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SINGULAR PERTURBATIONS WITH SEVERAL PARAMETERS

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1. Consider a system of the form

$$\varepsilon_k \frac{dy_k}{dt} = \sum_{j=0}^{N} A_{kj}(t) y_j \ , \quad k=0,1,\dots N$$

$$\varepsilon_0 = 1 \ , \quad \varepsilon_k > 0, \quad \lim_{\varepsilon \to 0} \frac{\varepsilon_{k+1}}{\varepsilon_k} = 0, \quad A_{kj} : I \longrightarrow \mathcal{L}(R^{n_j}, R^{n_k}), I \subset R;$$

the interesting case will be $I=(t_0,\infty)$.

Assume $A_{NN}(t)$ invertible for all $t \in I$, denote $A_{kj} = A_{kj}^N$, define $A_{kj}^{N-1} = A_{kj}^N - A_{kN}^N (A_{N,N}^N)^{-1} A_{Nj}^N$, $k,j=0,1,\dots,N-1$. Assume inductively $A_{\ell\ell}^{\ell}(t)$ invertible and define $A_{kj}^{\ell-1} = A_{kj}^{\ell} - A_{k\ell}^{\ell} (A_{\ell\ell}^{\ell})^{-1} A_{\ell j}^{\ell}$, $k,j=0,1,\dots,\ell-1$.
Let $\Gamma_k(t,s,\varepsilon)$, $k=0,1,\dots,N$ be the block columns of the fundamental matrix of the system.

THEOREM 1. Assume: a) $t \mapsto A_{kj}(t)$ are uniformly Lipschitz and uniformly bounded on I; b) $\mathrm{Re}\left[\sigma(A_{\ell\ell}^{\ell}(t))\right] \le -2\alpha_\ell < 0$ for all $t \in I$ $\ell = 1,\dots,N$; c) $|C_0(t,s)| \le c_0 e^{-2\alpha_0(t-s)}$; $C_0(t,s)$ is the fundamental matrix of the reduced system $\dot{y}_0 = A_{00}^0(t) y_0$.

Then for sufficiently small ε we have

$$| \Gamma_k(t,s,\varepsilon) | \le c \varepsilon_k \left[e^{-\alpha_0(t-s)} + \sum_{\ell=1}^{k} \frac{1}{\varepsilon_\ell} e^{-\alpha_\ell \frac{t-s}{\varepsilon_\ell}} \right]$$

for all $t \ge s$; c does not depend upon I if $\alpha_0 > 0$.

For the proof see [1].

We want to consider now the Cauchy problem

$$\varepsilon_k \frac{dy_k}{dt} = \sum_{j=0}^{N} A_{kj}(t) y_j \ , \quad y_k(t_0,\varepsilon) = \frac{1}{\varepsilon_k} y_k^0 .$$

For each $0 \le p \le N-1$ define reccurrently $y_k^{p,0} = y_k^{p+1,0} - A_{k,p+1}^{p+1}(t_0)(A_{p+1,p+1}^{p+1}(t_0))^{-1} y_{p+1}^{p+1,0}$, $k=0,1,\dots,p$, $y_k^{N,0} = y_k^0$, $k=0,1,\dots,N$.

Let $\tilde{y}(t,\varepsilon)$ be the solution of the Cauchy problem

$$\varepsilon_k \frac{dy_k}{dt} = \sum_{j=0}^{k} A_{kj}^k(t) y_j \ , \quad \tilde{y}_k(t_0,\varepsilon) = \frac{1}{\varepsilon_k} y_k^{k,0}$$

