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In: Michal Greguš (ed.): Equadiff 5, Proceedings of the Fifth Czechoslovak Conference on Differential Equations and Their Applications held in Bratislava, August 24-28, 1981. BSB B.G. Teubner Verlagsgesellschaft, Leipzig, 1982. Teubner-Texte zur Mathematik, Bd. 47. pp. 136--139.

Persistent URL: http://dml.cz/dmlcz/702277

THREE APPLICATIONS OF DIFFERENTIAL EQUATIONS IN TURBULENCE RESEARCH

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Assuming a homogeneous, incompressible, isothermal and Newtonian medium it is common use to deal with turbulent fluid flows on the basis of the NAVIER-STOKES-equations and the continuity condition. Unfortunately, with respect to practically interesting high REYNOLDS number problems for the time being this can not be done in a rigorous manner. Because of the very complicated turbulent flow structures some additional modelling must be introduced. In the following three examples taken from the work at the Institute for Mechanics at the Academy of Sciences of the GDR are summarized.

First of all the axisymmetric stationary and strongly swirling flow in a vaneless radial diffusor shall be considered. Approximate, but analytic expressions for the mean velocities and the two controlling components of the REYNOLDS stress tensor in the core region are wanted. For the modelling a plane turbulent vortex source is defined as a kind of basic flow. Introducing

$$\left.\begin{aligned} \overline{c_i}(r,z) &= \hat{c}_i(r) + \Delta\hat{c}_i(r,z) \,, \\ \overline{c_i' c_j'}(r,z) &= \widehat{c_i' c_j'}(r) + \Delta\widehat{c_i' c_j'}(r,z) \,, \end{aligned}\right\} (i,j = r,\varphi,z) \tag{1}$$

and assuming

$$\hat{c}_\varphi(r) \gg \hat{c}_r(r) \,;\, \Delta\hat{c}_r(r,z) \,;\, \Delta\hat{c}_\varphi(r,z) \tag{2}$$

the equations of motion can be linearized. Applying, furthermore, the usual boundary layer simplifications to $\Delta\hat{c}_i \,;\, \Delta\widehat{c_i' c_j'}$ it is possible to integrate the generated system of partial differential equations in a closed manner.

We obtain for instance

$$\Delta\hat{c}_r(r,z) = A_1(r)\cos(\lambda z) + A_2(r)\,\mathrm{ch}(\lambda z) + A_3(r)\sin(\lambda z) + A_4(r)\,\mathrm{sh}(\lambda z) \,, \tag{3}$$

and similar relations for $\Delta\hat{C}_\varphi$; $\Delta\hat{C}_z$; $\Delta\widehat{C_r'C_z'}$; $\Delta\widehat{C_\varphi' C_z'}$ with $\lambda = \lambda(r)$ provided that the basic flow is known. For that a simple description is given by

$$r h \hat{C}_r = K_Q \quad ; \quad \hat{C}_\varphi = D_1 r^{D_2} \; . \tag{4}$$

The constants K_Q ; D_1 ; D_2 can be taken from the inlet flow conditions. The unknown functions $A_p(r)$ $(p = 1, 2, 3, 4)$ created by the integration process must be determined by certain integral conditions and by the matching conditions with the wall layers. Comparisons with measurements show the utility of the found relations.
(Notation: r, φ, z - cylindrical co-ordinates; C_i - velocity component; $\overline{(\;)}$ - time mean value of the actual flow; $\hat{(\;)}$ - time mean value of the vortex source; $(\;)'$ - fluctuating quantity; $\Delta\hat{(\;)}$ - time mean value of the superposed flow; h - half channel width).

