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In: Michal Greguš (ed.): Equadiff 5, Proceedings of the Fifth Czechoslovak Conference on Differential Equations and Their Applications held in Bratislava, August 24-28, 1981. BSB B.G. Teubner Verlagsgesellschaft, Leipzig, 1982. Teubner-Texte zur Mathematik, Bd. 47. pp. 218--221.

Persistent URL: http://dml.cz/dmlcz/702292

OSCILLATION THEORY OF HIGHER-ORDER ORDINARY AND FUNCTIONAL DIFFERENTIAL EQUATIONS WITH FORCING TERMS

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1. Introduction

We are interested in comparing the oscillatory and asymptotic behavior of the differential equations

$(E_+)\qquad L_n x + F(t,x) = 0,\qquad\qquad (E_-)\qquad L_n x - F(t,x) = 0,$

with that of the corresponding forced equations

$(F_+)\qquad L_n x + F(t,x) = f(t),\qquad\qquad (F_-)\qquad L_n x - F(t,x) = f(t),$

where L_n is a disconjugate differential operator defined by

$$(1)\qquad L_n = \frac{1}{p_n(t)}\frac{d}{dt}\frac{1}{p_{n-1}(t)}\frac{d}{dt}\cdots\frac{d}{dt}\frac{1}{p_1(t)}\frac{d}{dt}\frac{\cdot}{p_0(t)}.$$

It is assumed throughout that $p_i:[a,\infty)\to(0,\infty)$, $0 \le i \le n$, are continuous; $\int_a^\infty p_i(t)dt = \infty$ for $1 \le i \le n-1$; $f:[a,\infty)\to R$ is continuous; $F:[a,\infty)\times R\to R$ is continuous and nondecreasing in the second variable; and $xF(t,x) > 0$ for $x \ne 0$.

We employ the notation:

$$(2)\qquad \begin{aligned} &D^0(x;p_0)(t) = \frac{x(t)}{p_0(t)},\\ &D^i(x;p_0,\dots,p_i)(t) = \frac{1}{p_i(t)}\frac{d}{dt}D^{i-1}(x;p_0,\dots,p_{i-1})(t),\ 1 \le i \le n. \end{aligned}$$

By a proper solution of $(E_+)[(E_-)$, (F_+) or $(F_-)]$ is meant a function $x:[T_x,\infty)\to R$ which satisfies $(E_+)[(E_-)$, (F_+) or $(F_-)]$ for all sufficiently large t and $\sup\{|x(t)|: t \ge T\} > 0$ for any $T \ge T_x$. We make the standing hypothesis that the above equations do possess proper solutions. A proper solution of one of the above equations is called oscillatory if it has arbitrarily large zeros; otherwise it is called nonoscillatory.

DEFINITION. Equation $(E_+)[(F_+)]$ is said to have property (A) if (i) for n even, all proper solutions of $(E_+)[(F_+)]$ are oscillatory, and (ii) for n odd, every nonoscillatory solution $x(t)$ of $(E_+)[(F_+)]$ satisfies

$$(3)\qquad \lim_{t\to\infty} D^0(x;p_0)(t) = 0.$$

Equation $(E_-)[(F_-)]$ is said to have property (B) if (i) for n odd, every nonoscillatory solution $x(t)$ of $(E_-)[(F_-)]$ satisfies

$$(4)\qquad \lim_{t\to\infty} |D^{n-1}(x;p_0,\dots,p_{n-1})(t)| = \infty,$$

and (ii) for n even, every nonoscillatory solution satisfies either (3) or (4).

There is much current interest in obtaining sharp criteria for the unforced equations (E_+) and (E_-) to have properties (A) and (B), respectively; see, for example, the papers [7,8,10,13,16]. A question naturally arises as to what will happen if a forcing term is added to (E_+) or (E_-), that is, we are led to the study of the oscillatory and asymptotic behavior of forced equations (F_+) and

(F_-). Speaking, for example, of (E_+) and (F_+), it is expected that (I) if the forcing term $f(t)$ oscillates and its amplitude is "large enough", then all proper solutions of (F_+) will be oscillatory regardless of oscillation or nonoscillation of the unforced equation (E_+); and(II) in case all proper solutions of (E_+) are oscillatory, the same will be true of the forced equation (F_+) provided the forcing term $f(t)$ is "small enough".

The purpose of this report is to show that such situations indeed occur not only for (E_+) and (F_+) but also for (E_-) and (F_-). Extensions to functional differential equations will also be discussed.

2. Oscillation generating forcings

It is natural to expect that a sufficiently large oscillating force exerted on a mechanical system with no oscillatory character may generate oscillation. That this is indeed the case is described in the following theorems which are formulated in terms of the repeated integrals defined below.

Let $h_i:[a,\infty) \to R$, $1 \le i \le N$, be continuous functions. We put for $t,s \in [a,\infty)$

$$I_0 = 1,$$
$$I_i(t,s;h_1,\dots,h_i) = \int_s^t h_1(r)I_{i-1}(r,s;h_2,\dots,h_i)dr, \quad 1 \le i \le N. \tag{5}$$

THEOREM 1. <u>Suppose that for any</u> $T > a$

$$\limsup_{t\to\infty} \frac{I_n(t,T;p_1,\dots,p_{n-1},p_nf)}{I_{n-1}(t,T;p_1,\dots,p_{n-1})} = +\infty, \tag{6}$$

$$\liminf_{t\to\infty} \frac{I_n(t,T;p_1,\dots,p_{n-1},p_nf)}{I_{n-1}(t,T;p_1,\dots,p_{n-1})} = -\infty. \tag{7}$$

(i) <u>All proper solutions of equation</u> (F_+) <u>are oscillatory</u>.

