MATRIX ANALYSIS OF CERTAIN DYNAMICAL SYSTEMS IN TECHNICS

(at linear, nonlinear, or random diff. and int. equations)

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The eigenvalue problem for a complex system of vibrating bars (turning axles) is chosen from ones treated in the lecture to be investigated here at last skitchily. Marguerre's, Egerváry's and author's papers [1-4] and their literatures may be recommended for more details. - The whole problematics of the lecture can be soon read in [5].

The eigenvalue problem of a vibrating (turning) system

1. As well known, the scalar differential equation (DE) of the unloaded vibrating bar (model) and one of the unloaded turning axle (model) too (namely in first approximation, with negligation of certain additional effects of turning [3]), then its equivalent vector form [5] is as follows:

$$\left.\begin{array}{l} y^{IV}-k^4 y = 0 \\ y(t_0)=y_0,\ldots,y'''(t_0)=y_0''' \\ y_0(t)=? \qquad y = \underline{e}^1\underline{y} \end{array}\right\| \begin{array}{l} \dot{\underline{y}} + \underline{\underline{P}}\,\underline{y} = \underline{0} \\ \underline{y}_0(t_0) = \underline{y}_0 \\ \underline{y}_0(t) = ? \end{array} \quad \left\langle \begin{array}{l} 0\leq t_0\leq t\leq 1;\ \underline{y}=[y,y',y'',y''']^*; \\ 0<k^4=\dfrac{\lambda^4}{1^4}=\dfrac{\mu\omega^2}{EI}=? \\ \underline{\underline{P}}=-\sum_{j=1}^{3}\underline{e}_j\underline{e}^{j+1}+k^4\underline{e}_4\,\underline{e}^4 \end{array}\right\rangle. \qquad (1\text{-}2)$$

Consequently, it can be treated as a linear, time-invariant system got in the state space $E_4=\{\underline{y}\}$.

A basic (solving functions) system of the DE (1a), then its Wronski-matrix and -determinant follow here (4,5):

$$\underline{y}^{(0)} = [\mathrm{chkt}, \mathrm{shkt}, \mathrm{coskt}, \mathrm{sinkt}] \equiv [c,\ s,\ C,\ S]_t \quad (3)$$

$$\underline{\underline{Y}}(t) \equiv [\underline{y}^{(i)}(t)] = \begin{bmatrix} c & s & C & S \\ ks & kc & -kS & kC \\ k^2c & k^2s & -k^2C & -k^2S \\ k^3s & k^3c & -k^3S & -k^3C \end{bmatrix}_t \quad (4)$$

$$(5)\quad Y(t) \equiv |\underline{\underline{Y}}(t)| \equiv \det \underline{\underline{Y}}(t) = 4k^6 > 0$$

By our matrix algorithm [5] DTA_{q+1}: $\underset{(D)}{\underline{\underline{Y}}}_{q+1} = \underset{(D)}{\underline{\underline{Y}}}_q - \frac{1}{y^{(q)}_{k_q k_q}} (\underline{y}_{k_q} - \underline{e}_{k_q})(\underline{y}^{k_q} + \underline{e}^{k_q})$

$$\langle q=0,1,2,3;\ k_q \in \{1,2,3,4\},\ k_q \neq k_p;\ \forall\, y^{(q)}_{k_q k_q} \neq 0;\ \underline{\underline{Y}}_0 = \underline{\underline{Y}}(t_0),\ \underline{\underline{Y}}_4 = \underline{\underline{Y}}^{-1}(t_0)\rangle \quad (6)$$

from $\underline{\underline{Y}}(t_0)$ its inverse $\underline{\underline{Y}}^{-1}(t_0)$, then by multiplication the Green -type matrix [4,5] $\bar{\underline{\underline{Y}}}(t-t_0) = \underline{\underline{Y}}(t)\underline{\underline{Y}}^{-1}(t_0)$ can be counted. With it, the <u>general solution</u> and the $\underline{y}_0$-conditioned <u>particular one</u> are the followings: (7)

$$\underline{y}(t) = \bar{\underline{\underline{Y}}}(t)\underline{c} = \bar{\underline{\underline{Y}}}(t-t_0)\underline{k}, \quad \underline{y}_0(t) = \bar{\underline{\underline{Y}}}(t-t_0)\underline{y}_0 \quad \langle \underline{y}_0(t_0) = \bar{\underline{\underline{Y}}}(0)\underline{y}_0 = \underline{\underline{E}}\underline{y}_0 = \underline{y}_0\rangle$$

