ON SOME SINGULAR BOUNDARY VALUE PROBLEMS FOR ORDINARY DIFFERENTIAL EQUATIONS

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In the present paper for the differential equation

$$u^{(n)} = f(t,u,u',\dots,u^{(n-1)}) \tag{1}$$

the following boundary value problems

$$u^{(i-1)}(0) = \varphi_{1i}(u(0),\dots,u^{(n-1)}(0)),\ u^{(j-1)}(a) = \varphi_{2j}(u(a),\dots,u^{(n-1)}(a)) \quad (i=1,\dots,n_0;\ j=1,\dots,n-n_0) \tag{2}$$

and

$$u^{(i-1)}(0) = \varphi_i(u(0),\dots,u^{(n-1)}(0)) \ (i=1,\dots,n_0),\ \int_0^{+\infty} t^{n-2n_0+1}[u^{(n_0)}(t)]^2 dt < +\infty \tag{3}$$

are considered.

These problems arose in connection with an attempt to answer two open questions of the qualitative theory of nonautonomous ordinary differential equations. The first of them deals with the existence of vanishing at infinity nonzero solutions of the linear differential equation

$$u^{(n)} = h(t)u \tag{4}$$

(see [1], [2]), and the second one - with the existence of so-called proper solutions of nonlinear differential equations with strongly increasing right-hand sides, which is e. g. the Emden-Fowler equation

$$u^{(n)} = h(t)\,|u|^{\lambda}\operatorname{sign} u \tag{5}$$

(see [3]).

Below denote by R^K the K - dimensional real Euclidean space, and by R_+ - the interval $[0,+\infty[$. In what follows it is assumed that $n \geq 2$, n_0 is the entire part of $\frac{n}{2}$, and

$\varphi_i: R^n \to R$, $\varphi_{1i}: R^n \to R$ and $\varphi_{2j}: R^n \to R$ $(i=1,\dots,n_0; j=1,\dots,n-n_0)$

are continuous functions, satisfying the conditions

$$\sup\{(1+|x_1|+\ldots+|x_n|)\,|\varphi_i(x_1,\ldots,x_n)|:(x_1,\ldots,x_n)\in R^n\}<+\infty;$$

$$\sup\{(1+|x_1|+\ldots+|x_n|)\,|\varphi_{1i}(x_1,\ldots,x_n)|:(x_1,\ldots,x_n)\in R^n\}<+\infty$$

and

$$\sup\{(1+|x_1|+\ldots+|x_n|)\,|\varphi_{2j}(x_1,\ldots,x_n)|:(x_1,\ldots,x_n)\in R^n\}<+\infty.$$

Consider first the problem on the finite interval, i. e. the problem (1), (2).

T h e o r e m 1. Let the function $f:[0,a]\times R^n\to R$ be continuous and

$$-h_1(t)\leq(-1)^{n-n_0-1}f(t,x_1,\ldots,x_n)\operatorname{sign}x_1\leq h_2(t,x_1,\ldots,x_{n_0}) \qquad (6)$$

where $h_1:[0,a]\to R_+$ and $h_2:[0,a]\times R^{n_0}\to R_+$ are continuous functions. Then the problem (1),(2) is solvable.

In the boundary conditions have the form

$$u^{(i-1)}(0)=0,\quad u^{(j-1)}(a)=0\quad (i=1,\ldots,n_0;\ j=1,\ldots,n-n_0) \qquad (2_0)$$

then it is possible to prove the existence theorem containing the case when the function $f:]0,a[\times R^n\to R$ has nonintegrable singularities when $t=0$ and $t=a$. In this case the solution of the problem (1),(2_0) will be sought in the class of n-times continuously differentiable functions $u:]0,a[\to R^n$ and under $u^{(i-1)}(0)$ and $u^{(j-1)}(a)$ we will mean

$$\lim_{t\to 0_+}u^{(i-1)}(t) \quad \text{and} \quad \lim_{t\to a_-}u^{(j-1)}(t).$$

T h e o r e m 2 . Let the function $f:]0,a[\times R^n\to R$ be continuous and satisfy the inequalities (6) where $h_1:]0,a[\to R_+$ is continuous and bounded, $h_2:]0,a[\to R_+$ is continuous, besides if $n=2n_0+1$ for any $\tau\in R_+$ we have

$$\sup\{(a-t)^{n_0}h_2(t,(a-t)^{n_0-1/2}x_1,\ldots,(a-t)^{1/2}x_{n_0}):|x_1|+\ldots+|x_{n_0}|\leq\tau\}<+\infty$$. Then the problem (1),(2) is solvable.

For the problem on the infinite interval, i. e. problem (1), (3) the following statements hold true.

