INTEGRAL AND ASYMPTOTIC EQUIVALENCE OF TWO SYSTEMS OF DIFFERENTIAL EQUATIONS

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The problem of approximation of solutions of a given differential equation with aid of solutions of another differential equation is not a new one;it is very important in the theory of differential equations as well as in the applications.It has already been investigated in great detail.These investigations gave birth to method of variation of constants,method of asymptotic integration,etc.The mentioned problem is also closely related to the notion of asymptotic equivalence and integral equivalence of two systems of differential equations.The problem of asymptotic equivalence was investigated by several authors,e.g. H.Weyl,N.Levinson,A. Wintner,V.A.Jakubovič,F.Bauer,J.S.Wong,R.Conti,M.Švec,N.Onuchic,P. Talpalaru,T.G.Hallam,T.Yoshizava,J.Kato,etc.

In this lecture we shall deal mainly with integral equivalence and with the relation between integral and asymptotic equivalence. Several of the results concerning integral equivalence presented here were obtained in cooperation with A.Haščák [1].

First,let's define basic notions required in the following:

Let be given two systems of differential equations

$$\text{(a)}\quad x' = F(t,x)\ , \qquad \text{(b)}\quad y' = G(t,y)$$

where x,y,F,G are n-vectors , $t \geq 0$.Suppose that F and G are such that the existence of solutions of (a) and (b) on the interval $[t_0,\infty)$,$t_0 \geq 0$, is guaranted.Let futher $\psi(t)$ be a positive continuous function on $[t_0,\infty)$.

<u>Definition 1.</u> We shall say that a vector function $z(t)$, $t \geq t_0$, is ψ- bounded,if there exists a constant $M>0$ such that

$$(1)\qquad |\psi^{-1}(t)z(t)| \leq M,\ t \geq t_0$$

where $|\cdot|$ denotes a suitable vector (matrix) norm.

<u>Remark 1.</u> Under a solution of a differential equation we shall understand a solution existing on some infinite interval $[t_0,\infty)$. The integral will be the Lebesgue integral.

<u>Definition 2.</u> We shall say that the systems (a) and (b) are ψ- asymptotically equivalent if for every solution $x(t)$ of (a) there is a solution $y(t)$ of (b) such that

$$(2)\qquad |\psi^{-1}(t)[x(t) - y(t)]| \to 0 \text{ as } t \to \infty$$

and conversely,for each solution $y(t)$ of (b) there is a solution

$x(t)$ of (a) such that (2) holds.

Definition 3. We shall say that systems (a) and (b) are (ψ, p)-integrally equivalent, $p > 0$, if to every solution $x(t)$ of (a) there is a solution $y(t)$ of (b) such that

(3) $$\psi^{-1}(t)[x(t) - y(t)] \in L_p([t_0, \infty))$$

and conversely, to each solution $y(t)$ of (b) there is a solution $x(t)$ of (a) such that (3) holds.

Here $L_p([t_0, \infty))$ denotes the space of all vector functions $z(t)$ mesurable and defined a.e. on $[t_0, \infty)$ such that $|z(t)|^p$ is Lebesgue integrable on $[t_0, \infty)$.

We start our considerations with special systems, i.e.

(4) $$x' = A(t)x + f(t),$$

(5) $$y' = A(t)y.$$

From the relation that $x(t) = y(t) + x_0(t)$, where $x_0(t)$ is a solution of (4) we have immediately

Theorem 1. The systems (4) and (5) are (ψ, p)-integrally equivalent iff there is a solution $x_0(t)$ of (4) such that $\psi^{-1}(t)x_0(t)$ belongs to $L_p([t_0, \infty))$.

We see that in this case the problem of (ψ, p)-integral equivalence turns into the problem of existence of solution $x_0(t)$ of (4) such that $\psi^{-1}(t)x_0(t) \in L_p([t_0, \infty))$.

