DUALITY METHODS IN THE THEORY OF OPTIMAL CONTROL

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Let Ω be a open bounded domain in E_2 with a sufficiently smooth boundary $\partial\Omega$. We define the state $u(e)$ of our system as the solution of the variational inequality

$$(\mathcal{P})\begin{cases} u(e) \in K(\Omega) \\ \int_\Omega \alpha(e)\nabla^2 u \, e \, \nabla^2(v - u(e))\, d\Omega \geq \\ \int_\Omega (f + B\, e)\,(v - u(e))\, d\Omega \qquad \text{for any} \quad v \in K(\Omega) \end{cases}$$

where

(1) $K(\Omega) = \{ v \in H_0^2(\Omega)\ v \geq \psi \}$ is a (non-empty) closed convex subset of $H_0^2(\Omega)$

$\alpha(e) > 0$ is the Lipschitz function

e = control function.

The cost function is given by

$$J(e) = \int_\Omega (u(e) - z_d)^2 d\Omega + N\|e\|^2_{L_2(\Omega)}$$

where z_d is given in $L_2(\Omega)$ and where N is a given positive number.

The problem of optimal control is now find

$\inf J(e)\ e \in U_{ad}(\Omega)$ = closed convex subset of $L_2(\Omega)$ (the set of admissable controls).

Then

there exists $e_0 \in U_{ad}(\Omega)$ such that

(2) $J(e_0) = \inf J(e)$

(the proof see Lions [2]; Hlaváček - Bock - Lovíšek [4]).

One can think to problem (2) as an optimal control related to control of free surfaces. In this respect a more realistic problem would be to try to find $e_0 \in U_{ad}(\Omega)$ minimizing the „distance" of the free surface in case $K(\Omega)$ is given by (1) to a given surface.

1. Duality methods .

Setting of the problem.

We consider two functions F and G from $H_0^2(\Omega)$ and $L_2(\Omega) \to R$ such that

$$\left\{\begin{array}{l} F \text{ and } G \text{ are lower semicontinuous and convex on } H_0^2(\Omega) \text{ and} \\ L_2(\Omega) \text{ respectively such that} \\ -\infty < F(v) \leq +\infty;\ -\infty < G(p) \leq +\infty;\ F \text{ and } G \text{ are not identi-} \\ \text{cally } +\infty. \end{array}\right.$$

Now we derive the dual formulation of the problem , which we write in the form

$$(\mathcal{P}) \quad \inf_{\substack{v \in H_0^2(\Omega) \\ e \in U_{ad}(\Omega)}} \left\{F(v,e) + G(\Lambda(e)v)\right\}$$

where

$$F(v,e) = -(f + B\,e, v)_{L_2(\Omega)} + \chi_{K(\Omega)}(v)$$

$\chi_{K(\Omega)}(v)$ is the indicator function of $K(\Omega)$.

We have

$$\langle A(e)u, v\rangle_{H_0^2(\Omega)} = a\,(e,u,v)$$

where

$$A(e) = \mathcal{L}(H_0^2(\Omega);\ H^{-2}(\Omega)) \quad , \text{ and } \quad A(e) = \nabla^2(\alpha(e)\nabla^2).$$

Then by using (second representation theorem) we can write

$$a(e,u,v) = (A^{1/2}(e)\,u\ ,\ A^{1/2}(e)\,v)_{L_2(\Omega)}.$$

Next we get

$$\Lambda(e) = A^{1/2}(e)\ ;\ (\Lambda^*(e) = (A^{1/2}(e)^* \in \mathcal{L}(L_2(\Omega)\ ;\ H^{-2}(\Omega)))$$

$$\Lambda(e) \in \mathcal{L}(H_0^2(\Omega)\ ; L_2(\Omega))\ ; \qquad \text{for any } e \in U_{ad}(\Omega)$$

$$G\,(p(e)) = \|p(e)\|^2_{L_2(\Omega)};\ p(e) \in L_2(\Omega)\ ; \qquad e \in U_{ad}(\Omega).$$

We shall formulate the dual problem $(\mathcal{P}^*)$ by

$$(\mathcal{P}^*) \sup_{\substack{p^* \in L_2(\Omega) \\ e \in U_{ad}(\Omega)}} \left\{ -F^*(\Lambda^*(e)\, p^*) - G^*(-p^*) \right\} =$$

$$= \sup_{\substack{p^* \in L_2(\Omega) \\ \Lambda^*(e)\, p^* + f + B\, e \leq 0 \\ e \in U_{ad}(\Omega)}} \left\{ -(\Lambda(e)\psi, p^*)_{L_2(\Omega)} - \langle \psi, f + B\, e \rangle_{V(\Omega)} - 1/2 \, \|p^*\|^2_{L_2(\Omega)} \right\}$$

Then it is known (cf. Ekeland - Teman [1]) the problem $(\mathcal{P}^*)$ has a unique solution.

