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BOUNDS FOR EIGENVALUES OF DIFFERENTIAL EQUATIONS

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A method due to B. KNAUER for obtaining lower bounds to eigenvalues of symmetric and positive differential operators L with discrete spectrum is extended to a class of eigenvalue problems of the more general type $Lu = \lambda Mu$ involving symmetric and positive ordinary or partial differential operators. Examples are given and numerical results are presented.

1. INTRODUCTION

In a real Hilbert space $E, (\ ,\ ), \|\ \|$ we consider eigenvalue problems $Lu = \lambda Mu$, $u \in D(L) \subset D(M) \subset E$ under the following assumptions:

i) L and M are symmetric and positive respectively in D(L) and D(M).

ii) There exists a sequence of eigenvalues $0 \le \lambda_1 \le \lambda_2 \le \dots$, $\lambda_n \to \infty$ as $n \to \infty$; the corresponding (orthogonalized) eigenfunctions $\varphi_1, \varphi_2, \dots$ form a total system in E.

As is well known, upper bounds to eigenvalues are obtained by the method of Rayleigh-Ritz: For any n-dimensional subspace $U_n \subset D(L)$ the corresponding Ritz eigenvalues $\Lambda_1 \le \dots \le \Lambda_n$ satisfy the inequalities $\lambda_k \le \Lambda_k$ for $k = 1, \dots, n$.

Several authors have shown how lower bounds to eigenvalues can be obtained in terms of the Ritz eigenvalues Λ_k, provided that some additional information is available. (See, for instance, [4] - [6], [10].) For problems $Lu = \lambda u$, B. KNAUER [4] proposed a method which is particularly simple from the numerical point of view. It is numerically stable as was pointed out by LÖBEL [7]. Numerical results are found in [4] and [8].

In this lecture, we consider two different extensions of KNAUER's method which apply to problems $Lu = \lambda Mu$. Numerical examples show that these procedures can yield - at least in particular cases - fairly good bounds in comparision with more sophisticated but also more laboreous methods as, for instance, A. WEINSTEIN's method of intermediate problems or G. FICHERA's method of orthogonal invariants (see [2], [3], [11] and the references given there).

2. EIGENVALUE PROBLEMS $Lu = \lambda Mu$

Let $U_n \subset D(L)$ be given. Firstly, one has to solve the problem

$$u \in U_n : (Lu - \Lambda Mu, v) = 0 \qquad \forall v \in U_n .$$

We need the Ritz eigenvalues $\Lambda_1 \leq \ldots \leq \Lambda_n$ as well as a system of Ritz eigenfunctions $u_1, \ldots, u_n$, orthonormalized with respect to the inner product $(u,v)_M = (Mu, v)$. In addition, we need reasonably good positive bounds α_{n+1} and β_{n+1} satisfying

$$\alpha_{n+1} \leq \inf_w (Lw,w) / (Mw, w) \;, \quad \beta_{n+1} \leq \inf_w (Mw,w) / (w, w)$$

where $w \in D(L) \ominus U_n$ (orthogonality with respect to $(\, , \,)_M$).

Now, for $k = 1, \ldots, n$, we consider matrices of the following type:

$$\left(\begin{array}{cccc|c} \Lambda_k & & & 0 & r_k \\ & \Lambda_{k+1} & & & r_{k+1} \\ & & \ddots & & \vdots \\ 0 & & & \Lambda_n & r_n \\ \hline r_k & r_{k+1} & \cdots & r_n & \alpha_{n+1} \end{array} \right) \qquad (1)$$

THEOREM 1. Let ν_k denote the lowest eigenvalue of (1) where

$$r_k = \|Lu_k - \Lambda_k^* M u_k\| / \sqrt{\beta_{n+1}} \;, \quad \Lambda_k^* = (Lu_k \, , Mu_k) / (Mu_k, Mu_k) \; .$$

Then the following inequalities hold:

$$\nu_k \leq \lambda_k \leq \Lambda_k \quad \text{for } k = 1, \ldots, n \; .$$

Proof: See [8].

