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MAXWELL'S EQUATIONS WITH INCIDENT WAVE AS A FIELD SOURCE

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1. Motivation

Plasma-vacuum interfaces $\Gamma^- = \{(x,y,z) \in R_3 : x = 0\}$ and $\Gamma^+ = \{(x,y,z) \in R_3 : x = a\}$ of a slab $\{(x,y,z) \in R_3 : 0 \le x \le a\}$ of plasma with a given density $\varrho = \varrho(x,y,z)$ are irradiated by a laser light. It is assumed that the waves of the light have the form $\exp i(\alpha x + \beta y)$, where $\beta(p/2\pi)$ is an integer (i.e. the waves are p-periodic in y). Moreover, we assume $\varrho = \varrho(x,y) = \varrho(x, y+p)$.

The aim is to calculate both, the electric and magnetic fields $\mathbf{E} = (E_1, E_2, E_3)$ and $\mathbf{B} = (B_1, B_2, B_3)$. Due to the assumptions above, the electromagnetic field depends on two spatial variables (namely x and y) and is p-periodic in y.

2. Formulation

We set $u = (u_1, u_2, u_3)$, where $u_1 = B_3$, $u_2 = E_2$, $u_3 = E_1$. Let the following data be given:

a) $\omega = \omega_1 + i\omega_2$ (a complex constant, which is related to the frequency of laser light and decay time); $i = \sqrt{-1}$, $\omega_1 > 0$, $\omega_2 > 0$.

b) $\varepsilon = \varepsilon(x,y)$ (a complex valued function, which depends on the plasma density ϱ) on $\Omega = \{(x,y) : 0 \le x \le a, 0 \le y \le p\}$ (Ω is cross-section of the plasma slab); ε is smooth and $\mathrm{Re}(-i\omega\varepsilon) \geq \text{constant} > 0$ on Ω.

c) $\mathbf{F} = (0, J_1, J_2)$; $J_i = J_i(x,y)$ are complex valued and smooth functions on Ω (J_i are components of the source current density on Ω).

d) $H^+ = H^+(y)$ and $H^- = H^-(y)$ on $\Gamma = \{y : 0 \le y \le p\}$ are the incident waves from the right and from the left; H^+ and H^- are smooth, $H^+(0) = H^+(p)$ and $H^-(0) = H^-(p)$.

In [1] , we justified the following

<u>Problem (P1)</u> Find $u \in \mathcal{C} = \{ v = (v_1, v_2, v_3)$: each component $v_i = v_i(x,y)$ is smooth, complex valued function on Ω ; $v_i(x,0) = v_i(x,p)$ for $0 \le x \le a \}$ such that

(1) $\Lambda u + A u = F$ on Ω

(2) $T_1^+ u + B T_2^+ u = H^+$ on Γ

(3) $T_1^- u - B T_2^- u = H^-$ on Γ ,

where

$$\Lambda u = \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \frac{\partial u}{\partial x} + \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \frac{\partial u}{\partial y} , \quad A u = -i\omega \begin{pmatrix} 1 & 0 & 0 \\ 0 & \varepsilon & 0 \\ 0 & 0 & \varepsilon \end{pmatrix} u \ ;$$

$(T_i^+ u)(y) = u_i(a,y)$, $(T_i^- u)(y) = u_i(0,y)$ for $y \in \Gamma$ and $i = 1, 2$. The operator B is defined as follows: For each $\varphi = \varphi(y)$ smooth on Γ , $\varphi(0) = \varphi(p)$, we set

$$B\varphi = (B\varphi)(y) = \frac{\omega}{2} \int_{-\infty}^{+\infty} H_0^{(1)}(\omega |y - y'|)\, \tilde{\varphi}(y')\, dy'$$

for each $y \in \Gamma$, where $\tilde{\varphi}$ is the p-periodical extension of φ on $\mathbb{R}_1$ and

$$H_0^{(1)}(\omega r) = -\frac{2i}{\pi} \int_1^{+\infty} \frac{\exp(i\omega r t)}{\sqrt{t^2 - 1}}\, dt$$

for $r > 0$.

