1, N_2), s = (s_1, s_2) \in \mathbb{R}^2$, where $w_0 = s_1\varphi_1 + s_2\varphi_2$, $A_0\varphi_0 = 0$, $A_0\varphi_1 = \varphi_0$, then (4.1) and (4.7) yield

$$\begin{aligned} \partial_{s_1} N_1 + s_1 \partial_{s_2} N_1 &= 0 \\ \partial_{s_1} N_2 + s_1 \partial_{s_2} N_2 &= N_1 . \end{aligned} \tag{4.8}$$

Observe that the first equation implies

$$N_1 = \varphi(s_1, \mu) ,$$

where φ_1 is a polynominal in s_1 of degree k.

In view of the relation $N_1 R_0 = -R_0 N_1$ we obtain $N_1 = 0$, since $\varphi(s_1, \mu) = -\varphi(s_1, \mu) = 0$. Similarly, the second equation of (4.8) shows now that N_2 is independent of s_2. Therefore, the normal form of (4.4)(set $R_k = 0$ in (4.6)) reads

$$\begin{aligned} \partial_x s_1 &= s_2 \\ \partial_x s_2 &= \varphi(s_1, \mu) = \tfrac{3}{3b-1}(\mu s_1 - \tfrac{3}{2}s_1^2 + 0(\mu^2 s_1 + s_1^3)) , \end{aligned} \tag{4.9}$$

where the explicit constants follow by setting $h^0 = 0$ in (4.5)(c.f.[]). Let ϕ be a primitive of φ, then $s_2^2 - 2\phi(s_1, \mu)$ is an integral of (4.9) of each μ. If μ is positive, then $s_2^2 - 2\phi(s_1, \mu) = 0$ describes a homoclinic orbit connecting the saddle point

$s_1 = s_2 = 0$ with itself. It is unique up to shifts in x; moreover $s_1(x) > 0$ for all $x \in \mathbb{R}$. Since $s_1(x) = -Z(x)$ holds in first approximation, this homoclinic orbit leads to a solitary wave of depression.

Observe that the scaling $s_1(x) = \mu S_1(\mu^{1/2}x), s_2(x) = \mu^{3/2}S_2(\mu^{1/2}x)$ leads to the limiting equation

$$S_1' = S_2 \quad S_2' = \frac{3}{3b-1}(S_1 - \frac{3}{2}S_1^2 + 0(\mu)). \tag{4.9'}$$

This gives S_j in lowest order for $\mu = 0$. Moreover it shows the robustness of (4.9′) to reversible pertubations. First, observe that R_1 in (4.6) must be odd in s_2, R_2 even. Therefore $\sigma_2 = s_2 + R_2(s_1, s_2), \sigma_1 = s_1$ is invertible near $s_1 = s_2 = 0$ and yields (4.6) with $R_1 = 0$ and s_j replaced by σ_j. After scaling as above, we obtain for (4.6) the system (4.9).

Let q denote the (unique) even homoclinic solution s_1 of (4.9′) and Q its scaled version. Write $S_j = Q * Z_j$ then (4.9′) yields

$$Z_1'' - a_1Z_1 = 2a_2QZ_1 + r(Z_1, Z_1', \mu), \tag{4.10}$$

where $r = 0(Z_1^2 + Z_1'^2 + \mu), a_1 = 1/(b - \frac{1}{3}), a_2 = -\frac{3}{2}a_1$. Observe that (4.10) leaves even functions Z_1 even. Moreover $Z_1'' - a_1Z_1$ has an inverse in $C_b^0(\mathbb{R})$ which denote by $K(t)$. Therefore (4.10) can be written

$$Z_1 = 2a_2K * (QZ_1) + K * r(Z_1, Z_1', \mu), \tag{4.11}$$

where $*$ denotes the convolution.$K * (QZ_1)$ defines a linear compact operator in $C_b^0(\mathbb{R})$ in view of the exponential decay of Q (like $\exp-(a_1)^{1/2}|x|)$). Since $2a_2K * (QZ_1)$ has not the eigenvalue 1 (the only function qualifiying is Z_1', which is odd), the implicit function theorem implies the local solvability of (4.9') near $Z_1 = 0; \mu = 0$.