The second example is devoted to the interaction between the disturbed core flow and the boundary layer in the inlet region of a pipe flow. The axial momentum equation takes the simplified form

$$\bar{C}_z \frac{\partial \bar{C}_z}{\partial z} + \bar{C}_r \frac{\partial \bar{C}_z}{\partial r} = -\frac{1}{\rho}\frac{\partial \bar{p}}{\partial z} + \frac{1}{r}\frac{\partial}{\partial r}\left[r(\nu+\nu_t)\frac{\partial \bar{C}_z}{\partial r}\right] \tag{5}$$

and the continuity condition yields

$$\frac{\partial \bar{C}_z}{\partial z} + \frac{1}{r}\frac{\partial (r\bar{C}_r)}{\partial r} = 0. \tag{6}$$

(p - static pressure, ν - kinematic viscosity, ν_t - eddy viscosity, ρ - density).
At the pipe wall the boundary conditions $\bar{C}_z = \bar{C}_r = 0$ hold and at the inlet section a starting profile is given. LAUNDER and SPALDING developed an additional system of transport equations to determine the eddy viscosity ν_t . In that way the entire problem can be based on the iterative numerical solution of the following symbolic equation for ϕ :

$$\bar{C}_z^{(m)} \frac{\partial \phi^{(m+1)}}{\partial z} + \bar{C}_r^{(m)} \frac{\partial \phi^{(m+1)}}{\partial r} = \frac{1}{r}\frac{\partial}{\partial r}\left[r B^{(m)} \frac{\partial \phi^{(m+1)}}{\partial r}\right] + S^{(m)} \tag{7}$$

Since $\overline{C}_z^{(m)}$ and $B^{(m)}$ are strongly positive, (7) is parabolic and can be solved by a marching procedure. For practical calculations a finite difference scheme exploiting upstream difference approximation formulas was adapted. The static pressure follows from the condition, that the overall mass-flow rate must be the same at each cross section of the pipe. The numerical results confirm typical properties of disturbed and undisturbed pipe inlet flows detected by measurements.

The third application contains comparisons of various closure hypotheses for the time dependent decay of homogeneous isotropic turbulence. The dynamical equation can be written as

$$\frac{\partial E(k,t)}{\partial t} = - \frac{\partial W(k,t)}{\partial k} - 2\nu k^2 E(k,t). \tag{8}$$

(E - energy spectrum, W - energy transfer function, k - wavenumber, t - time). In (8) the energy spectrum as well as the transfer term are unknown functions. Proposals providing a connection between E and W were made - among others - by HEISENBERG, OBUKHOFF, MIJAKODA/OGURA, and KOVASZNAY. To analyse the properties of these four mentioned hypotheses the dynamical equation was solved as an initial value problem with the aid of numerical procedures. Two types of equations appear, namely an integro - differential equation and a quasi-linear first-order differential equation. An implicit finite difference scheme was applied to the first one and the RUNGE-KUTTA-method for solving the characteristic system to the second one. By means of these methods numerical computations of energy spectra were carried through for different initial spectra and different closures. Among others, it was detected that the application of the OBUKHOFF hypothesis to an initial spectrum, already evaluated by TOLLMIEN on the basis of the HEISENBERG closure yields results which contradict previous findings.

Details on the considered examples are described in

[1] M. Hoffmeister:
Modell einer unsymmetrischen Radialdiffusorströmung mit starker Umfangskomponente.
Bericht des Zentralinstituts für Mathematik und Mechanik bei der Akademie der Wissenschaften der DDR (Dezember 1979, unveröffentlicht).

[2] D. Ullrich:
Theoretische Untersuchung von Zweiparametermodellen der Turbulenz und ihre Anwendung auf Rohreinlaufströmungen.
Dissertation, Akademie der Wissenschaften der DDR, eingereicht 1981.

[3] B. Dreßler:
Theoretische und numerische Untersuchungen zum zeitlichen Verhalten des Energiespektrums in der homogenen isotropen Turbulenz.
Dissertation, Akademie der Wissenschaften der DDR, 1980.

Acknowledgement: The authors thank Dr. G. Seifert for helpful discussions in the course of the work.