$k=0,1,\dots,N.$

THEOREM 2. Under the same assumptions as in Theorem 1

$$y_0(t,\varepsilon)=C_0(t,t_0)y_0^{0,0}+\sum_{\ell=1}^{N} A_{0,\ell}^{\ell}(t_0)(A_{\ell,\ell}^{\ell}(t_0))^{-1}C_\ell(t,t_0,\varepsilon)y_\ell^{\ell,0}+$$
$$+\sum_{\ell=1}^{N}\varepsilon_\ell w_0^{\ell-1}(t,\varepsilon)$$

$$y_k(t,\varepsilon)=\tilde{y}_k(t,\varepsilon)+\frac{1}{\varepsilon_k}\sum_{\ell=k+1}^{N} A_{k,\ell}^{\ell}(t_0)(A_{\ell,\ell}^{\ell}(t_0))^{-1}C_\ell(t,t_0,\varepsilon)y_\ell^{\ell,0}+$$
$$+\tilde{w}_k(t,\varepsilon)+\sum_{\ell=k}^{N}\varepsilon_\ell w_k^{\ell-1}(t,\varepsilon)$$

$$|\tilde{w}_k(t,\varepsilon)|\le\tilde{c}_k e^{-\frac{\alpha_k(t-t_0)}{2\varepsilon_k}}\;,\;|w_k^j(t,\varepsilon)|\le c_j\rho_j(t,t_0,\varepsilon)$$

$$\rho_j(t,t_0,\varepsilon)=\frac{1}{\varepsilon_j}\Big[\sum_{m,q=0}^{j}\frac{1}{\varepsilon_q\varepsilon_m}\int_{t_0}^{t}e^{-\frac{\alpha_m}{\varepsilon_m}(t-s)}\big(e^{-\frac{\alpha_q}{\varepsilon_q}(s-t_0)}-e^{-\frac{\alpha_{j+1}}{\varepsilon_{j+1}}(s-t_0)}\big)ds$$
$$+\sum_{q=0}^{j}\frac{1}{\varepsilon_q}\big(e^{-\frac{\alpha_q}{\varepsilon_q}(t-t_0)}-e^{-\frac{\alpha_{j+1}}{\varepsilon_{j+1}}(t-t_0)}\big)\Big];$$

C_ℓ is the fundamental matrix associated with $\frac{1}{\varepsilon_\ell}A_{\ell\ell}^{\ell}$.

Remark that the problem considered may be obtained from

$$\varepsilon_N\frac{dx_k}{dt}=\sum_{j=0}^{N}\frac{\varepsilon_N}{\varepsilon_j}A_{kj}(t)x_j\;,\quad x_j(t_0)=y_j^0$$

by taking $y_j=\frac{1}{\varepsilon_j}x_j$.

For N=1 such problem corresponds to the critical case considered by Vasilieva in 1975 [2].

The estimates for ρ_j give the boundary layer behavior if $t>t_0$ and show ρ_j are zero for $t=t_0$.

3. We consider now systems that are nonlinear with respect to the slow variables. The stimulating example describes the dynamics of a synchronous machine [3]

$$\frac{d\delta}{dt}=\omega s$$

$$T\frac{ds}{dt}+Ds=\frac{C_{mec}}{C_{mec_n}}-\frac{1}{P_n}\left[\frac{e_d''U\cos\delta}{x_q''}+\frac{e_q''U\sin\delta}{x_d''}-\left(\frac{1}{x_q''}-\frac{1}{x_d''}\right)\frac{U^2}{2}\sin 2\delta\right]$$

$$T_{d_0}'\frac{de_q'}{dt}=e_f-e_q'\frac{x_d-x_d''}{x_d'-x_d''}+e_q''\frac{x_d-x_d'}{x_d'-x_d''}$$

$$T_{d_0}''\frac{de_q''}{dt}=e_q'-\frac{x_d'}{x_d''}e_q''+\frac{x_d'-x_d''}{x_d''}U\cos\delta$$

$$T_{q_0}''\frac{de_d''}{dt}=-\frac{x_q}{x_q''}e_d''-\frac{x_q-x_q''}{x_q''}U\sin\delta$$

Here the slow variables are δ and s ; e''_q and e''_d are faster than e'_q .