– The former (geometrical) figurations will be transformed – advantageously – into their <u>technical variants</u> [1,3] (8a–e)

$$\underline{z} \equiv [y, l\varphi, -Ml^2/C, -Vl^3/C] = [ly, ly', l^2y'', l^3y'''] = \langle 1, l, l^2, l^3\rangle \underline{y} \equiv \underline{\underline{L}}\underline{y},$$

$$\underline{\underline{Z}}(t) = \underline{\underline{L}}\,\underline{\underline{Y}}(t) \equiv [l^k y_j^{\ k}(t)], \quad Z(t) \equiv |\underline{\underline{Z}}(t)| = |\underline{\underline{L}}|\,|\underline{\underline{Y}}(t)| = l^6 Y(t) = 4k^6 l^6 = 4\lambda^6 > 0$$

$$\bar{\underline{\underline{Z}}}(t-0) = \underline{\underline{Z}}(t)\underline{\underline{Z}}^{-1}(0) = \underline{\underline{L}}\,\underline{\underline{Y}}(t)\underline{\underline{Y}}^{-1}(0)\underline{\underline{L}}^{-1} = \underline{\underline{L}}\,\bar{\underline{\underline{Y}}}(t-0)\underline{\underline{L}}^{-1} = \begin{bmatrix} \Gamma_0 & \Sigma_1 & \Gamma_2 & \Sigma_3 \\ \lambda^4\Sigma_3 & \Gamma_0 & \Sigma_1 & \Gamma_2 \\ \lambda^4\Gamma_2 & \lambda^4\Sigma_3 & \Gamma_0 & \Sigma_1 \\ \lambda^4\Sigma_1 & \lambda^4\Gamma_2 & \lambda^4\Sigma_3 & \Gamma_0 \end{bmatrix}_t,$$

where $\Gamma_0 = \frac{1}{2}(c+C)_t$, $\Sigma_1 = \frac{1}{2\lambda}(s+S)_t$ $\left(\begin{matrix}\text{at } t=l \\ kt = \frac{\lambda}{l}l = \lambda\end{matrix}\right)$

$\Gamma_2 = \frac{1}{2\lambda^2}(c-C)_t$, $\Sigma_3 = \frac{1}{2\lambda^3}(s-S)_t$

<u>The transit of the technical state vector</u> $\underline{z}(t)$ from a $t_0 \in T$ to a $t \in T$, or just from $t_0 = 0$ to $t = l$ is given by the linear transform [1,4] (9a,b)

$$\underline{z}_t \equiv \underline{z}(t) = \bar{\underline{\underline{Z}}}(t-t_0)\underline{z}(t_0) \equiv \bar{\underline{\underline{Z}}}_t\underline{z}_0, \quad \text{or} \quad \underline{z}_1 \equiv \underline{z}(1) = \bar{\underline{\underline{Z}}}(1)\underline{z}(0) \equiv \bar{\underline{\underline{Z}}}_1\underline{z}_0 \quad \langle \underline{z}_0 = \underline{\underline{L}}\,\underline{y}\rangle$$

with values of the technical $\underline{z}(t)$ conditioned with arbitrary $\underline{z}_0$ (as well with $\underline{z}_1$ too). – <u>E.g.</u> at the <u>boundary conditions</u> $\underline{z}_0 = [z, 0, z_3, 0]^*_0$ and $\underline{z}_1 = [z_1, 0, z_3, 0]^*_1$, the <u>homogeneous part</u> of (9b) and its <u>determinant</u> (=0 for non-trivial solutions) are formed so: ... ... (10a) (10b,c)

$$\underline{z}_{,1} \equiv \begin{bmatrix} 0 \\ 0 \end{bmatrix}_1 = \begin{bmatrix} \Gamma_0 & \Gamma_2 \\ \lambda^4\Gamma_2 & \Gamma_0 \end{bmatrix} \begin{bmatrix} z_1 \\ z_3 \end{bmatrix}_0 \equiv \bar{\underline{\underline{Z}}}_{,1}\underline{z}_{,0}, \quad Z_{,1}(\lambda) = \Gamma_0^2 - \lambda^4\Gamma_2^2 \equiv \mathrm{ch}\lambda\cos\lambda = 0,$$