Proposition 4.1 (c.f.[1])
For given $b > \frac{1}{3}$, there exists a right neighbourhood of $\lambda = 1$ such that (4.4) has a unique even homoclinic solution, decaying to zero at infinity. Therefore, the original problem (2.1),(2.2) has a solitary wave solution for these parameter values. It is unique up to translations in ξ and it is a wave of depression.

5 Local forcing

It is remarkable that the reduction works also when the vectorfield in (3.2) is nonautonomous (depends explicitly on x). The only assumption needed is, that the linearization A at $\underline{w} = 0$ is autonomous. For the proof see Mielke [10]. In

(4.3) we have already given the general form of the reduction function h, and h^1 represents the effect of nonautonomy, in our case the external pressure ϵp_0. This term will in general break the reversibility. We assume here that p_0 has compact support, thus, the vectorfield is asymptotically autonomous. This again implies that $h^1 = 0\ (exp(-\Delta|x|))$ at $|x| = \infty$ for any $\Delta > 0$ (c.f.[11]).

The explicit calculations, being relatively straight forward, are suppressed. For details see [7],p. 153f. We obtain ($\partial s'_x = s'_1$)

$$\begin{aligned} s'_1 &= s_2 \\ s'_2 &= \tfrac{3}{3b-1}(\mu s_1 - \tfrac{3}{2}s_1^2 + R_1(s_1, s_2, \mu)) \\ &\quad - \tfrac{3\epsilon}{3b-1}(p_0(x) + R_{21}(x, s_1, s_2, \epsilon, \mu)) \ , \end{aligned} \tag{5.1}$$

where the remainder terms R_2, R_{21} are of the order $\mathcal{O}(\mu s_1^2 + s_2^2 + s_1^3)$ resp. $\mathcal{O}(\epsilon)$ uniformly in $x \in I\!R$ and in bounded sets of μ, s_1, s_2. Moreover, we have the estimate

$$|h^1(\mu, \epsilon, x, s)| \ \le |\epsilon| c_0(\Delta, \gamma) exp(-\Delta|x|)$$

for any Δ and max $(|\mu|, |\epsilon|) < \gamma$. Therefore, since h^1 determines R_{21}, it follows

$$|R_{21}(x, s, \epsilon, \mu)| \le |\epsilon| c_1(\Delta, \gamma)(|\mu| + |s_1| + |s_2|) exp(-\Delta|x|) \, . \tag{5.2}$$

We assume as before that μ is positive. Let us scale as in the preceding section, and set

$$\eta = \epsilon \mu^{-3/2}(b - \frac{1}{3})^{1/2} , \quad \xi = \mu^{1/2} x \ .$$

Then (5.1) yields

$$S_1'' - S_1 + \frac{3}{2}S_1^2 - \eta \frac{P_0}{< P_0 >} = R_0 + R_1 \ , \tag{5.3}$$

where $R_0 = 0(\mu)$ and $R_1 = 0(\mu^{1/2}\eta\, exp(-\Delta|\xi|\mu^{-1/2})$ and $P_0(\xi) = p_0(x), < P_0 >= \int_{-\infty}^{\infty} P_0(\xi)\delta\xi$, which we assume to be nonzero.

Since $P_0(\xi)/ < P_0 >$ converges pointwise to 0 for $\xi \neq 0$ and its mean value is one, it is natural to consider as limiting equation for $\mu \to 0$.

$$S_1'' - S_1 + \frac{3}{2}S_1^2 - \eta\delta_0 = 0 \ , \tag{5.4}$$

where δ_0 is the Dirac functional concentrated at 0. As will be shown, smooth solutions of (5.3) can be constructed as pertubations of (5.4). For a discussion of the complete solution picture see [11].