We will discuss this problem in the case that $A(t) = A$ is a constant matrix. Suppose that A has the Jorden canonical form. Let be $\mu_1 < \mu_2 < \dots < \mu_s = \lambda$ distinct real parts of eigenvalues $\lambda_i(A)$ of A and let be m_i the maximum order of those blocks in A which correspond to eigenvalues with real part μ_i. Denote $m_s = m$. Let be a real number. Then let $\ell = m_j$ if $\mu_j = \mu$ and $\ell = 1$ if no μ_j equals μ. Suppose that $A = \mathrm{diag}(A_1, A_2)$, where A_1 and A_2 are square matrices such that $\mathrm{Re}\,\lambda_i(A_1) < \mu$, $\mathrm{Re}\,\lambda_i(A_2) \geq \mu$ for all i. Then $Y(t) = \mathrm{diag}\,(\exp tA_1, \exp tA_2)$ is the fundamental matrix of (5) with $Y(0) = I$ (identity matrix) and

$$Y_1(t) = \mathrm{diag}\,(\exp tA_1, 0), \quad Y_2(t) = \mathrm{diag}\,(0, \exp tA_2)$$

and such that

(6) $$Y(t) = Y_1(t) + Y_2(t), \quad Y(t)Y^{-1}(s) = Y_1(t)Y_1^{-1}(s) + Y_2(t)Y_2^{-1}(s),$$
$$Y_i(t)Y_i^{-1}(s) = Y(t-s), i = 1,2$$

and there exist numbers $c_1 > 0$, $c_2 > 0$ such that

(7) $$|Y_1(t)| \leq c_1 \exp\,(\mu - \delta)\ \chi_{m^*}(t),$$
$$|Y_2^{-1}(t)| = |Y_2(-t)| \leq c_2 \exp\,(-\mu t)\ \chi_{\ell}(t), \quad t \geq 0$$

where $-\delta = \max[\mathrm{Re}\,\lambda_i(A_1) - \mu] < 0$, $m^* = m_i$ if $\mu_i - \mu = -\delta$ and

$$\chi_k(t) = \begin{cases} t^{k-1}, & t \geq 1, \\ 1, & 0 \leq t \leq 1 \end{cases}$$

We are now able to represent the solution $x(t)$ of (4)in the form

$$(8) \quad x(t) = Y_1(t)x_0 + \int_0^t Y_1(t-s)f(s)ds - \int_t^\infty Y_2(t-s)f(s)ds$$

using the formula of variation of constants,the assumption that $f(t)$ is such that $|\int_0^\infty Y_2^{-1}(s)f(s)ds| < \infty$ and putting $x_0 = -\int_0^\infty Y_2(t-s)f(s)ds$. Taking $\mu = 0$ we have the majorants of the three terms on the right in (8):

$$|Y_1(t)x_0| \leq |x_0| e^{-\delta t}, \quad |\int_0^t Y_1(t-s)f(s)ds| \leq c_1 \int_0^t e^{-\delta(t-s)} |f(s)| ds,$$

$$|\int_t^\infty Y_2(t-s)f(s)ds| \leq c_2 \int_t^\infty \chi_\ell(t-s)|f(s)| ds .$$

Thus,we have to guarantee that

$$\int_0^t e^{-\delta(t-s)}|f(s)| ds \in L_p([0,\infty)), \quad \int_t^\infty \chi_\ell(t-s)|f(s)| ds \in L_p([0,\infty)).$$

The following lemmas will be useful (see [1]):

<u>Lemma 1.</u> Let σ be a positive constant and let be $g(t) \geq 0$, $g(t) \in L_1([0,\infty))$.Then

$$\int_0^t e^{-\sigma(t-s)} g(s)ds \in L_p([0,\infty)) \text{ for all } p \geq 1 .$$

<u>Lemma 2</u>. Let be $\int_0^\infty s|f(s)| ds < \infty$ .Then $\int_t^\infty |f(s)| ds \in L_p([0,\infty))$ for all $p \geq 1$.

Application of these lemmas on (8) gives

<u>Theorem 2.</u> Let A be a constant square matrix.Let $f(t)$ be continuous on $[0,\infty)$ and let

$$(9) \qquad \int_0^\infty t^\ell |f(t)| dt < \infty .$$

Then systems (4) and (5) are (1,p)-integrally equivalent, $p \geq 1$.