A similar boundary value problem can be formulated for the components E_3 , B_2 , B_1 of the electromagnetic field.

<u>Remark</u> (1) is the reduced system of steady Maxwell's equations; (2) and (3) are boundary conditions on plasma-vacuum interfaces.

3. <u>Weak formulation</u>

We proved (see [1]), that $B\psi_k = \lambda_k \psi_k$ on Γ for each integer k, where $\psi_k = \psi_k(y) = p^{-1/2} \exp\left(\frac{2k\pi i}{p} y\right)$ and $\lambda_k = p\omega(p^2\omega^2 - (2k\pi)^2)^{-1/2}$; we mean the branch $\pi > \arg \sqrt{\cdot} \ge 0$.

For each real α, we define $\mathcal{V}^{\alpha}$ to be the closure of $\{v = v(y) : v$ is p-periodic and infinitely differentiable on $\mathbb{R}_1\}$ in the norm of the "fractional" Sobolev space $W^{\alpha,2}(\Gamma)$. Using the spectral properties of B, one can verify that $B : \mathcal{V}^{\alpha} \longrightarrow \mathcal{V}^{\alpha+1}$ is linear and bounded operator (for each real α). If $\alpha = -1/2$ then B is dissipative.

<u>Graph of the operator Λ :</u> We set

$$\mathcal{L} = \{ v = (v_1, v_2, v_3) : \Big(\sum_k \int_\Omega v_k \bar{v}_k \, dx\Big)^{1/2} \equiv \| v \| < +\infty \}$$

and define $\mathcal{G}$ to be the closure of $\mathcal{C}$ in the norm $\| \cdot \|_{\mathcal{G}}$, $\| v \|_{\mathcal{G}} = (\| v \|^2 + \| \Lambda v \|^2)^{1/2}$. It is possible to prove (see [1]) that there exist continuous extensions $T_1^+ : \mathcal{G} \longrightarrow \mathcal{V}^{1/2}$, $T_1^- : \mathcal{G} \longrightarrow \mathcal{V}^{1/2}$, $T_2^+ : \mathcal{G} \longrightarrow \mathcal{V}^{-1/2}$, $T_2^- : \mathcal{G} \longrightarrow \mathcal{V}^{-1/2}$; moreover, the operators above are surjective.

<u>Problem (P)</u> Data: $F \in \mathcal{L}$, $H^+ \in \mathcal{V}^{1/2}$, $H^- \in \mathcal{V}^{1/2}$. Find $u \in \mathcal{G}$: $\Lambda u + A u = F$ a.e. on Ω, $T_1^+ u + B T_2^+ u = H^+$ a.e. on Γ, $T_1^- u - B T_2^- u = H^-$ a.e. on Γ. □

<u>Theorem</u> There exists one and only one solution to Problem (P).

<u>Proof</u> see [1] .

4. Approximation

We considered finite difference approximation of the system (1), while the boundary conditions (2) and (3) were discretised by means of Galerkin method. Namely, the operator B was approximated by ΠB, where Π was a projection on a space S of piecewise constant functions. The operator $\Pi B : S \longrightarrow S$ was represented as an $m \times m$ matrix, where m was a number of nodal points on Γ. The matrix can be explicitly evaluated; moreover, we know (explicitly) its eigenvectors and eigenfunctions.

For the details concerning the discretisation of Problem (P), convergence results and the algorithm solving the discrete scheme, we refer to our report [1] .

5. References

[1] V.Janovský,I.Marek,J.Neuberg : Maxwell´s equations with incident wave as a field source (Mathematical and numerical analysis), Technical Report KNM-0105057/81, Charles University of Prague (Faculty of Mathematics and Physics).

[2] C.Müller : Foundations of the mathematical theory of electromagnetic waves, Grundlehren der Math.Wiss., Vol.155, Springer Verlag, Berlin 1969.