Solutions of (5.4) are obtained, by solving (5.4) separately for $\xi > 0$ and $\xi < 0$, yielding S_1^+ and S_1^-. We add the additional constraint

$$S_1^-(-\infty) = 0, \qquad S_1^+(0) - S_1^-(0) = \eta \ .$$

Thus we can construct all bounded solutions of (5.4) by intersecting the (identical) phase portraits $P^{\pm}$, where P^{+} is shifted, relative to P^{-}, by the amount of η parallel to the S_1'-axis. We obtain piecewise continously differentiable functions, which may have jumps in the derivative at $\xi = 0$ only. For an example see Figure 3.

The robustness of these functions to perturbations is proved in the Banachspace.

$$
\begin{aligned}
Y &= \{a \in C_b^0(\mathbb{R}) / (e^{\xi/2} a)_- \in C_b^2(-\infty, 0], a_+ \in C_b^2[0, \infty)\} \\
\|a\|_Y &= |a|_0 + \sum_{j=1}^{2} \{ |a^{(j)}|_{0,+} + |a^{(j)} e^{\xi/2}|_{0,-} \} \ ,
\end{aligned}
$$

where

$$
\begin{aligned}
|a|_0 &= \sup_{\mathbb{R}} |a(\xi)|, \qquad |a|_{0,\pm} = \sup_{\overline{\mathbb{R}}_\pm} |a(\xi)| \\
a_\pm &= a|_{\overline{\mathbb{R}}_\pm} \ .
\end{aligned}
$$

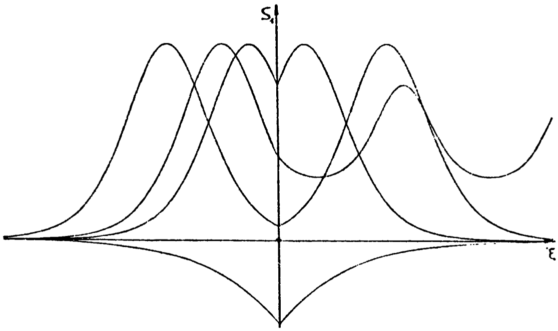

Figure 3

Assume that $S^0 \in Y$ is a solution of (5.4) for some $\eta \neq 0$. Since it vanishes at $-\infty$, it coincides in $(-\infty, 0)$ either with part of the homoclinic solution constructed in section 4 or with the other part of the unstable manifold of 0. In particular $S_-^0 = S_{|\mathbb{R}^-}$ is of one sign.

Lemma 5.1

Let $S^0 \in Y$ be as described above and assume $S^0(+\infty) = 0$. Then there exists a unique solution $S_1 \in Y$ of (5.3) satisfying

$$
\|S_1 - S^0\|_Y = \mathcal{O}(\mu^{1/2}) \ .
$$

<u>Proof:</u> Observe that $S_1 \in Y$ implies $S_2 = \partial_x S_1 \in C_b^0(\overline{\mathbb{R}}_\pm)$. It may have a discontinuity at 0. Therefore, R_j in (5.3) have the same property. We define

$K(x) = -exp(-|x|)/2$, then $Kf \in Y$ if $f \in C_b^0(\bar{R}_\pm)$, where Kf denotes the convolution. Moreover we have $K\delta_0 = -exp(-|\xi|)/2$. Therefore, S_1 solves

$$S_1 = -\frac{\eta}{2}e^{-|\xi|} + \eta T_0 - \frac{3}{2}KS_1^2 + T(S_1;\mu)\,, \tag{5.5}$$

where

$$\begin{aligned} T_0 &= \frac{1}{< P_0 >}KP_0 + \frac{1}{2}e^{-|\xi|} \\ T_1 &= K(R_0 + R_1)\,. \end{aligned}$$