We note that in the paper [2],Theorem 2 ,we had the condition

$$(10) \qquad \int_0^\infty t^{\ell-1} |f(t)| dt < \infty$$

as sufficient for the asymptotic equivalence of (4) and (5).It seems that the integral equivalence implies the asymptotic equivalence.We shall see later that this is not true in general.

The motivation which we explained to get Theorem 2 gives us some ideas how to proceed by establishing the (φ,p)-integral equivalence between

$$(11) \qquad x' = A(t)x + f(t,x),$$

$$(12) \qquad y' = A(t)y .$$

There are three things to be used: formula of variation of constants, decomposition of fundamental matrix $Y(t)$ of (12) into two matrices $Y_1(t)$ and $Y_2(t)$ exhibiting similar properties as (6) and (7), estimation and growth of $f(t,x)$. The last will be facilitated if we know the apriori estimation of the solutions of (11) and (12).

Using supplementary projections P_1 and P_2 we get that, if $y(t)$ is a solution of (12), for the solution $x(t)$ of (11) the integral equation

$$(13)\quad x(t)=y(t)+\int_{t_0}^{t} Y(t)P_1Y^{-1}(s)f(s,x(s))ds-\int_{t}^{\infty} Y(t)P_2Y^{-1}(s)f(s,x(s))ds$$

holds. To prove that $\psi^{-1}(t)[x(t)-y(t)]\in L_p([t_0,\infty))$, it suffices to prove that the second and third terms on the right in (13) multiplied by $\psi^{-1}(t)$ belong to $L_p([t_0,\infty))$. To this aim serve Lemma 2 and

Lemma 3. ([1]) Let $\psi(t)$ and $\varphi(t)$ be positive functions for $t\geq 0$, $Y(t)$ be a nonsingular matrix and P a projection. Let further be

$$(14)\qquad \int_0^t |\psi^{-1}(t)Y(t)PY^{-1}(s)\ \varphi(s)|^p ds \leq K \text{ for } t\geq 0,\ p>0$$

and

$$(15)\qquad \int_0^\infty \exp\left\{-K^{-p}\int_0^t \varphi^p(s)\,\psi^{-p}(s)ds\right\}dt<\infty .$$

Then

$$(16)\qquad \lim |\psi^{-1}(t)Y(t)P| = 0 \text{ as } t\to\infty$$

and

$$(17)\qquad |\psi^{-1}(t)Y(t)P| \in L_p([0,\infty)).$$

Using Schauder's fixed-point theorem, Lemma 2 and Lemma 3 we can prove

Theorem 3. ([1]) Let $Y(t)$ be a fundamental matrix of (12) and let $\psi(t)$ and $\varphi(t)$ be positive continuous functions for $t\geq 0$. Suppose that :

a) there exist supplementary projections P_1, P_2 and constants $K>0$ and $2\leq p<\infty$ such that

$$\int_0^t |\psi^{-1}(t)Y(t)P_1Y^{-1}(s)\ \varphi(s)|^p\ ds + \int_t^\infty |\psi^{-1}(t)Y(t)P_2Y^{-1}(s)\,\varphi(s)|^p\ ds \leq K^p \text{ for } t\geq 0;$$

b) there exists $g:[0,\infty)\times[0,\infty)\to[0,\infty)$ such that

(i) $g(t,u)$ is nondecreasing in u for each fixed $t\in[0,\infty)$ and integrable on compact subsets of $[0,\infty)$ for fixed $u\in[0,\infty)$;

(ii) $\int_0^\infty s g^{p'}(s,c)ds<\infty$ for any constant $c\geq 0$, where $\frac{1}{p}+\frac{1}{p'}=1$;

(iii) for each $x\in R^n$, $|f(t,x)|\leq \varphi(t)g(t,\psi^{-1}(t)|x|)$ a.e. on $[0,\infty)$

c) $\int_0^\infty \exp\{ -K^{-p} \int_0^t \varphi^p(s) \psi^{-p}(s)ds\}dt < \infty$;

d) $\int_0^\infty |P_1Y^{-1}(s)\varphi(s)g(s,c)|ds < \infty$, $c \geq 0$.

