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DUALITY METHODS IN THE THEORY OF OPTIMAL CONTROL

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Let  $\Omega$  be a open bounded domain in  $E_2$  with a sufficiently smooth boundary  $\partial\Omega$ . We define the state  $u(e)$  of our system as the solution of the variational inequality

$$(P) \left\{ \begin{array}{l} u(e) \in K(\Omega) \\ \int_{\Omega} \alpha(e) \nabla^2 u \cdot e \nabla^2 (v - u(e)) \, d\Omega \geq \\ \int_{\Omega} (f + B e) (v - u(e)) \, d\Omega \end{array} \right. \quad \text{for any } v \in K(\Omega)$$

where

$$(1) \quad K(\Omega) = \{v \in H_0^2(\Omega) \mid v \geq \psi\} \quad \text{is a (non-empty) closed convex subset of } H_0^2(\Omega)$$

$\alpha(e) > 0$  is the Lipschitz function  
 $e$  = control function.

The cost function is given by

$$J(e) = \int_{\Omega} (u(e) - s_d)^2 \, d\Omega + N \|e\|_{L_2(\Omega)}^2$$

where  $s_d$  is given in  $L_2(\Omega)$  and where  $N$  is a given positive number.

The problem of optimal control is now find  
 $\inf J(e) \mid e \in U_{ad}(\Omega)$  = closed convex subset of  $L_2(\Omega)$  (the set of admissible controls).

Then

$$(2) \quad J(e_0) = \inf J(e) \quad \text{there exists } e_0 \in U_{ad}(\Omega) \text{ such that}$$

(the proof see Lions [2]; Hlaváček - Bock - Lovíšek [4]).

One can think to problem (2) as an optimal control related to control of free surfaces. In this respect a more realistic problem would be to try to find  $e_0 \in U_{ad}(\Omega)$  minimizing the "distance" of the free surface in case  $K(\Omega)$  is given by (1) to a given surface.

1. Duality methods .

Setting of the problem.

We consider two functions  $F$  and  $G$  from  $H_0^2(\Omega)$  and  $L_2(\Omega) \rightarrow \mathbb{R}$  such that

$$\left\{ \begin{array}{l} F \text{ and } G \text{ are lower semicontinuous and convex on } H_0^2(\Omega) \text{ and } \\ L_2(\Omega) \text{ respectively such that} \\ -\infty < F(v) \leq +\infty; -\infty < G(p) \leq +\infty; F \text{ and } G \text{ are not identi-} \\ \text{cally } +\infty. \end{array} \right.$$

Now we derive the dual formulation of the problem , which we write in the form

$$(\mathcal{P}) \inf \left\{ F(v, e) + G(\Lambda(e) v) \right\}$$

$$v \in H_0^2(\Omega)$$

$$e \in U_{ad}(\Omega)$$

where

$$F(v, e) = -(f + B e, v)_{L_2(\Omega)} + \chi_{K(\Omega)}(v)$$

$\chi_{K(\Omega)}(v)$  is the indicator function of  $K(\Omega)$  .

We have

$$\langle \Lambda(e) u, v \rangle_{H_0^2(\Omega)} = a(e, u, v)$$

where

$$\Lambda(e) = \mathcal{L}(H_0^2(\Omega); H^{-2}(\Omega)) \quad , \text{ and } \Lambda(e) = \nabla^2(\alpha(e)\nabla^2) .$$

Then by using (second representation theorem) we can write

$$a(e, u, v) = (\Lambda^{1/2}(e) u, \Lambda^{1/2}(e) v)_{L_2(\Omega)} .$$

Next we get

$$\Lambda(e) = \Lambda^{1/2}(e) ; (\Lambda^*(e) = (\Lambda^{1/2}(e))^* \in \mathcal{L}(L_2(\Omega); H^{-2}(\Omega)))$$

$$\Lambda(e) \in \mathcal{L}(H_0^2(\Omega); L_2(\Omega)) ; \quad \text{for any } e \in U_{ad}(\Omega)$$

$$G(p) = \|p(e)\|_{L_2(\Omega)}^2 ; p(e) \in L_2(\Omega) ; \quad e \in U_{ad}(\Omega) .$$

We shall formulate the dual problem ( $\mathcal{P}^*$ ) by

$$(\mathcal{P}^*) \sup_{\substack{p^* \in L_2(\Omega) \\ e \in U_{ad}(\Omega)}} \left\{ -F^*(\Lambda^*(e) p^*) - G^*(-p^*) \right\} =$$

$$- \sup_{p^* \in L_2(\Omega)} \left\{ \left( \Lambda^*(e) p^* \right)_{L_2(\Omega)} - \langle \psi, f + B e \rangle_{V(\Omega)} - 1/2 \|p^*\|_{L_2(\Omega)}^2 \right\}$$

$$\Lambda^*(e) p^* + f + B e \leq 0$$

$$e \in U_{ad}(\Omega)$$

Then it is known (cf. Ekeland - Teman [1]) the problem ( $\mathcal{P}^*$ ) has a unique solution.

