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ON BOUNDARY ELEMENT METHODS FOR SOLVING
ELLIPTIC BOUNDARY VALUE PROBLEMS

Wolfgang L. Wendland
Technische Hochschule Darmstadt, Bundesrepublik Deutschland

Introduction: Here we give a short review on the asymptotic error analysis of approximations of boundary integral equations by finite elements. These boundary element methods became recently popular in providing additional numerical methods for boundary value problems. Here we consider only elliptic problems. Their reformulation as boundary integral relations is by no means unique but always requires the fundamental solution restricting their practical capability to mainly constant coefficient problems. On the other hand desired physical quantities often are just those to be computed on the boundary. (A comparison with usual finite elements can be found in [18].) All the results here are obtained in joint work with G. C. Hsiao, E. Stephan, D. Arnold and several other friends. Some of them, however, have also been achieved by Mme Le Roux, J. Nedelec and their colleagues [10,11,12]. The literature on this subject has already grown so rapidly that our references here are by no means appropriate (see [6,17,18]).

Our error analysis rests on a Gårding-type coerciveness inequality providing convergence of optimal order for Galerkin's method with finite elements [5,14], duality arguments prove even super approximation [9]. An appropriate numerical integration in Galerkin's stiffness matrix yields our Galerkin-collocation [5,17]. For equations on curves and odd splines we also present recent results for ordinary collocation [3]. For the special case of singular integral equations and linear splines convergence has also been proved by S. Prössdorf and G. Schmidt [13] who even proved the necessity of strong ellipticity for convergence in this case.

Many of our results have been extended to problems involving singularities as mixed boundary conditions and corners. For a review and references see [18].

1. Strongly elliptic integral equations

Here we consider linear boundary integral equations of the form

$$(1.1)\qquad Au = f \text{ on } \Gamma \text{ or } \{Au-\omega = f \text{ on } \Gamma \text{ and } \Lambda u = B\}\ ,$$

where $\Gamma\subset\mathbb{R}^n$ is a given sufficiently smooth (n-1)-dimensional compact manifold, $u,f \in (H^{s\pm\alpha}(\Gamma))^p$; $\omega,B \in \mathbb{R}^p$ (resp. $\mathbb{C}^p$);
A,Λ : $(H^{s+\alpha})^p \to (H^{s-\alpha})^p$, $\mathbb{R}^p$ (or $\mathbb{C}^p$) respectively, continuously.
$H^\sigma(\Gamma)$ denotes the Sobolev-Slobodetzkii space of order $\sigma\in\mathbb{R}$ and $\|\cdot\|_\sigma$ the corresponding norm; f and B are given; u and ω are unknowns. $\alpha\in\mathbb{R}$ is given and fixed. A is a given matrix of pseudo-differential operators on Γ [15, Chap. 1.5]. $a_0(x,\xi)$ denotes the principal symbol of A subject to a fixed finite covering of Γ by local charts. We shall mainly deal with $n = 2$, i.e. Γ a closed smooth curve and s the arc length. The entries of $a_0(x,\xi)$ are homogeneous in ξ for $|\xi| \geq 1$ of degree 2α . (For more general systems see [14,18]) Throughout the paper A is supposed to be <u>strongly</u> <u>elliptic</u>, i.e. there exists a $C^\infty(\Gamma)$ complex p×p matrix $\Theta(x)$ and $\gamma>0$ such that

$$(1.2) \qquad \mathrm{Re}\, \zeta^T\Theta(x)a_0(x,\xi)\bar{\zeta} \geq \gamma|\zeta|^2 \quad \text{for all } x\in\Gamma,\ |\xi|=1, \xi\in\mathbb{R},\ \zeta\in\mathbb{C}^p .$$

Strong ellipticity of (1.1) implies coercivity of ΘA [9. loc. cit. [14]], i.e. there exists a compact bilinear form $k[u,v]$ on $H^\alpha\times H^\alpha$ and $\gamma_0>0$ such that

$$(1.3) \qquad \mathrm{Re}(\Theta Av,v)_{L_2(\Gamma)} \geq \gamma_0\|v\|_\alpha^2 - |k[u,v]| \quad \text{for all } v\in H^\alpha(\Gamma) .$$

We further assume that (1.1) is <u>uniquely</u> <u>solvable</u>. The above class of equations is very rich [14,18] containing much more than the following two examples.