The general form is

$$\varepsilon_k \frac{dy_k}{dt} = \sum_{\ell=1}^{N} A_{k\ell}(y_0) y_\ell + a_k(y_0) \ , \ k = 0, 1, \ldots, N.$$

Define reccurrently $a_k^p(y_0)$ and $A_{k\ell}^p(y_0)$ by

$$a_k^p(y_0) = a_k^{p+1}(y_0) - A_{k,p+1}^{p+1}(y_0)(A_{p+1,p+1}^{p+1}(y_0))^{-1} a_{p+1}^{p+1}(y_0)$$

$$A_{k\ell}^p(y_0) = A_{k,\ell}^{p+1}(y_0) - A_{k,p+1}^{p+1}(y_0)(A_{p+1,p+1}^{p+1}(y_0))^{-1} A_{p+1,\ell}^{p+1}(y_0)$$

$a_k^N(y_0) = a_k(y_0)$, $A_{k\ell}^N(y_0) = A_{k\ell}(y_0)$;assume $\operatorname{Re}\sigma(A_{pp}^p(y_0)) \leq -2\alpha_p < 0$ for all y_0 .Assume also that the reduced system $\dot{y}_0 = a_0^0(y_0)$ admits a constant solution $\hat{y}_0$ such that $\frac{\partial a_0^0}{\partial y_0}(\hat{y}_0)$ is Hurwitz.Let y_0^0 be in a compact contained in the domain of attraction of $\hat{y}_0$ and let $\tilde{y}_0$ be the corresponding solution of the reduced system.Define $\tilde{y}_\ell$ from the problem

$$\varepsilon_\ell \frac{dy_\ell}{dt} = a_\ell^\ell(\tilde{y}_0(t)) + \sum_{j=1}^{\ell} A_{\ell j}^\ell(\tilde{y}_0(t)) y_j \ , \ y_\ell(t_0) = y_\ell^0$$

where y_ℓ^0 belongs to a compact $M_\ell \subset R^{n_\ell}$.

Then $y_k(t,\varepsilon) = \tilde{y}_k(t,\varepsilon) + \eta z_k(t,\varepsilon)$, $|z_k(t,\varepsilon)| \leq c e^{-\alpha(t-t_0)}$ for $|\varepsilon| < \varepsilon_0$ where c and ε_0 depend upon the diameters of the compacts M_k .

It is proved also that

$$|\tilde{y}_\ell(t,\varepsilon) - \hat{y}_\ell(t)| \leq \sum_{j=1}^{\ell} c_j e^{-\frac{\alpha_j}{\varepsilon_j}(t-t_0)} |\tilde{y}_j(t_0,\varepsilon) - \hat{y}_j(t_0)| + c_0 \varepsilon_1 e^{-\hat{\alpha}_0(t-t_0)} |y_0^0 - \hat{y}_0| \ , \ t \geq t_0 ,$$

where $\hat{y}_\ell$ are defined by

$$\hat{y}_\ell(t) = -[A_{\ell\ell}^\ell(\tilde{y}_0(t))]^{-1}[a_\ell^\ell(\tilde{y}_0(t)) + \sum_{j=1}^{\ell} A_{\ell j}^\ell(\tilde{y}_0(t))\hat{y}_j(t)].$$

It is easy to see that all assumptions are satisfied in the case of a synchronous machine.

Remark that systems linear with respect to the fast variables have been considered also by Chow [4] who studied the stability by using Liapunov functions.

References

1. V.Dragan , A.Halanay: Singular perturbations with several parameters.
Preprint series in mathematics, INCREST,Bucharest,38/1980.

2. A.B.Vasilieva: Singularly perturbed systems containing indeterminacy when degenerating (Russian).
VII. Internationale Konferenz über Nichtlineare Schwingungen
8-13. September 1975 in Berlin, Band I,2, 355-362.
Abhandlungen der Akad. der Wissenschaften der DDR
Abt. Math. ,Naturwissenschaften,Technik, Nr. 4N, 1977
Akademie Verlag, Berlin.

3. A.Halanay: Stability problems for synchronous machines
Ibidem, Band II,1 407-421.

4. J.H.Chow: Asymptotic stability of a class of nonlinear singularly perturbed systems.
Journal of the Franklin Inst. vol. 305, (1978) , 275-282.