Then between the set of ψ - bounded solutions of (11) and ψ - bounded solutions of (12) there is ψ - asymptotic equivalence and also (ψ ,p)- integral equivalence.

In this theorem the assumptions are concentrated mainly to the function g(t,u).It is possible to change the assumptions in such a way that we will assume more about the expression on the left side of the inequality in a) and less about the function g(t,u).It holds

Theorem 4. Assume that the following hypotheses from the Theorem 3 are satisfied: a),b) (i),(iii).Instead of b) (ii) let be satisfied only : $\int_0^\infty g^{p'}(t,c)dt < \infty$, $0 < c < \infty$; instead of c) let be satisfied: $\int_0^\infty \varphi^p(t) \psi^{-p}(t)dt = \infty$ .Finally,let the left side of the inequality a) belong to $L_1([0,\infty))$.Then the conclusions of the Theorem 3 are still valid.

The proof of Theorem 4 can be made in the same manner as that of Theorem 3.The difference is only at the end by proving that $\psi^{-1}(t)[x(t) - y(t)] \in L_p([0,\infty))$.In fact,we get in both cases that

$$\psi^{-1}(t)[x(t) - y(t)] = \int_{t_0}^t \psi^{-1}(t)Y(t)P_1Y^{-1}(s)f(s,x(s))ds - \\ - \int_t^\infty \psi^{-1}(t)Y(t)P_2Y^{-1}(s)f(s,x(s))ds.$$

Using the Hölder's inequality we get

$$|\psi^{-1}(t)[x(t) - y(t)]| \leq \\ \left(\int_0^t |\psi^{-1}(t)Y(t)P_1Y^{-1}(s)\varphi(s)|^p ds\right)^{1/p} \left(\int_0^t g^{p'}(s,2\rho)ds\right)^{1/p'} + \\ + \left(\int_t^\infty |\psi^{-1}(t)Y(t)P_2Y^{-1}(s)\varphi(s)|^p ds\right)^{1/p} \left(\int_t^\infty g^{p'}(s,2\rho)ds\right)^{1/p'}$$

where 2ρ is the ψ - bound of both solutions x(t) and y(t).Now, we can proceed either as it was done in the proof of Theorem 3 or we can get

$$|\psi^{-1}(t)[x(t) - y(t)]| \leq \left\{ \left(\int_0^t |\psi^{-1}(t)Y(t)P_1Y^{-1}(s)\varphi(s)|^p ds\right)^{1/p} + \\ \left(\int_t^\infty |\psi^{-1}(t)Y(t)P_2Y^{-1}(s)\varphi(s)|^p ds\right)^{1/p} \right\} \left(\int_0^\infty g^{p'}(s,2\rho)ds\right)^{1/p'}$$

which completes the proof of Theorem 4.

We note that the hypotheses of Theorem 4 were used by T.G.Hallam ([3]) .He proved that to each solution x(t) of (11) such that $\varphi^{-1}(t)x(t) \in L_p([t_0,\infty)) \cap L_\infty([t_0,\infty))$ there exists such a solution y(t) of (12) that $\varphi^{-1}(t)y(t) \in L_p([t_0,\infty)) \cap L_\infty([t_0,\infty))$ and conversely.

Remark 2. If we substitute in Theorem 3 the condition b) (ii) by the condition : $(\int_t^\infty g^{p'}(s,c)ds)^{1/p'} \in L_p([0,\infty))$ and for p we assume that $1 < p < \infty$,then the conclusions of Theorem 3 hold.

To complete the problem investigated in Theorem 3 it is necessary to investigate the cases when $p = 1$ ($p' = \infty$) and $p' = 1$ ($p = \infty$).We get the following corollaries:

Corollary 3.1. ([1]) Let $p = 1$ ($p' = \infty$).Let the assumptions of Theorem 3 be satisfied except b) (ii),which let be substituted by the condition

$$\lim_{t\to\infty} \gamma_c(t) = 0 \text{ for each } c \geq 0 \text{ and } \gamma_c(t) \in L_1([0,\infty))$$

where $\gamma_c(t) = \sup_{s\leq t} g(s,c)$.Then the conclusions of Theorem 3 still hold.