<u>Example 1 [6]:</u> First approximation of an exterior viscous two-dimensional flow around Γ :

$$(1.4) \qquad \begin{aligned} Au-\omega &= -\int_\Gamma \log|x-y|u(y)ds_y + \int_\Gamma L(x,y)u(y)ds_y - \omega = f(x) = 0 , \\ \Lambda u &= \int_\Gamma u\,ds = (0,1) , \end{aligned}$$

$L_{\ell,k} = |x-y|^{-2}(x_\ell-y_\ell)(x_k-y_k) + (e-1-\log 4)\delta_{\ell k}$; $\ell,k = 1,2$; $p = 2$; $x,y \in \mathbb{R}^2$, e Euler's constant [7,6]; $a_0(x,\xi) = \pi|\xi|^{-1}\delta_{k\ell}, \alpha = -\frac{1}{2}$.
Further applications: Plane elasticity [6,11] , plate bending [6], conformal mapping with $L\equiv 0$ [6,17 loc. cit. [19,71,72,75]].

<u>Example 2 [13]:</u>

$$(1.5) \qquad \begin{aligned} Au = a(x)u(x) &+ \frac{1}{\pi i}\int_\Gamma \frac{b(x,y)u(y)}{(y_1-x_1)+i(y_2-x_2)}(dy_1+idy_2) \\ &+ \int_\Gamma L(x,y)u(y)ds_y = f , \end{aligned}$$

a,b,L are given smooth complex p×p matrices
$a_0(x,\xi) = a(x) + b(x,x)\cdot\mathrm{sign}\,\xi$, $\alpha = 0$.

In [13] it is shown that strong ellipticity of (1.5) is equivalent to

(1.6) $\det(a(x) + \lambda b(x,x)) \neq 0$ for all $x \in \Gamma$, $-1 \leq \lambda \leq 1$.

2. Galerkin's method with finite elements

For simplicity, let $n=2$, i.e. Γ a curve given by a 1-periodic representation $x = \underset{\sim}{x}(t)$. Let $\tilde{H} \subset C^{m-1}$ be the family of spaces of splines of degree m subordinate to partitions $0<t_1<\ldots<t_N = 1$, divided by $|\underset{\sim}{x}'(t)|$, $h := \max\limits_{j=1,\ldots,N} (t_j - t_{j-1})$. Then we have the approximation property [4, Theorem 4.12] for $-\infty<t\leq s\leq m+1$, $t<m+\frac{1}{2}$:

(2.1) $\inf\limits_{\chi \in \tilde{H}} \| u-\chi \|_t \leq ch^{s-t} \| u \|_s$. (c denotes a generic constant.)

The Galerkin approximation of (1.1) is to find $u \in \tilde{H}, \omega_h$ such that

(2.2) $\forall \chi \in \tilde{H} : (\chi, \theta A u_h)_{L_2} = (\chi, \theta f)_{L_2(\Gamma)}$ or

$$\{(\chi, \theta A u_h - \theta\omega)_{L_2} = (\chi, \theta f)_{L_2}, \; \Lambda u_h = B\} .$$

For (2.2), Céa's lemma, approximation property (2.1) and the Aubin-Nitsche lemma yield:

<u>Theorem 2.1 [9,3]:</u> <u>The Galerkin equations</u> (2.2) <u>are uniquely solvable for all</u> $0<h\leq h_0$ <u>with some</u> $h_0>0$. <u>The Galerkin solutions</u> u_h <u>satisfy for</u> $\alpha<m+\frac{1}{2}$, $2\alpha-m-1\leq t\leq \alpha \leq s\leq m+1$

(2.3) $\| u-u_h \|_t \leq ch^{s-t} \| u \|_s$, $|\omega-\omega_h| \leq ch^{s+m-2\alpha} \| u \|_s$.

For special cases see also [10,12,8].

<u>Remark:</u> If the partitions are quasiuniform and $\tilde{H}$ provide the inverse assumption [4] then (2.3) holds also for $\alpha<t<m+\frac{1}{2}$, $t\leq s\leq m+1$.

3. Galerkin-collocation

In order to reduce the computing time for the numerical integrations in the influence matrix of (2.1), for (1.4) in [5] and more general equations in [17] we have utilized the additional assumptions that the principal part of θA is a convolution, i.e.

(3.1)
$$\theta A u = Du + Ku = \int_{\mathbb{R}} p(t-\tau) u(t) |\underset{\sim}{x}'(t)| dt,$$
$$p(\eta) = p_1(\eta) + \log|\eta| p_2(\eta),$$

$p_1(\eta)$ and $p_2(\eta)$ are homogeneous of degree $-1-2\alpha$, and that the partitions are uniform i.e.

