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NUMERICAL APPROXIMATION OF A NONLINEAR STURM-LIOUVILLE PROBLEM ON AN INFINITE INTERVAL

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Let $V = H_0^1(\mathbb{R}_+) = \{f: \mathbb{R}_+ \to \mathbb{R} \mid f, f' \in L^2(\mathbb{R}_+),\ f(o) = o\}$ be with norm $\|f\| = (\int_0^\infty (|f|^2 + |f'^2|))^{\frac{1}{2}}$. For $(\lambda, u) \in \mathbb{R} \times V$, we consider the equation

$$- u''(t) + \alpha(t)u(t) + \beta(t)\theta(u(t))u^2(t) + \lambda u(t) = o\ ,\ t \in \mathbb{R}_+, \tag{1}$$

where $\alpha \in L^\infty(\mathbb{R}_+) \cap L^1(\mathbb{R}_+)$, $\beta \in L^\infty(\mathbb{R}_+) \cap L^2(\mathbb{R}_+)$, $\theta \in C^p(\mathbb{R})$, $p \geq 4$. We are interested by the approximation of solution branches of (1) in the neighborhood of a solution (λ_0, u_0) of (1). By Naimark [4], p 301, $u_0 \neq o$ implies that $\lambda_0 > o$; consequently, *in the following we shall restrict ourselves to positive values of* λ_0.

Let $\{V_h\}_h$ be a family of finite-dimensional subspaces of V, dense in the limit in V when h tends to zero. The Galerkin approximation of (1) consists in finding $(\lambda, u) \in \mathbb{R} \times V_h$ such that

$$\int_0^\infty \{u'(t)\, w'(t) + (\alpha(t)u(t) + \beta(t)\theta(u(t))\, u^2(t) + \lambda u(t))\, w(t)\}\, dt = o \quad \forall\, w \in V_h. \tag{2}$$

For $h > o$, let $N = N(h)$ be the integer part of $h^{-\delta}$ where $\delta > 1$ is some given number; as a particular choise of V_h, we can consider

$V_h = \{v \in V \mid v$ is piecewise linear with respect to the mesh $\{ih\}$, $v(t) = o$ for $t \geq Nh\}$; (3)

by using furthermore the numerical formula of quadrature $\int w(t) \doteq h \sum_i w(ih)$, Problem (2) becomes equivalent to the classical difference scheme :

$$u_{i-1} - 2u_i + u_{i+1} - h^2(\alpha(ih) + \beta(ih)\theta(u_i)\, u_i + \lambda)u_i = o \quad i = 1,2\ldots N-1, \tag{4}$$

where $u_i = u(ih)$, $u_0 = u_N = o$.

Consider the following auxiliary eigenvalue problem : find $(\mu, \phi) \in \mathbb{R} \times V$ such that

$$- \phi''(t) + \alpha(t)\phi(t) + \beta(t)(\theta'(u_0(t))u_0^2(t) + 2\theta(u_0(t))\, u_0(t))\phi(t) + \lambda_0\phi(t) = \mu\phi(t)\ ,\ t \in \mathbb{R}_+\ ; \tag{5}$$

since $\lambda_0 > o$, by [2] or [4], then $\mu = o$ either belongs to the resolvent set or is an isolated eigenvalue of multiplicity 1 of the operator L, where $L\phi$ denotes the left member of (5).

In the following we shall consider branches of solutions of (1) or (2) of the form $(\lambda, u(\lambda)) \in \mathbb{R} \times V$ or of the form $(\lambda(\xi), u(\xi)) \in \mathbb{R} \times V$; $u^{(k)}$ will then denote the k - th derivative with respect to λ or ξ.

Theorem 1. (Regular point) We suppose that $\mu = o$ is not an eigenvalue of (5). Then, in a neighborhood of (λ_0, u_0), Problem (1) possesses an unique branch of solutions which can be parametrized in the form $(\lambda, u(\lambda))$, $|\lambda - \lambda_0| < \varepsilon$; for h small enough, Problem (2) possesses a corresponding unique branch of solutions which can be parametrized in the form $(\lambda, u_h(\lambda))$, $|\lambda - \lambda_0| < \varepsilon$; $u(\lambda)$ and $u_h(\lambda)$ are of class C^p and for $o \leq k \leq p-1$, $|\lambda - \lambda_0| < \varepsilon$, we have the error estimate :

$$||u^{(k)}(\lambda) - u_h^{(k)}(\lambda)|| \leq c \sum_{e=o}^{k} \inf_{v \in V_h} ||u^{(e)}(\lambda) - v|| \tag{6}$$

where c is independent of h and λ.