Corollary 3.2. ([1]) Let $p = \infty$ ($p' = 1$) and let the assumption a) of Theorem 3 be replaced by

$$\sup_{0\leq s\leq t} |\varphi^{-1}(t)Y(t)P_1Y^{-1}(s)\ \varphi(s)| + \sup_{t<s<\infty} |\varphi^{-1}(t)Y(t)P_2Y^{-1}(s)\ \varphi(s)| \leq K$$

and let

$$|\varphi^{-1}(t)Y(t)P_1| \in L_v([0,\infty)) \ , \ 0 < v < \infty$$

and let the other assumptions of Theorem 3 be valid.Then between the φ - bounded solutions of (11) and those of (12) there is (φ,v)-integral equivalence.

Theorem 5. ([1]) Let $\varphi(t)$, $\alpha(t)$ and $\beta(t)$ be positive continuous functions for $t \geq t_0 \geq 0$ with $\lim \varphi^{-1}(t) = 0$ as $t\to\infty$ and $\beta(t)$ bounded on $[t_0,\infty)$.Let Y(t) be a fundamental matrix of (12). Let further w: $[t_0,\infty)xJ \to J$, $J = [0,\infty)$, be such that

a) $|f(t,x)| \leq w(t,|x|)$ for $t \geq t_0$, $x \in R^n$; w(t,r) is nondecreasing in r for each fixed $t \geq t_0$; $w(t,c\,\varphi(t))$ is integrable on compact subsets of $[t_0,\infty)$ for each $c \geq 0$;

b) $\int_{t_0}^\infty s\,\alpha(s)w(s,c\varphi(s))ds < \infty$ for each $c \geq 0$;

c) $\int_{t_0}^t \beta(t-s)\alpha(s)w(s,c\varphi(s))ds \in L_p([t_0,\infty))$ for each $c \geq 0$;

d) Let exist two supplementary projections P_1 and P_2 and a constant $c > 0$ such that

$$|Y(t)P_1Y^{-1}(s)\alpha^{-1}(s)| \leq c\beta(t-s) \quad \text{for } t_0 \leq s \leq t ,$$

$$|Y(t)P_2Y^{-1}(s)\alpha^{-1}(s)| \leq c \quad \text{for } t_0 \leq t \leq s < \infty .$$

Then between the set of all ψ - bounded solutions of (11) and the set of all ψ - bounded solutions of (12) holds (1,p)-integral equivalence, $p \geq 1$.

As a special case of Theorem 5 we get

Theorem 6. ([1]) Let $\ell, m, \delta, m^*, \lambda$ be defined as before (at the beginning). Suppose that there exists w: $J \times J \to J$ such that

a) $w(t,r)$ is nondecreasing in r for each $t \in J$ and $w(t, ce^{\lambda t}\chi_m(t))$ is integrable on compact subsets of J for each $c \geq 0$;

b) $|f(t,x)| \leq w(t,|x|)$ a.e. on J for each $x \in R^n$;

c) (i) $\int_0^\infty t^\ell w(t, ce^{\lambda t}\chi_m(t))dt < \infty$ for each $c \geq 0$ if $\lambda \geq 0$;

(ii) $\int_0^\infty e^{-\lambda t} w(t, ce^{\lambda t}\chi_m(t))dt < \infty$ for each $c \geq 0$ if $\lambda < 0$;

d) $\lim \frac{1}{c} \int_{t_0}^\infty e^{-\lambda t} w(t, ce^{\lambda t}\chi_m(t))dt = 0$ as $t_0 \to \infty$ uniformly with respect $c \in [1,\infty)$;

e) $\int_{t_0}^t e^{-\delta(t-s)}\chi_{m^*}(t-s)t^{\ell-1} w(t, ce^{\lambda t}\chi_m(t)))dt \in L_p([t_0,\infty))$, $p \geq 1$.

Then the systems (11) and (12) are asymptotically equivalent and also (1,p)-integrally equivalent.

We note that the hypotheses b), c) (ii) d) guarantee the existence of each solution $x(t)$ on $[t_0,\infty)$ and the validity of the estimate $|x(t)| \leq D \exp\{\lambda(t-t_0)\}\chi_m(t-t_0)$, $0 \leq t_0 \leq t$.(See [2], Theorem 5.) This is the fact which leads to the asymptotic and (1,p)-integral equivalence between all solutions of (11) and all solutions of (12).