Example 1. (Buckling of a uniform beam , $a \geq 0$)

$$u^{iv} - a[(1-x)u']' = -\lambda u'' \;, \quad u(0) = u''(0) = u'(1) = u'''(1) = 0.$$

Using the subspace U_n spaned by first $n = 10$ eigenfunctions of $u^{iv} = -\lambda u''$ (under the same boundary conditions) we obtained with $\alpha_{11} = \beta_{11} = (21\pi / 2)^2$ the following bounds (for details see [9]):

a = 50			a = 100		
k	ν_k	Λ_k	k	ν_k	Λ_k
1	31.13060	31.13062	1	50.30137	50.30141
2	52.00702	52.00714	2	84.89723	84.89750
3	88.10059	88.10073	3	117.2274	117.2281
4	146.5671	146.5674	4	173.2422	173.2431
5	225.2525	225.2528	5	251.2003	251.2012
6	323.8163	323.8168	6	349.4366	349.3860
7	442.1765	442.1772	7	467.6156	467.6185
8	580.3039	580.3075	8	605.6299	605.6455
9	738.1839	738.1904	9	763.4267	763.4633
10	915.7321	916.3402	10	940.2857	943.1005

3. PROBLEMS IN VARIATIONAL FORM

Let the eigenvalue problem now be given in variational form

$$u \in D_a : \quad a(u, \varphi) = \lambda\, b(u, \varphi) \qquad \forall\, \varphi \in D_a \tag{2}$$

where $a(\,,\,)$ and $b(\,,\,)$ are symmetric and positive bilinear forms defined in D_a and D_b respectively ($D_a \subset D_b \subset E$) . Let $U_n \subset D_a$ be given. Then the Ritz eigenvalue problem can be written in the form

$$u \in U_n : \quad a(u, \varphi) = \lambda\, b(u, \varphi) \qquad \forall\, \varphi \in U_n \quad .$$

Let $\Lambda_1 \le \dots \le \Lambda_n$ denote the Ritz eigenvalues and $u_1, \dots, u_n$ corresponding Ritz eigenfunctions, orthonormalized with respect to the inner product $b(\,,\,)$ and the related norm $\|\ \|_b$. Suppose we know a reasonable good positive bound α_{n+1} ,

$$\alpha_{n+1} \le a(w, w) / b(w, w) \qquad \forall\, w \in D_a \ominus U_n$$

(orthogonality with respect to $b(\,,\,)$). Suppose further that for each u_k we know a function $v_k \in D_b$ satisfying

$$a(u_k, \varphi) = b(v_k, \varphi) \qquad \forall\, \varphi \in D_a \quad . \tag{3}$$

<u>THEOREM 2</u>. Let ν_k denote the lowest eigenvalue of (1) where

$$r_k = \| v_k - \Lambda_k u_k \|_b \quad .$$

Then the following inequalities hold:

$$\nu_k \le \lambda_k \le \Lambda_k \qquad \text{for } k = 1, \dots, n \quad .$$

<u>Proof</u>: See [8] .

<u>Example 2</u>. (Buckling of a uniform beam. See also [1, p.404])

$$u^{iv} - (x u')' = -\lambda u''$$

$$u(0) = u'(0) = u''(1) = 0 \quad , \quad -u'''(1) + u'(1) = \lambda u'(1) \quad . \tag{4}$$

Here, the eigenparameter λ appears in one of the boundary conditions. The variational formulation (2) is given by

$$u \in D_a : \quad \int_0^1 (u'' \varphi'' + x u' \varphi')\, dx = \lambda \int_0^1 u' \varphi'\, dx \qquad \forall\, \varphi \in D_a$$

where we may choose: $D_a = \{ C^2[0, 1] \mid u(0) = u'(0) = 0 \}$ and $D_b = \{ C^1[0, 1] \mid u(0) = 0 \}$. Now, when u, v is a pair of functions satisfying (3) it follows immediately that $u^{iv} - (x u')' = - v''$ in $[0, 1]$ and $u''(1) = 0$, $-u'''(1) + u'(1) = v'(1)$, provided u is in $C^4[0, 1]$. Suppose that the Ritz eigenfunctions are in $C^4[0, 1]$ and satisfy $u_k''(1) = 0$. Now we define v_k by

$$v_k' = -u_k''' + x u_k' \quad , \quad v_k(0) = 0$$

so that the pairs u_k, v_k satisfy (3) . Using the subspace U_n , spaned by the first two eigenfunctions of $u^{iv} = -\lambda u''$ together with the (modified) conditions (4), we obtained with $\alpha_{n+1} = (5\pi/2)^2$ the bounds

$$3.16792 \leq \lambda_1 \leq 3.167945 \quad , \quad 22.7301 \leq \lambda_2 \leq 22.7313 \quad .$$

These bounds are in agreement with those obtained by BAZLEY et al. [1, p. 405] using the first $n = 12$ eigenfunctions for the Ritz bounds and constructing intermediate problems for getting lower bounds:

$$3.16793 \leq \lambda_1 \leq 3.167932 \quad , \quad 22.73018 \leq \lambda_2 \leq 22.73018 \quad .$$

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