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# BLOW UP FOR A NONLINEAR DEGENERATE PARABOLIC EQUATION

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It is well known that the existence or nonexistence of global solutions to the problem

$$\begin{aligned} &u_t = \Delta u + u^p && x \in \Omega, t > 0 \\ &u(x,t) = 0 && x \in \partial\Omega, t > 0 \\ &u(x,0) = u_0(x) && x \in \Omega, \end{aligned} \tag{1}$$

with $\Omega \subset R^N$ bounded, $u^p := |u|^{p-1}u$, $p > 1$ depends on the initial datum $u_0$. Namely, there exist choices of initial data for which the corresponding solutions exist globally, and other choices for which the solutions break down by becoming unbounded in finite time. In the latter case, there exists a positive time $t_{max} = t_{max}(u_0) < \infty$ such that the solution $u(t, u_0)$ of (1) exists on $[0, t_{max})$ ,

$$\lim_{t \to t_{max}} ||u(t, u_0)||_{L^\infty(\Omega)} = \infty \tag{2}$$

and we shall say that the solution $u$ *blows up in finite time.* See e.g. [1] and references therein .

We give here a survey of some results from [3 – 9] , mainly as applied to the problem

$$\begin{aligned} &u_t = \Delta u^m + u^p - au && x \in \Omega,\ t > 0 \\ &u(x,t) = 0 && x \in \partial\Omega, t > 0 \\ &u(x,0) = u_0(x), && u_0^m \in L^\infty(\Omega) \cap H_0^1(\Omega), \end{aligned} \tag{3}$$

where $0 < m < \infty$, $a \geq 0$ and

$$p > \max\{1 , m\} . \tag{4}$$

Problems related to (3) have been extensively studied (see e.g. [14] and references there), and it is known that the lack of regularity for $m \neq 1$ gives rise to certain mathematical difficulties. What concerns the global solvability for "small" initial data and $p, m$ satisfying (5) below, see [7].

Here we shall deal with the following three questions:

*(i)* For which $u_0$'s does blow up in finite time occur ?
*(ii)* How does $\int_\Omega |u(x,t; u_0)|^q dx$ behave, as $t \to t_{max}$ , for $q > 0$ ?
*(iii)* Is blow up in infinite time possible ?

Before we state our results, let us introduce the notion of solution to Problem (3) and some notations.

By a solution of Problem (3) on $[0,T]$ we mean a function $u \in L^\infty(Q), Q = \Omega \times (0,T)$ such that $u^m \in L^\infty(0,T;H_0^1(\Omega))$ , $(u^{(m+1)/2})_t \in L^2(Q)$, satisfying

$$\int_\Omega u(x,\tau)\psi(x,\tau)dx - \int_\Omega u_0(x)\psi(x,0)dx =$$
$$\int_0^\tau \int_\Omega (u\psi_t - \nabla u^m \nabla\psi + (u^p - au)\psi)dxdt$$

for all $\psi \in L^1(0,T;H_0^1(\Omega)), \psi_t \in L^1(Q)$ and almost all $\tau \in (0,T)$ .

Let

$$\mathcal{J}(w) := \frac{1}{2}\int_\Omega |\nabla w(x)|^2 dx - \frac{m}{m+p}\int_\Omega |w(x)|^{(m+p)/m} dx +$$
$$+\frac{a}{m+1}\int_\Omega |w(x)|^{(m+1)/m} dx ,$$
$$\mathcal{K}(w) := \frac{d}{d\lambda}\mathcal{J}(\lambda w)|_{\lambda=1}$$

and

$$\mathbf{d} := k \inf_{w\in H_0^1(\Omega)} \left( \frac{\left(\int_\Omega (|\nabla w(x)|^2 + a|w(x)|^{1+1/m})dx\right)^{1/2}}{\left(\int_\Omega |w(x)|^{(m+p)/m}dx\right)^{m/(m+p)}} \right)^{2(p+m)/(p-m)}$$

where

$$p > m , \quad k = \min\left\{\frac{1}{2}, \frac{m}{m+\chi_a}\right\} - \frac{m}{m+p}$$

and $\chi_a$ equals one if $a > 0$ and zero if $a = 0$ .

**Theorem 1**[5]. Assume that

$$\frac{p}{m} < \frac{N+2}{N-2} \quad \text{if } N \geq 3 \tag{5}$$

and let

$$\mathbf{B} := \{w \in H_0^1(\Omega) : \mathcal{J}(w) < \mathbf{d} \text{ and } \mathcal{K}(w) < 0 \}.$$

Then $\mathbf{d} > 0$ , $\mathbf{B}$ is invariant and $t_{\max}(u_0) < \infty$ provided $u_0^m \in \mathbf{B}$ .

This theorem (for $m = 1$ and $a = 0$ it can be found in [12]) generalizes the previous blow up result from [2], which establishes blow up of $u(t,u_0)$ if $\mathcal{J}(u_0^m) \leq 0$, $u_0 \neq 0$, $a = 0$. The main difficulty in the proof of our result is caused by the lack of regularity of the solutions. For $m = p$ , a similar result was established in [7].

**Theorem** 2[8]. If

$$q > (p-m)\max\left\{1, \frac{N}{2}\right\}, \tag{6}$$

then

$$\|u(t,u_0)\|_{L^\infty(\Omega)} \le c\left(1+\|u_0\|_{L^\infty(\Omega)}\right)\left(1+\sup_{0\le s\le t}\int_\Omega |u(x,s;u_0)|^q dx\right)^{\vartheta}$$

for $t \in [0, t_{\max})$ and some positive constants $c, \vartheta$ which depend only on $m, p, q, \Omega$.

As an immediate consequence we obtain, that

$$\limsup_{t\to t_{max}} \int_\Omega |u(x,t;u_0)|^q dx = \infty$$

provided (6) holds and $t_{\max} < \infty$.

The last statement was again previously known for m=1, see [13]. The condition (6) is optimal for $m = 1$, $N > 1$, see [10].

For an analogous recent result concerning a problem which includes

$$\begin{aligned} &u_t = \Delta u && x \in \Omega, t > 0 \\ &\frac{\partial u}{\partial \nu} = u^p - au && x \in \partial\Omega, t > 0 \\ &u(x,0) = u_0(x). && \end{aligned} \tag{7}$$

see [9].

**Theorem** 3[3]. Let (5) hold and let $u(t,u_0)$ be a global solution ($t_{\max}(u_0) = \infty$) of Problem (3). Then

$$\sup_{t\ge 0}\|u^m(t,u_0)\|_{H_0^1(\Omega)} < \infty, \quad \sup_{t\ge 0}\|u(t,u_0)\|_{L^\infty(\Omega)} < \infty.$$

This theorem generalizes in several directions the corresponding result in [11]. The restriction (5) is optimal, see [11]. For an analogous result concerning Problem (7) see [4].

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