(ii) <u>All proper solutions of equation</u> (F_-) <u>such that</u>

$$x(t) = O(p_0(t)I_{n-1}(t,a;p_1,\dots,p_{n-1})) \quad \text{as} \quad t \to \infty \tag{8}$$

<u>are oscillatory</u>.

THEOREM 2. <u>Suppose that for any</u> $T > a$

$$\limsup_{t\to\infty}\int_T^t p_n(s)f(s)ds = +\infty, \qquad \liminf_{t\to\infty}\int_T^t p_n(s)f(s)ds = -\infty, \tag{9}$$

<u>and</u>

$$I_n(t,T;p_1,\dots,p_{n-1},p_nf) = O(I_{n-1}(t,a;p_1,\dots,p_{n-1})) \tag{10}$$

<u>as</u> $t \to \infty$ <u>uniformly with respect to</u> T.

(i) <u>All proper solutions of equation</u> (F_+) <u>are oscillatory</u>.

(ii) <u>All proper solutions of equation</u> (F_-) <u>satisfying</u> (8) <u>are oscillatory</u>.

Theorems 1 and 2 generalize some results of [3] and [11] for second order equations. For the proof of Theorem 1 see [2]. Theorem 2 seems to be new. For other related results we refer to [15].

3. Oscillation preserving forcings

Suppose that equation (E_+) is oscillatory in the sense that all of its proper solutions are oscillatory. Then, to what extent does a forcing term added affect the oscillatory behavior of (E_+)? It is likely that in this case the forced equation (F_+) is oscillatory provided the forcing term $f(t)$ is "sufficiently small". That such a situation really occurs has been shown by many authors including [1,4-6,14]. We refer in particular to the paper [6] in which it is proved that if $f(t)$ is "small" and oscillatory, then the equation $x^{(n)} + F(t,x) = f(t)$ is oscillatory if and only if the equation $x^{(n)} + F(t,x) = 0$ is oscillatory. This result can be extended as follows.

THEOREM 3. Let n be even and suppose there exists an oscillatory function $\varphi(t)$ such that

$$(11) \qquad L_n\varphi(t) = f(t) \quad \text{and} \quad \lim_{t\to\infty} D^0(\varphi;p_0)(t) = 0.$$

Then all proper solutions of (E_+) are oscillatory if and only if all proper solutions of (F_+) are oscillatory.

Theorem 3 is due to [12]. The odd order case can be discussed similarly and combining the result with Theorem 3, we have the following theorem on the preservation of property (A).

THEOREM 4. Suppose there exists an oscillatory function $\varphi(t)$ satisfying (11). Then equation (E_+) has property (A) if and only if equation (F_+) has property (A).

It can be shown that property (B) of equation (E_-) is preserved under the effect of a slightly different class of small forcings.

THEOREM 5. Suppose there exists an oscillatory function $\psi(t)$ such that

$$(12) \qquad L_n\psi(t) = f(t) \quad \text{and} \quad \lim_{t\to\infty} D^i(\psi;p_0,\ldots,p_i)(t) = 0 \quad \text{for} \quad 0 \le i \le n-1.$$

Then equation (E_-) has property (B) if and only if equation (F_-) has property (B)

4. Extensions to functional differential equations

Let us now consider the functional differential equations

$$(H_\pm) \qquad L_nx(t) \pm F(t,x(g_1(t)),\ldots,x(g_N(t))) = 0,$$

$$(I_\pm) \qquad L_nx(t) \pm F(t,x(g_1(t)),\ldots,x(g_N(t))) = f(t),$$

where L_n and $f(t)$ are as before, $g_i:[a,\infty) \to R$, $1 \le i \le N$, are continuous, $\lim_{t\to\infty} g_i(t) = \infty$, $1 \le i \le N$, $F:[a,\infty) \times R^N \to R$ is continuous, $F(t,x_1,\ldots,x_N)$ is nondecreasing in each x_i, and $x_1F(t,x_1,\ldots,x_N) > 0$ if $x_1x_i > 0$, $1 \le i \le N$.

All the definitions and terminologies given in Introduction for ordinary differential equations also apply to $(H_+),\ldots,(I_-)$, and the results stated in the preceding sections can be extended to these equations without much difficulty. For example, Theorems 4 and 5 allow the following extensions.

THEOREM 4*. Suppose there exists an oscillatory function $\varphi(t)$ satisfying (11). Then equation (H_+) has property (A) if and only if equation (I_+) has property (A).

THEOREM 5*. Suppose there exists an oscillatory function $\psi(t)$ satisfying (12). Then equation (H_-) has property (B) if and only if equation (I_-) has property (B).

Very recently the present author [9] has proved that the even order equation

(13) $$y^{(n)}(t) - p^n y(t - n\sigma) - q^n y(t + n\tau) = 0,$$

where p, q, σ and τ are positive constants, is oscillatory if $pe\sigma > 1$ and $qe\tau > 1$. It is then natural to ask under what conditions on $f(t)$ the equation

(14) $$y^{(n)}(t) - p^n y(t - n\sigma) - q^n y(t + n\tau) = f(t)$$

is oscillatory. A more general question is: Is it possible to indicate a class of forcings $f(t)$ which make equation (I_-) remain oscillatory when the unforced equation (H_-) with both retarded and advanced arguments is known to be oscillatory?

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