(3.2) $\mu_j(t) = \mu(\frac{t}{h} - j + 1) |\underset{\sim}{x}'|^{-1}$, $j=1,\ldots,N = \frac{1}{h}$

form a basis to $\tilde{H}$. Then the Galerkin weights $(D\mu_j, \mu_k)_{L_2}$ form a Toeplitz matrix whose entries can be easily computed from two vectors with elements being independent of Γ and h which can be computed in advance and tabelized once for all for further use. For the smooth remaining kernels and the right hand sides of (2.1) we develop specific integration formulas integrating polynomials of degree $\le 2M+1$ exactly and using only the nodal points $t_j+h\cdot n$, $n=\frac{m+1}{2}$. The method is called Galerkin-collocation. Here the Strang lemma yields:

Theorem 3.1[17,6]: For $2\alpha\le m+1$, $2\alpha<2m+1$, $0\le s\le m+1$, $\alpha\le s$, $-1-\alpha' := -1-\min\{0,\alpha\}<M$ and $\frac{1}{2}<\sigma\le 2M+2$ the Galerkin-collocation solutions $\underset{\sim}{u}_h$, $\underset{\sim}{\omega}_h$ provide

$$\| u-\underset{\sim}{u}_h\|_{L_2} \le c_1 h^s \| u\|_s + c_2 h^{2M+2+2\alpha'}\| u\|_0 + c_3 h^{\sigma+2\alpha'}\| f\|_\sigma ,$$

$$|\omega-\underset{\sim}{\omega}_h| \le c_1' h^{s+m-2\alpha}\| u\|_s + c_2' h^{2M+2}\| u\|_0 + c_3' h^{\sigma}\| f\|_\sigma .$$

For (1.4) numerical experiments in [6] with 27 different cases revealed very accurate results and the following orders:

	m=0, M=0	m=1, M=1	m=2, M=1
exact order	2	4	4
experimental order	2.045	4.04	4.05

4. Collocation with splines

For the collocation we restrict the splines to only odd degrees $m = 2n-1 > 2\alpha$, $n\in\mathbb{N}$. Then collocation of (1.1) at the grid points t_j read for $u_c\in\tilde{H}$ as

$$\begin{aligned} Au_c(t_j) - \omega_c &= f(t_j)\ , \quad j = 1,\dots,N \quad\text{and} \\ \Lambda u_c &= B\ . \end{aligned} \tag{4.1}$$

Theorem 4.1 [3] : The collocation equations (4.1) are uniquely solvable for all $0<h\le h_0$ with some $h_0>0$. For $2\alpha\le t\le n+\alpha\le s\le m+1$, $t < m+\frac{1}{2}$ we have

$$\| u-u_c\|_t \le ch^{s-t}\| u\|_s \quad\text{and}\quad |\omega-\omega_c| \le ch^{s-2\alpha-1/2}\| u\|_s . \tag{4.2}$$

Remarks: For quasiuniform partitions providing the inverse assumption [4] for $\tilde{H}$, (4.2) also holds for $n+\alpha < t$.

In case of strongly elliptic singular integral equations (1.5) with $\alpha=0$, $t=0$ and $m=1$ our result can be obtained from [13]. For example (1.4) and the special choice $t=0$, $m=1$, (4.2) might be obtained from [16]. Comparison of (2.3) with (4.2) for (4.1) shows that (4.2) is valid only for a much smaller range of indices t and s . For smooth data we have in particular $|\omega-\omega_h| = O(h^{sm+1-2\alpha})$ but only $|\omega-\omega_c| = O(h^{m-1/2-2\alpha})$. For further details see [3].

If $b\equiv 0$ then (1.5) is a system of Fredholm integral equations of the second kind. In this case collocation with splines is well established [17 loc. cit. [3,58]].

5. Collocation and numerical integration

Again, the Strang lemma can be used for comparing (4.1) with corresponding equations involving numerical integration If A has convolutional principal part, this leads to estimates similar to Theorem 3.1. But this is yet to be done. For Fredholm integral equations of the second kind, i.e. $b\equiv 0$ in (1.5) these results are well known from [9 loc. cit. [5,6,10]]. For the special system (1.4) however there are only preliminary results available [1,2 p. 273 ff.].

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