Theorem 2. (Turning point). We suppose that $\mu = o$ is an eigenvalue of (5) with eigenvector $\phi_0 \in V$ and that $\int^{\infty} u_0 \cdot \phi_0 \neq o$. Then, in a neighborhood of (λ_0, u_0), Problem 1 possesses an unique branch of solutions which can be parametrized in the form $(\lambda(\xi), u(\xi))$, $|\xi| < \varepsilon$, where $\lambda(\xi)$ and $u(\xi)$ are C^p mappings with $\lambda(o) = \lambda_0$, $u(o) = u_0$, $\lambda'(o) = o$, $u'(o) \neq o$. Suppose furthermore that $\lambda''(o) \neq o$. For h small enough, Problem (2) possesses a corresponding unique branch of solution which can be parametrized in the form $(\lambda_h(\xi), u_h(\xi))$, $|\xi| < \varepsilon$; $\lambda_h(\xi)$ and $u_h(\xi)$ are of class C^p and there exists an unique ξ_h, $|\xi_h| < \varepsilon$ such that $\lambda_h^{\cdot}(\xi_h) = o$; for $o \leq k \leq p-1$ and $|\xi| < \varepsilon$, we have the error estimates:

$$|\lambda^{(k)}(\xi) - \lambda_h^{(k)}(\xi)| + ||u^{(k)}(\xi) - u_h^{(k)}(\xi)|| \leq c \sum_{e=o}^{k} \inf_{v \in V_h} || u^{(e)}(\xi) - v||, \tag{7}$$

$$|\lambda_0 - \lambda_h(\xi_h)| \leq c \{ \inf_{v \in V_h} ||\phi_0 - v||^2 + \inf_{v \in V_h} || u_0 - v||^2 \}, \tag{8}$$

where c is independent of h and ξ.

Theorem 3. (Bifurcation from the trivial branch). We suppose that $u_0 = o$ and $\mu = o$ is an eigenvalue of (5) with eigenvector $\phi_0 \in V$. Then, in the neighborhood of (λ_0, o), Problem (1) possesses an unique non-trivial branch of solutions which can be parametrized in the form $(\lambda(\xi), u(\xi))$, $|\xi| < \varepsilon$ where $\lambda(\xi)$ and $u(\xi)$ are C^{p-2} mappings with $\lambda(o) = \lambda_0$, $u(o) = o$, $u'(o) \neq o$. For h small enough, Problem (2) possesses a corresponding unique non-trivial branch of solutions which can be parametrized in the form $(\lambda_h(\xi), u_h(\xi))$, $|\xi| < \varepsilon$ where $\lambda_h(\xi)$ and $u_h(\xi)$ are C^{p-2} mappings with $u_h(o) = o$. For $o \leq k \leq p-3$, we have the error estimates :

$$\sup_{|\xi|<\varepsilon_0} \{ |\lambda^{(k)}(\xi) - \lambda_h^{(k)}(\xi)| + ||u^{(k)}(\xi) - u_h^{(k)}(\xi)|| \} \leq c \sum_{e=o}^{k+1} \sup_{|\xi|<\varepsilon_0} (\inf_{v\in V_h} ||u^{(e)}(\xi) - v||). \tag{9}$$

Remark 1. Theorems 1, 2 and 3 can be proven by applying the results of [3] which generalize in several directions those of [1]. Use of [1] would suppose a property of compactness which is missing in our example : if L denotes the operator defined by the left member of (5), the inverse L^{-1}, when it exists, is non compact since (1) is considered on an infinite interval.

Remark 2. Due to properties of exponential decay of solutions of (1) when $t \to \infty$ (see [4]), it is easy to verify that for V_h given by (3), the right members of (6), (7) and (9) are $O(h)$ whereas the right member of (8) is $O(h^2)$. By using the standard arguments relative to the introduction of numerical integration in Galerkin methods, it is possible to verify that the same estimates are valid for the method defined by (4), if $\alpha(t)$ and $\beta(t)$ are sufficiently regular.

REFERENCES.

[1] F. Brezzi, J. Rappaz, P.A. Raviart. Finite Dimensional Approximation of Nonlinear Problems. Part I: Branches of Nonsingular Solutions; Num. Math. 36, 1 - 25 (1980). Part II: Limit Points; Num. Math. 37, 1 - 28 (1981). Part III: Simple Bifurcation Points; to appear in Num. Math.

[2] J. Descloux. Essential Numerical Range of An Operator With Respect To A Coercive Form And The Approximation Of Its Spectrum By The Galerkin Method. To appear in SIAM J. Num. Anal.

[3] J. Descloux, J. Rappaz. On Numerical Approximation Of Solution Branches of Nonlinear Equations. Report of the Math. Dept. of the EcolePolytechnique Fédérale de Lausanne; to appear.

[4] M. A. Naimark. Linear Differential Operators. Part 2. Linear Differential Operators in Hilbert Spaces. G. Harrap, London, 1968.