Rudolf Kodnár

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# ON ONE SYSTEM OF NONLINEAR EQUATIONS

Rudolf Kodnár
Bratislava, ČSSR

In this paper, some problems of the system of differential equations connected with the theory of nonlinearly elastic plates (Lepik [1] and Jeršov [2]) will be analysed.

Let $\Omega$ be a bounded domain in $R^2$ with boundary $\partial\Omega$. We consider the system

$$\begin{aligned}(1)\quad &\delta_1 w_{xxxx}+2\delta_2 w_{xxyy}+\delta_3 w_{yyyy}-[(\alpha w_{xx})_{xx}+(\alpha w_{yy})_{yy}+2(\alpha w_{xy})_{xy}]-\\ &-\tfrac{1}{2}[(\alpha w_{yy})_{xx}+(\alpha w_{xx})_{yy}-2(\alpha w_{xy})_{xy}]=\lambda[F_0,w]+\{k,F\}+[F,w]+g_1,\end{aligned}$$

$$(2)\quad \delta_4 F_{xxxx}+2\delta_5 F_{xxyy}+\delta_6 F_{yyyy}=-\tfrac{1}{2}[w,w]+\{k,w\}+g_2.$$

In the system (1),(2) $\delta_i (i=1,2,\dots,6)$ are positive constants and $\alpha(x,y)$, $F_0(x,y)$, $k(x,y)$, $g_1(x,y)$, $g_2(x,y)$ are sufficiently regular real functions. $\lambda$ is a real parameter. Further, in (1),(2) we use the following notation

$$[\xi,\eta]=\xi_{xx}\eta_{yy}+\xi_{yy}\eta_{xx}-2\xi_{xy}\eta_{xy},\qquad \{\xi,\eta\}=\gamma_1\xi_{xx}\eta_{yy}+\gamma_2\xi_{yy}\eta_{xx}-2\gamma_3\xi_{xy}\eta_{xy},$$

where $\gamma_1,\gamma_2,\gamma_3$ are nonnegative constants.

The existence of the solution (1),(2) is considered in the space $W=H_1\times H_2$. The spaces $H_1,H_2$ are generated by the boundary conditions. Let the boundary conditions have such a form, that $H_i (i=1, 2)$ have the properties of the space $W_2^2$. Let $g_1\in H_1'$, $g_2\in H_2'$, where $H_i' (i=1,2)$ are dual spaces to $H_i$.

<u>Definition</u>. The pair $\langle w,F\rangle\in W$ will be called the weak solution of the system (1),(2) under given boundary conditions, if

$$\begin{aligned}(3)\quad \forall_{\varphi\in H_1}:\ &(w,\varphi)_{H_1}-\int_\Omega \alpha D(w,\varphi)\,d\Omega-\tfrac{1}{2}\int_\Omega \alpha[w,\varphi]\,d\Omega=\lambda([F_0,w],\varphi)_{L_2}+\\ &+(\{k,F\},\varphi)_{L_2}+([F,w],\varphi)_{L_2}+\langle g_1,\varphi\rangle_{H_1},\end{aligned}$$

$$(4)\quad \forall_{\psi\in H_2}:\ (F,\psi)_{H_2}=-\tfrac{1}{2}([w,w],\psi)_{L_2}-(\{k,w\},\psi)_{L_2}+\langle g_2,\psi\rangle_{H_2},$$

where $D(w,\varphi)=w_{xx}\varphi_{xx}+2w_{xy}\varphi_{xy}+w_{yy}\varphi_{yy}$,

$\langle g_1,\varphi\rangle_{H_1}$, $\langle g_2,\psi\rangle_{H_2}$ are dualities.

<u>Theorem 1</u>(topological method). Let exist such constants $C_1\geq 0$, $C_2>0$, that inequalities

$$(5)\quad \begin{aligned}&1> C_D\max_\Omega|\alpha|+\lambda C_F+C_{1H_1}\|g_2\|+C_1+C_2,\\ &C_k\leq 2\sqrt{2C_1}\end{aligned}$$

hold, where $C_D, C_F, C_{1H_1}, C_k$ are positive constants (as follows from the estimates in the proof). Let T is the operator generated by the form (3)+(4) (addition). Then there exists in W a solution of the equation

$$(T\langle w,F\rangle, \langle \varphi,\psi\rangle)_W = 0.$$

If the solution is unique in the bounded domain Q ($\theta \in Q$), then the Galerkin approximations converge in the norm W.