(10d)

which last one furnishes the <u>eigen values</u> $\lambda_k = (2k+1)\pi/2$ $(k = 0, 1, \dots)$

2. Now let be treated a <u>symmetrical system of unloaded turning axles</u> (vibrating bars), whose (left) <u>half</u> consists of a) a <u>massive rotor</u> of constants $l_0, \mu_0, EI_0 = C_0$ $(t_1 - 0 = t'_0 \geq t \geq t_0 = 0,\ i = 0)$ and of b) <u>unmassive axle stubs</u> of constants $l_i, \mu_i, EI_i = C_i$ $(t_{i+1} - 0 = t'_i \geq t \geq t_i;\ i = 1, 2, \dots, n)$ connected serially into the (left) half of a sole elastic axle (with whole length $2L = 2\sum_i l_i$), as e.g. a turbine's axle [2] . . . . . (11a,b)

– Some <u>stub-rotor ratios</u> and formulas (12a–e)

$$\alpha_i = EI_i/EI_0 \quad \beta_i = l_i/l_0, \qquad \tilde{Z}_i(l_i) \approx \begin{bmatrix} 1 & 1 & 1/2 & 1/6 \\ 0 & 1 & 1 & 1/2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\sigma_i = \mu_i l_i/\mu_0 l_0 \approx 0, \qquad \lambda_i^4 = \lambda_0^4 \cdot \sigma_i \beta_i^3/\alpha_i \approx 0,$$

express [2,3] the <u>negligation</u> of stub masses (the stubs remain only as elastic connectina elements), then the <u>normalization</u> of z_i [1,3] from the stub's measures (l_i, I_i) to the rotor's one (l_0, I_0) by the linear transform $\mathring{z}_i = \underline{\underline{A}}\, z_i = \langle 1, 1/\beta_i, \alpha_i/\beta_i^2, \alpha_i/\beta_i^3 \rangle \cdot z_i$ to have at the border points $B_{i+1} = (t_{i+1}, t'_i)$ instead of the spring: $z_{i+1} \neq z'_i = \tilde{Z}_i(l_i) z_i$ the continuity:

$$\mathring{z}_{i+1} = \mathring{z}_i{}' = \underline{\underline{A}}_i \tilde{\underline{\underline{Z}}}_i(l_i) \underline{\underline{A}}_i^{-1} \mathring{z}_i \equiv \check{\underline{\underline{Z}}}_i \mathring{z}_i \text{ with } \check{\underline{\underline{Z}}}_i = \begin{bmatrix} 1 & \beta & \beta^2/2\alpha & \beta^3/6\alpha \\ 0 & 1 & \beta/\alpha & \beta^2/2\alpha \\ 0 & 0 & 1 & \beta \\ 0 & 0 & 0 & 1 \end{bmatrix}_i . \quad (12a\text{–}e)$$

Just these last ones furnishes the <u>connectional conditions</u> of $\mathring{z}$ at B_{i+1} (for $i = 0, 1, \dots, n$).

<u>The transit of the normed technical vector</u> $\mathring{z}(t)$ on the n stubs and on the rotor given by n+1 linear equations

$$\mathring{z}_{n+1} = \check{Z}_n \mathring{z}_n, \dots, \mathring{z}_{i+1} = \check{Z}_i \mathring{z}_i, \dots, \mathring{z}_2 = \check{Z}_1 \mathring{z}_1,\ z_1 = \bar{Z}_0 z_0 \quad (13a)$$

can be <u>united</u> [1,3] to one on the whole axle given a sole linear equation

$$\mathring{z}_{n+1} = \check{Z}_n \dots \check{Z}_i \dots \check{Z}_1 \cdot \bar{Z}_0 \mathring{z}_0 \equiv \check{Z}_{n1} \bar{Z}_0 z_0. \quad (13b)$$