T h e o r e m 3 . Let the function $f: R_+ \times R^n \to R$ be continuous and satisfy the inequalities (6) where $h_1 : R_+ \to R_+$ and $h_2 : R_+ \times R^{n_0} \to R_+$ are continuous functions and

$$\int_0^{+\infty} t^{n-n_0} \left[1+(2n_0+1-n)\ln^{1/2}(1+t) \right] h_1(t)dt < +\infty .$$

Then the problem (1),(3) is solvable.

T h e o r e m 4 . Let the function f be continuous and satisfy the inequalities

$$h_0(t)|x_1|^{\lambda} - h_1(t) \le (-1)^{n-n_0-1} f(t, x_1, \dots, x_n) \operatorname{sign} x_1 \le$$

$$h_2(t, x_1, \dots, x_{n_0})$$ where $\lambda \ge 1$, $h_k : R_+ \to R_+$ $(k=0,1)$ and $h_2 : R_+ \times R^{n_0} \to R_+$ are continuous functions,

$$\int_0^{+\infty} t^{n-1/2} h_1(t)dt < +\infty$$ and for a certain $\varepsilon > 0$

$$\lim_{t\to+\infty} t^{n-\frac{\lambda-1}{2}(1+\varepsilon)} h_0(t) = +\infty .$$

Then the problem (1),(3) has a solution u such that

$$\lim_{t\to+\infty} t^{1/2+i} u^{(i)}(t) = 0 \quad (i=0,1,\dots,n_0-1). \tag{7}$$

Let the function $f: R_+ \times R^n \to R$ be continuous and $f(t,0,\dots 0) = 0$. A solution u of the equation (1) defined in a certain neighborhood of $+\infty$ is said to be a p r o p e r one if for any sufficiently large t

$$\sup\{|u(s)| : t \le s < +\infty\} > 0.$$

A proper solution is said to be an o s c i l l a t o r y one if it has a sequence of zeros tending to $+\infty$.

By means of Theorems 3 and 4 the conditions of the existence of proper oscillatory solutions of the equation (1) and of a n_0 - dimensional family of vanishing at the infinity solutions of the equation (4) are derived. By this the answer is given to those two questions which were posed above.

For any $\lambda \neq 1$ set

$$\mu(n,\lambda) = n - 1 + \frac{1-(-1)^{n_0}}{2}(\lambda - 1) \qquad \text{for } \lambda > 1 ,$$

$$\mu(n,\lambda) = n - n_0 + (n_0-1)\lambda \qquad \text{for } \lambda < 1 .$$

Theorem 5. Let $n \geq 4$ and let the function f satisfy the inequalities

$$h_0(t)\omega(x_1) \leq (-1)^{n-n_0-1} f(t,x_1,\dots,x_n)\operatorname{sign} x_1 \leq h(t,x_1,\dots,x_{n_0})$$

where $\omega : R \to R_+$, $h_0 : R_+ \to R_+$ and $h : R_+ \times R^{n_0} \to R_+$ are continuous functions

$$\omega(x) > 0 \quad \text{for} \quad x \neq 0, \ \liminf_{|x| \to +\infty} \frac{\omega(x)}{|x|^{\lambda}} > 0 , \ \lambda \neq 1$$

and

$$\int_1^{+\infty} t^{\mu(n,\lambda)} h_0(t)dt = +\infty .$$

Let in addition the Cauchy problem for the equation (1) with the zero initial conditions have only the zero solution. Then there exists a continuum of proper oscillatory solutions of this equation.

Corollary. If $n \geq 4$, $\lambda > 1$

$$(-1)^{n-n_0-1} h(t) \geq 0 \quad \text{for} \quad t \in R_+ \quad \text{and} \int_0^{+\infty} t^{\mu(n,\lambda)} |h(t)| dt = +\infty ,$$

then the equation (5) has a continuum of proper oscillatory solutions.

Consider the equation (4) with a continuous coefficient

$h: R_+ \to R$.

Denote by $U_h^{n_o}$ the set of all solutions of this equation satisfying the condition

$$\int_0^{+\infty} \left[u^{(n_o)}(t)\right]^2 dt < +\infty \quad .$$

T h e o r e m 6 . If $(-1)^{n-n_o-1} h(t) \geq 0$ for $t \in R_+$, then $U_h^{n_o}$ is a n_o - dimensional linear space. Moreover for any $t_o \in R_+$ and $c_i \in R$ $(i-1, \dots, n_o)$ there exists the unique $u \in U_h^{n_o}$ satisfying the initial conditions

$$u^{(i-1)}(t_o) = c_i \qquad (i=1, \dots, n_o) \quad .$$

T h e o r e m 7 . If

$$\lim_{t \to +\infty} (-1)^{n-n_o-1} t^n h(t) = +\infty$$

then $U_h^{n_o}$ is a n_o - dimensional linear space whose arbitrary element satisfies the conditions (7) .

R e f e r e n c e s

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