Theorem 2(variational method). Let $\gamma_1 = \gamma_2 = \gamma_3 = 1$. Let G is a operator generated by the form (3)-(4) (subtraction). Then G is a potential operator. If

$$(6) \quad 1 > C_D \max_{\Omega} |\alpha| + C_F \lambda + C_2,$$

then there exists in W a solution of the equation

$$(G\langle w,F\rangle, \langle \varphi,\psi\rangle)_W = 0$$

and the Ritz approximations represent the minimizing sequence.

Theorem 3(compactness method). Let inequalities (5) hold. Then there exists solution of the system (3),(4) and subsequence of the Galerkin approximations converges to the solution in the norm W. Proofs of the theorems are in [3].

Remark 1. The existence of the solution for equations of type [1] under the Dirichlet boundary conditions is considered in [4].

Theorem 4(bifurcation theorem). Let $g_1 \equiv g_2 \equiv 0, \gamma_1 = \gamma_2 = \gamma_3 = 1$. Let inequality (6) holds. $\lambda_0$ is the bifurcation point of the system (3), (4) if and only if it is the eigenvalue of the problem

$$\forall_{\varphi \in H_1}: (w,\varphi)_{H_1} - \int_\Omega \alpha [D(w,\varphi) - \tfrac{1}{2}[w,\varphi]] d\Omega - ([k,F],\varphi)_{L_2} = \lambda ([F_0,w],\varphi)_{L_2},$$

$$\forall_{\psi \in H_2}: (F,\psi)_{H_2} = -([k,w],\psi)_{L_2}.$$

Proof is given in [3].

It seems [5] that in the course of the numerical solution of the nonlinear problems of the type [1] by the Kačanov's method, the difficulties may arise due to the complexity of the process realisation. These difficulties can be removed by properly modifying the Kačanov's method [5]. Then we obtain a sequence of problems each of them being a special case of (1),(2). When solving these problems by Galerkin method, the original problem is converted to that one of finding solution to the systems of nonlinear algebraic equations of the form

$$(7) \quad \forall_i: (A_i - \sigma_c B_i) w_i + \sigma_c \lambda^2 B_i w_i = C_{ijk} w_j F_k + g_{1i},$$

$\forall_p : D_{pq}F_q = E_{pml}w_m w_l + g_{2p}$,

with $A_i, B_i, C_{ijk}, D_{pq}, E_{pml}, g_{1i}, g_{2p}, \delta_c$ being constants; $w_i$ and $F_q$ are unknowns and $\lambda$ is a parameter. The indices i,j,m,l take value from the one set and indices p,q,k (in general) from the other set of indices. Special cases of the systems (7) may be successfully solved by using a perturbation method [5], as it is shown in [6], where also the proper interpretation of the results is presented.

When using any approximate method it is convenient to have an aposteriori information on the solution. This information is of the utmost importance, when the use of an iterative method of the Kačanov's type for problems of form [1] is under consideration. In the case of linear potential problem, the Slobodianskij's approach [7] may be used provided proper conditions are satisfied. In a more general case the following procedure may be used [8]:

Let B is a real Banach space and $\Psi$ a functional defined on B. We are seeking

(8) $\min_B \Psi(u)$ .

Let the existence of minimum in (8) is guaranteed by the assumption on $\Psi$. Let us suppose further, that

$$\Psi(u) = \sum_{i=1}^{n} \Psi_i(u)$$

and that

$$\forall_i : \operatorname{grad} \Psi_i$$

exist. Let us denote

(9) $J = \sum_{i=1}^{n} \Psi_i(u_i) + \sum_{i=1}^{n-1} \langle \chi_i, u_{i-1} - u_1 \rangle$,

where $\chi_i \in B'$ and $\langle \, , \rangle$ is duality on B. Let the problem

$$\min_B J$$

is well defined. Let the necessary conditions of the minimization of individual functionals from J are also the sufficient ones.

<u>Theorem 5</u>. Let $u_i$, $\chi_i$ (i=1,2,...,n) fulfill the relations

$$\forall_{i>1} : \operatorname{grad} \Psi_i(u_i) + \chi_{i-1} = 0,$$

$$\operatorname{grad} \Psi_1(u_1) = \sum_{i=1}^{n-1} \chi_i .$$

Then $J \leq \min_B \Psi$ holds.

<u>Remark 2</u>. If the Slobodianskij's assumptions are satisfied, then

our construction of the lower estimate is identical to that one of Slobodianskij.

Remark 3. Let in (9) $n=2$ and $\Psi_1$ is a quadratic functional. Then

$$\Psi - J = \Psi_1(u_2 - u_1),$$

where $u_2$ is an arbitrary function from B.

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