Since the support of P_0 is contained in an interval of length $O(\mu^{1/2})$, we have

$$\|T_0\|_Y \le C_1\mu^{1/2} \quad \text{and also} \quad \|T_1(S_1;\mu)\|_Y \le C_1\mu$$

uniformly in compact sets of η, μ and S_1. As a whole, T is Lipschitz continous in Y with a Lipschitz constant $O(\mu^{1/2})$. Set $S_1 = S^0 + Z$, then we obtain

$$Z + 3KS^0Z = -\frac{3}{2}KZ^2 + \eta T_0 + T_1(S^0 + Z;\mu)\,. \tag{5.6}$$

It suffices to show, since T_0 and T_1 vanish for $\mu = 0$, that $id + 3KS^0$ has a bounded inverse in Y. Since $S^0(\pm\infty) = 0$, we have $S^0 = 0(exp(-|\xi|)$ for large ξ, and thus $3KS^0$ is compact in Y. It remains to be shown that -1 is not an eigenvalue of $3KS^0$.

Observe that $S^0(\pm\infty) = 0$ implies that S^0 is an even function. Moreover, $S^{0\prime}$ is the only solution in Y of $Z'' + 3S^0Z = 0$. Multiplication by $S^{0\prime}$ and integration yields

$$(S^{0\prime}Z' - S^{0\prime\prime}Z)(\xi) = 0$$

for all $\xi \neq 0$. Therefore, -1 is no spectral point of $3KS^0$ and the Lemma is proved.

The solution S_1 just constructed solves (5.3) in the classical sence for $\xi \neq 0$. Hence S'' is continous, except at $\xi = 0$. Therefore $S_1' \in C_b^0(\mathbb{R})$ for $\mu > 0$, which implies, via a bootstrapping-argument, that $S_1 \in C_b^k(\mathbb{R})$ for any $k \in \mathbb{N}$ given, whenever $\mu > 0$.

The only case remaining is $S^0(+\infty) \neq 0$. Then $S_+^0 = S_{\mathbb{R}+}^0$ is periodic, as is easily seen from the phase diagram of (5.4). Again we set $S_1 = S^0 + Z$ and argue as for (5.5) and (5.6). It remains to be shown that $Z + 3KS^0Z$ has a continous inverse in Y. To see this we consider for $Z_\pm = Z_{|\mathbb{R}_\pm}$ and invert separately on $\mathbb{R}_-$ and $\mathbb{R}_+$. Define $Z_- = \delta S_-^0 + U$ where $\delta \in \mathbb{R}$ is an additional parameter. We obtain

$$\begin{aligned} U'' - U + 3S_-^0U &= F - \tfrac{3}{2}\delta(S_-^0)^2 \\ U(-\infty) = U(0) &= 0\,, \end{aligned} \tag{5.7}$$

where $F \in C_b^0(-\infty, 0)$ with $F = 0(exp(\xi/2))$ as $\xi \to -\infty$. The only solution of the homogeneous problem ($F = \delta = 0$) is $\partial_x S_-^0$. The range of the left side is closed, since $K_- S_-^0$ is compact in $C_b^0(-\infty, 0)$, where

$$K_-(\xi, t) = -e^t \sin h\xi, \quad \text{for } t < \xi$$

and $K_-(t, \xi) = K_-(\xi, t)$. Therefore, we can apply Fredholm's alternative and conclude existence if

$$\delta = \frac{2}{(S_-^0(0))^3} \int_{-\infty}^0 F(S_-^0)' d\xi$$

and uniqueness if

$$\int_{-\infty}^0 U(S_-^0)' d\xi = 0 \,.$$

To conclude the solvability in $[0, \infty]$, observe that S_+^0 is periodic with period $2\pi/\omega_0$. To normalize the period to 2π we set $\omega\xi = t$ and $S_+^0(\xi) = s_0(t)$, similarly $Z_+(\xi) = z(t)$. Define $z(t) = \delta s_0(t) + s_0'(t) + v(t)$ and obtain

$$\begin{aligned} \omega^2 v'' - v + 3s_0 v &= f - \tfrac{3}{2}\delta s_0^2 \\ v(0) = 0, \quad & v \quad 2\pi - \text{periodic} \,. \end{aligned} \tag{5.8}$$