Next we take

(3) $$\mathcal{J}(e,v) = 1/2 \, \|\Lambda(e)\, u(e)\|^2_{L_2(\Omega)} - (f + B\, e, v)_{L_2(\Omega)}$$

($\mathcal{J}(e,v)$ is a convex $\ell.s.c$ function of $K(\Omega)$ into R).

Let $\Lambda(\Omega)$ be a closed convex cone of $L_\infty(\Omega)$ defining a partial ordering relation $\leq$ and let $\tilde{\Lambda}^*(\Omega)$ be its closed in $(L_\infty(\Omega))^*$.
We set $\left\{ B = \psi - I \mid K(\Omega) \to L_\infty(\Omega) \right\}$.
For each $p^* \in (L_\infty(\Omega))^*$, $p^* \geq 0$ the mapping $v \to \langle p^*, B\, v \rangle_{L_\infty(\Omega)}$ of $K(\Omega)$ into R is $\ell.s.c$.
$\left\{ v \in K(\Omega) \mid B\, v \leq 0 \right\} \neq 0$.
Then the dual problem is

$$(\mathcal{P}^*_\psi) \sup_{\substack{p^* \leq 0 \,;\\ e \in U_{ad}(\Omega)}} \inf_{v \in K(\Omega)} \left\{ \mathcal{J}(e,v) - \langle p^*, B\, v \rangle_{L_\infty(\Omega)} \right\}.$$

We define the Lagrangian function (for any $p^* \in \tilde{\Lambda}^*(\Omega)$; $p^* \leq 0$)
$$\mathcal{L}(v, p^*, e) = \mathcal{J}(e,v) - \langle p^*, B\, v \rangle_{L_\infty(\Omega)}.$$

Next by using idea Begis - Glowinski [3] ;
we have the problem $(\mathcal{P}_\mathcal{L})$

$$\begin{cases} \mathcal{L}(u(p^*,e) \,;\, p^*, e) \leq \mathcal{L}(v, p^*, e) & \text{for any } v \in K(\Omega) \\ J(e_0(p^*);p^*) \leq J(e(p^*);p^*) & \text{for any } e \in U_{ad}(\Omega) \\ \left(\text{where } J(e,p^*) = \|u(p^*,e) - z_d\|^2_{L_2(\Omega)} + N(e,e)_{L_2(\Omega)}\right) & \end{cases}$$

and finally we search $\bar{p}^* \in \tilde{\Lambda}^*(\Omega)$ such that
$\langle \bar{p}^* - p^*;\ B\, u(\bar{p}^*;\ e(\bar{p}^*))\rangle_{\tilde{\Lambda}(\Omega)} \geq 0$ for any $p^* \in \tilde{\Lambda}^*(\Omega)$.

Theorem

Let us assume, that

1^o there exists $b \in H^{-2}(\Omega)$ and $\beta \in R$ such that for any $p^* \in \tilde{\Lambda}^*(\Omega)$ and $v \in H_0^2(\Omega)$ we have $\langle p^*;\ B\, v\rangle_{\tilde{\Lambda}(\Omega)} = \langle b\ ,\ v\rangle_{H_0^2(\Omega)} + \beta$

2^o when

$\bar{p}_n^* \to \bar{p}^*$ in $\tilde{\Lambda}^*(\Omega)$ weakly star and $u_n \to u$ in $H_0^2(\Omega)$ weakly, then $\liminf \langle \bar{p}_n^*,\ B\, u_n\rangle_{\tilde{\Lambda}(\Omega)} \geq \langle \bar{p}^*;\ B\, u\rangle_{\tilde{\Lambda}(\Omega)}$ $(\bar{p}_n^*,\ \bar{p}^* \in \tilde{\Lambda}^*(\Omega))$.

Then optimal problem $(\mathcal{P}_\chi)$ has at least one solution .

References

1 I. Ekeland and R. Teman : Convex analysis and variational problems. North-Holland 1976 .

2 Y. L. Lions : Remarks on the theory of optimal control of distributed systems. Control theory of systems governed by partial differential equations. Academic Press, Inc. 1977 , 1 - 103 .

3 Y. P. Yvon : These. L' université Paris VI , 1973 .

4 I. Hlaváček, I. Bock, J. Lovíšek : Optimal control of a variational inequality with applications to structural analysis problems . To appear .