In almost all our Theorems we had the following situation: one part of assumptions has guaranteed the asymptotic equivalence and if we have added some further assumptions we obtained also integral equivalence. It might seem that integral equivalence implies asymptotic equivalence. We are going to demonstrate that this is not true in general.

Lemma 4. There exists a (nonnegative) function $f(t)$ defined and continuous on $[0,\infty)$ such that scalar equations

$$x' + ax = f(t)\ , \quad y' + ay = 0, \quad a > 0$$

are (1,p)-integrally equivalent, but they are not asymptotically equivalent.

Proof. We have $x(t) = ce^{-at} + \int_0^t e^{-a(t-s)}f(s)ds$, $y(t) = ce^{-at}$.

We are going to seek such $f(t) > 0$, that $\int_0^\infty \left(\int_0^t e^{-a(t-s)}f(s)ds\right)^p dt$ exists for $1 \leq p < \infty$ and $\limsup_{t\to\infty} \int_0^t e^{-a(t-s)}f(s)ds > 0$. Put $g(t) = \int_0^t e^{-a(t-s)}f(s)ds$. Then $g(t)$ has to satisfy : $\int_0^\infty g^p(t)dt < \infty$ and $\limsup_{t\to\infty} g(t) > 0$. Such functions exist and may be even unbounded. The construction of such a function $g(t)$ does not present any problem. Then for f we get : $f(t) = g'(t) + ag(t)$.

Let us now make some observations concerning the problem of sufficient conditions for the integral equivalence to imply the asymptotic equivalence. We shall need the following lemma:

Lemma 5. Let $f(t) \in L_p([0,\infty))$ for $1 \leq p < \infty$ and $|f(t)|'$ be bounded on $[0,\infty)$. Then $\lim f(t) = 0$ as $t \to \infty$.

The proof of this lemma is similar to that in [4], Lemma 6. The condition of boundedness of $|f(t)|'$ can be relaxed by uniform continuity of $f(t)$ on $[0,\infty)$. (See [5], exercise 13.31.)

Theorem 6. Let $A(t) = A$ be a square matrix such that $\mathrm{Re}\,\lambda_i(A) < -a < 0$ for all i. Let $f(t,x)$ be bounded for $0 \leq t$, $|x| < \infty$ and let the systems (11) and (12) be (1,p)-integrally equivalent. Then they are also asymptotically equivalent.

Proof. Let $x(t)$ be a solution of (11) and let $y(t)$ be a solution of (12) and such that they are (1,p)-integrally equivalent. Then $u(t) = x(t) - y(t)$ is a solution of the equation

$$u = Au + f(t,u + y(t)) \tag{18}$$

and

$$\int_0^\infty |u(t)|^p dt < \infty . \tag{19}$$

Using the method of variation of constants we have

$$u(t) = X(t)c + \int_0^t X(t-s)f(s,u(s)+y(s))ds \tag{20}$$

where $X(t)$ is a fundamental matrix of (12) and following the assumption and (7) $|X(t)| \leq c_1 \exp\{-at\}$, $t \geq 0$. Then $|X(t)c| \leq D$ for $t \geq 0$. Further there exists $K > 0$ such that $|f(t,x)| \leq K$ for $t \geq 0$ and $|x| < \infty$. Therefore from (20) we have

$$|u(t)| \leq D + Kc_1 \int_0^t e^{-a(t-s)}ds \leq D_1 \quad \text{for} \quad t \geq 0.$$

Thus $u(t)$ is bounded. Then from (18) an easy calculus gives that $|u(t)|' \leq AD_1 + K$. From this and from (19), using Lemma 5, we have that $\lim u(t) = 0$ as $t \to \infty$.

Remark 3. The negativity of real parts of the characteristic roots of A and the boundedness of $f(t,x)$ does not guarantee the asymptotic equivalence of (11) and (12). As an example we give the following: $x' = -ax + k$, $y' = -ay$, $a > 0$. These two equations are neither asymptotically nor (1,p)-integrally equivalent.