We use ω as a parameter which is close to ω_0. Observe that s_0' solves uniquely the homogeneous problem (5.8) for $\omega = \omega_0$. Applying again the Fredholm alternative we conclude the unique solviability of (5.8) under the following conditions

$$-(\omega^2 - \omega_0^2) \int_0^{2\pi} s_0''^2 dt = \int_0^2 \pi f s_0' dt, \quad \int_0^{2\pi} v s_0' dt = 0 \,.$$

This determines ω and v. Since $z(0) = Z_+(0) = \delta S_+ 0 = Z_-(0)$ holds, in view of the continuity of S^0, we have shown

Theorem

Given any solution S^0 of (5.4) in Y for $\eta \neq 0$, then there exists a unique solution $S^1 \in Y$ of (5.3) satisfying $\|S_1 - S^0\|_Y = \mathcal{O}(\mu^{1/2})$ as μ tends to $+0$. Moreover $S_1 \in C_b^2(I\!R)$ for $\mu > 0$.

Thus we justified the limit equation (5.4). The further reaching question, whether all solutions of small amplitude can be constructed in this way, has been answered affirmatively by Mielke. We refer the reader to [11].

6 References

[1] Amick, C.J., and Kirchgässner, K. (1989). A theory of solitary water–waves in the presence of surface tension. Arch. Rat. Mech. Anal.105, 1–49.

[2] Amick, C.J. and Toland, J.F. (1981). On solitary water–waves of finite amplitude. Arch. Rat. Mech. Anal. 76, 9–95.

[3] Beckert, H. (1963). Existenzbeweis für permanente Kapillarwellen einer schweren Flüssigkeit. Arch. Rat. Mech. Anal.13, 15–45.

[4] Elphick, C., Tirapegui, E., Brachet, M. E., Coullet, P., Iooss, G.(1987). A simple global characterization for normal forms of singular vector fields. Physica 29D, 95–127.

[5] Friedrichs K. O., and Hyers D. H. (1954). The existence of solitary waves. Comm. Pure Appl. Math. 7, 517–550.

[6] Kirchgässner, K. (1982). Wave solutions of reversible systems and applications. J. Diff. Eqn. 45, 113-127.

[7] Kirchgässner, K. (1988). Nonlinearly resonant surface waves and homoclinic bifurcation. Adv. Appl. Mech. 26, 135–181.

[8] Lavrentiev, M. A. (1943). On the theory of long waves. Amer. Math. Soc. Tranl. No. 102.

[9] Levi–Civita, T. (1926). Détermination rigoureuse des ondes permanentes d'ampleur finie. Math. Amn. 95, 264–314.

[10] Mielke, A. (1986). A reduction principle for nonautonomous systems in infinite–dimensional spaces. J. Diff. Eqn. 65, 68-88.

[11] Mielke, A. (1986). Steady flows of inviscid fluids under localized pertubations. J. Diff. Eqn. 65, 89 116.

[12] Mielke, A. (1988). Reduction of quasilinear elliptic equations in cylindrial domains with applications. Math. Meth. Appl. Sci. 10, 51–66.

[13] Mielke, A. (1988) Saint–Venant's problem and semi-inverse solutions in nonlinear elasticity. Arch. Rat. Mech. Anal. 102, 205–229.

[14] Nekrasov, A. I. (1967). The exact theory of steady waves on the surface of a heavy liquid. Math. Research Center Report No.813, University of Wisconcin, Madison.

[15] Ter–Krikorov, A.M. (1963) Théorie exacte des ondes longues stationaires dans un liquide hétérogène. J. de Mécanique 2, 351–376.

[16] Vanderbauwhede A. and Iooss G., (1990). Center manifold theory in infinite dimensions. To appear in: Dynamics Reported.

[17] Zeidler, E. (1971). Beiträge zur Theorie und Praxis freier Randwertaufgaben. Akademie Verlag.