This last one will be <u>detailed</u> and filled with <u>boundary conditions</u> being similar to (10a) in the form [2,3] (13c)

$$\left\langle B_j = f_j(\alpha_i, \beta_i),\quad j = 1,2,3,4,5;\quad i = 1,2,\dots,n \right\rangle$$

$$\overset{\circ}{\underset{\sim}{z}}_{n+1} \equiv \begin{bmatrix} 0 \\ z_2 \\ 0 \\ z_4 \end{bmatrix}_L = \begin{bmatrix} 1 & B_1 & B_2 & B_3 \\ 0 & 1 & B_4 & B_5 \\ 0 & 0 & 1 & B_1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \Gamma_o & \Sigma_1 & \Gamma_2 & \Sigma_3 \\ \lambda^4\Sigma_3 & \Gamma_o & \Sigma_1 & \Gamma_2 \\ \lambda^4\Gamma_2 & \lambda^4\Sigma_3 & \Gamma_o & \Sigma_1 \\ \lambda^4\Sigma_1 & \lambda^4\Gamma_2 & \lambda^4\Sigma_3 & \Gamma_o \end{bmatrix} \begin{bmatrix} z_1 \\ 0 \\ z_3 \\ 0 \end{bmatrix}_o \equiv \check{\underset{=}{Z}}_{n1} \bar{\underset{=}{Z}}_o \underset{-}{z}_o ,$$

whose geometrical dates B_j are counted in our [3,4], with corrections of [2] in B_2 and B_3. The homogeneous part of this linear equation

$$\underset{\sim}{z}_{n+1} \equiv \begin{bmatrix} 0 \\ 0 \end{bmatrix}_L = \begin{bmatrix} 1 & B_1 & B_2 & B_3 \\ 0 & 0 & 1 & B_1 \end{bmatrix} \begin{bmatrix} \Gamma_o & \Gamma_2 \\ \lambda^4\Sigma_3 & \Sigma_1 \\ \vdots & \vdots \end{bmatrix} \begin{bmatrix} z_1 \\ z_3 \end{bmatrix}_o \equiv \check{\underset{\sim}{Z}}_{n1} \bar{\underset{\sim}{Z}} \underset{\sim}{z}_o \equiv \underset{\sim}{Z} \underset{\sim}{z}_o \qquad (14a)$$

has the 0-determinant (for non-trivial solutions), as the equation of eigenvalues – with the "geometrical coordinates" $X = B_1$, $Y = 3(B_1B_2 - B_3)$ (from [2] and corrected by [3,4] in Y)– (14b)

$$c \cdot \underset{\sim}{Z}(\lambda) \equiv -1 + (\mathrm{tg}\lambda - \mathrm{th}\lambda) \cdot X + \lambda^2 \mathrm{tg}\lambda \, \mathrm{th}\lambda \cdot X^2 + Y \cdot \frac{\lambda^3}{6} (\mathrm{tg}\lambda \, \mathrm{th}\lambda) = 0.$$

On this base, one can draw the parabola-set of "technical parameter" λ (given by [2])

$$Y = \frac{6\left(\frac{\mathrm{ctg}\lambda}{\lambda} - X\right)\left(\frac{\mathrm{cth}\lambda}{\lambda} + X\right)}{(\mathrm{ctg}\lambda + \mathrm{cth}\lambda)} \qquad \left(\lambda^4 = \frac{\mu\omega^2 l_o}{EI_o}\right) \qquad (14c)$$

and to the "point" (X_1, Y_1) of the given axle (rotor + stubs) can read out of the nomogram the value λ_1 of carrier parabola, finally can count simply the (first, minimal) s.c. critical angular velocity of axle (circular frequency of bar)

$$\omega_1 = (\lambda_1 / l_o)^2 \cdot \sqrt{EI_o / \mu_o} \qquad (\mathrm{sec}^{-1}). \qquad (15)$$

Further investigations (e.g. finer models, better approximations, additional effects of turning etc.) can be read in our [3].

3. References: [1] K.Marguerre: Vibration and stability problems of beams treated by matrices. J.Math. & Phys. (1956). – [2] E.Egerváry: The Rayleigh method applied to count a turning system's critical angular velocity. <Hung. lang.> Mat. Lapok 1 (1949). – [3] F.Fazekas: Beiträge zur kleinsten kritischen Drehzahl des Rotors usw. I. Int. Koll. f. Math.,Veröff.,Weimar, 1961. – [4] F.Fazekas: Ordinary differential equations, Part II/A. Book series MMGy (ed. Fazekas), tome B.VII, part II <Hung. lang.>, Tankönyvkiadó 1969 (sec.ed.). – /5/ F.Fazekas: Matrix analysis of differential and integral equations I-II-III. Bulletins for Applied mathematics /BAM/, TUB 1978-81. – [6] F.Fazekas: Matrix analysis of certain dynamical systems in technics (the whole matter of this lecture). Prepared for the BAM.–