In the same way as Theorem 6 we can prove

Theorem 7. Let $A(t) = A$ and let

(21) $$f(t,x) = \lambda(t)w(|x|)$$

where $\lambda(t)$ is a positive bounded function, $w(r)$, $r \geq 0$, a real positive function. Let there exist (1,p)-integral equivalence between the sets of all bounded solutions of (11) and of (12), respectively. Then there is 1-asymptotic equivalence between these sets of solutions.

Theorem 8. Let $A(t) = A$ and let all solutions of (12) be bounded. Let (21) hold with $\lambda(t)$ bounded and integrable on $[0,\infty)$ and let $w(r)$, $r \geq 0$, be bounded, $w(r) \leq D$. Let the systems (11) and (12) be (1,p)-integrally equivalent, $1 \leq p < \infty$. Then the systems (11) and (12) are also 1-asymptotically equivalent.

Proof. Let $Y(t)$, $Y(0) = I$, be fundamental matrix of (12). Then from the boundedness of all solutions of (12) it follows that $|Y(t)| \leq C$, $t \geq 0$. Using the method of variation of constants we have for the solution $x(t)$ of (11) the representation

$$x(t) = Y(t)x(0) + \int_0^t Y(t-s)f(s,x(s))ds.$$

From this we get

$$|x(t)| \leq C|x(0)| + C\int_0^\infty \lambda(s)w(|x(s)|)ds \leq C|x(0)| + CD\int_0^\infty \lambda(s)ds = K.$$

Thus all solutions of (11) are bounded. Let now $x(t)$ and $y(t)$ be solutions of (11) and (12), respectively, which are (1,p)-integrally equivalent. Then $u(t) = x(t) - y(t)$ is bounded and an easy calculus gives that $|u(t)|' \leq |A||u(t)| + \lambda(t)w(|x(t)|)$. From this it follows that $|u(t)|'$ is bounded. Because $u(t) \in L_p([0,\infty))$ the use of Lemma 5 gives that $\lim u(t) = 0$ as $t \to \infty$.

Turn now our attention to the problem whether the φ-asymptotic equivalence implies (φ,p)-integral equivalence for some $p \geq 1$. The following example demonstrates that it is not true in general.

Let $t_k = \sum_{i=1}^{k} (3/2)^{i(i-1)}$, $k=1,2,\ldots$ Evidently $\lim t_k = \infty$ as $k \to \infty$. Define the function $f(t)$ as follows: $f(t_k) = (1/2)^k$, $f(t)$ is linear in interval $[t_k, t_{k+1}]$, $k=1,2,\ldots$ An easy calculus gives that

$\lim f(t) = 0$ as $t \to \infty$, $\int_1^{\infty} f^p(t)dt = \infty$ for every $1 \leq p < \infty$. Let us modify this function such that $f'(t)$ exists and the above properties continue to hold. Then define $z(t) = f'(t) + af(t)$, $a > 0$ and consider the equations : $x' + ax = z(t)$, $y' + ay = 0$. Then

$$x(t) - cy(t) = \int_1^t e^{-a(t-s)} z(s)ds = f(t).$$

Evidently these two equations are asymptotically equivalent but not $(1,p)$-integrally equivalent for $1 \leq p < \infty$.

After all, it is not without the interest the question, how many functions as $f(t)$ do exist ? If we denote by $C_0([t_0,\infty))$ the set of all continuous functions $g(t)$ on $[t_0,\infty)$ and such that $\lim g(t) = 0$ as $t \to \infty$, then the problem is to characterize the set $H = C_0([t_0,\infty)) - \bigcup_{p>0} L_p([t_0,\infty))$. As it was told me by T.Šalát, to whom I have posed this problem, this set is of the second Baire cathegory. It means that „ the majority " of the functions of $C_0([t_0,\infty))$ behave as our function $f(t)$.

At the end I want to note that I investigated here the systems (11) and (12) to facilitate the interpretation. All these problems can be discussed for the equations with deviating argument, for integral and integro-differential equations and others. The Lemmas introduced here will be helpful in those investigations.

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