Next we take

$$(3) \quad \mathcal{J}(e, v) = 1/2 \|\Lambda(e) u(e)\|_{L_2(\Omega)}^2 - (f + B e, v)_{L_2(\Omega)}$$

( $\mathcal{J}(e, v)$  is a convex i.s.c function of  $K(\Omega)$  into  $\mathbb{R}$ ).

Let  $\tilde{K}(\Omega)$  be a closed convex cone of  $L_\infty(\Omega)$  defining a partial ordering relation  $\leq$  and let  $\tilde{K}^*(\Omega)$  be its closed in  $(L_\infty(\Omega))^*$ .

We set  $\{ B = \psi - I \mid K(\Omega) \rightarrow L_\infty(\Omega) \}$ .

For each  $p^* \in (L_\infty(\Omega))^*$ ,  $p^* \geq 0$  the mapping  $v \rightarrow \langle p^*, B v \rangle_{L_\infty(\Omega)}$  of  $K(\Omega)$

into  $\mathbb{R}$  is i.s.c.

$$\{ v \in K(\Omega) \mid B v \leq 0 \} \neq \emptyset.$$

Then the dual problem is

$$(\mathcal{P}^*) \sup_{\substack{p^* \leq 0 \\ v \in K(\Omega) \\ e \in U_{ad}(\Omega)}} \inf \left\{ \mathcal{J}(e, v) - \langle p^*, B v \rangle_{L_\infty(\Omega)} \right\}.$$

We define the Lagrangian function (for any  $p^* \in \tilde{K}^*(\Omega)$ ;  $p^* \leq 0$ )

$$\mathcal{L}(v, p^*, e) = \mathcal{J}(e, v) - \langle p^*, B v \rangle_{L_\infty(\Omega)}.$$

Next by using idea Begis - Glowinski [3];

we have the problem ( $\mathcal{P}_\varepsilon$ )

$$\left\{ \begin{array}{ll} \mathcal{L}(u(p^*, e); p^*, e) \leq \mathcal{L}(v, p^*, e) & \text{for any } v \in K(\Omega) \\ J(e, p^*; p^*) \leq J(e, p^*; p^*) & \text{for any } e \in U_{ad}(\Omega) \end{array} \right.$$

(where  $J(e, p^*) = \|u(p^*, e) - z_d\|_{L_2(\Omega)}^2 + H(e, e)_{L_2(\Omega)}$ )

and finally we search  $\bar{p}^* \in \tilde{\Lambda}^*(\Omega)$  such that  
 $\langle \bar{p}^* - p^*; B u(\bar{p}^*) - e(\bar{p}^*) \rangle_{\tilde{\Lambda}(\Omega)} \stackrel{\approx}{=} 0$  for any  $p^* \in \tilde{\Lambda}^*(\Omega)$ .

**Theorem**

Let us assume, that

1° there exists  $b \in H^{-2}(\Omega)$  and  $\beta \in \mathbb{R}$  such that for any  $p^* \in \tilde{\Lambda}^*(\Omega)$  and  $v \in H_0^2(\Omega)$  we have  $\langle p^*; B v \rangle_{\tilde{\Lambda}(\Omega)} < \langle b, v \rangle_{H_0^2(\Omega)} + \beta$

2° when

$\bar{p}_n^* \rightarrow \bar{p}^*$  in  $\tilde{\Lambda}^*(\Omega)$  weakly star and  $u_n \rightarrow u$  in  $H_0^2(\Omega)$  weakly,  
then  $\liminf \langle \bar{p}_n^*, B u_n \rangle_{\tilde{\Lambda}(\Omega)} \leq \langle \bar{p}^*, B u \rangle_{\tilde{\Lambda}(\Omega)}$  ( $\bar{p}_n^*, \bar{p}^* \in \tilde{\Lambda}^*(\Omega)$ ).

Then optimal problem  $(P_x^*)$  has at least one solution.

**References**

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