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THE SOLUTION OF THE INITIAL VALUE PROBLEM FOR THE GENERAL DYNAMIC EQUATIONS IN NONLINEAR ELASTICITY UNDER DAMPING CONDITIONS

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Let G_0 be an elastic body in equilibrium under fixed boundary conditions with a stress distribution $\sigma^c_{\mu\nu}(x^0) \subset C^\infty(G_0)$. Starting with an initial impulse and initial displacements:

$$(1) \quad \frac{\partial u_i^0(x^0)}{\partial t} = \gamma_i^0(x^0)\ , \quad u_i^0\,(x^0) = \chi_i^0(x^0) \subset C^\infty(G^0)$$

we follow the resulting movement of the Lagrangean points x^0 of the body G_0. In a fixed position (x,t) this movement can be described by the nonlinear dynamic equations in Euler form :

$$(2) \quad \varrho(x,t)\frac{d v_i}{dt}(x,t) + \alpha\, v_i(x,t) = X_i(x) + \sigma_{ji,j}(x,t)\ ,$$

$v_i(x,t)$, $\varrho(x,t)$ and α expressing the velocity vector, the density and the constant damping coefficient in the position (x,t). The difficulty of the initial value problem (2),(1) arises from the fact that the divergence $\sigma_{ji,j}(x,t)$ of the stress tensor depends upon the history of the path of the Lagrangean point x^0 to x during the time interval $[0,t]$. Another difficulty is that in general discontinuities will be caused by reflection from the boundary. In our theory we consider initial datas(1) with compact support in G_0 and follow the deformation as long as it arrives at the boundary within the time interval, say $0 \leq t \leq t^+$. Here we construct regular solutions of our intial value problem (2),(1). Global regular solutions for all times cannot be expected as in the case of the linear wave equations because of the complicated interaction of changing stresses in the area of reflection. If in a small boundary layer the movement of the particles is stopped by strong damping material, our solutions can be set forward for $t > t^+$ by the same procedure as for $0 \leq t \leq t^+$ above. The strong regularity assumptions (1) can be reduced considerably by standard approximation methods. In [5] E. Trefftz derived the expression of the second variation of the elastic potential U_x with respect to the deformed body G_x

$$(3) \quad Q_2(w,w) = \frac{1}{2}\int_{G_x} \sigma_{\mu\nu}(x)\frac{\partial w_i}{\partial x_\mu}\frac{\partial w_i}{\partial x_\nu} + 2\, a(w,w)\, dx$$

under the assumption that the resulting stress distribution $k_{\mu\nu}(y)$ over a state arising by admissable displacements:

$G_x \to G_y \; : \; x_i + w_i(x_i) = y_i \quad , \; i = 1,2,3$

can be split into the sum

$$(4) \qquad k_{\mu\nu}(y) = \sigma_{\mu\nu}(x) + \tau_{\mu\nu}(x)$$

in which the additional stresses $\tau_{\mu\nu}(x)$ are to be calculated within the framework of the linear theory :

$$(5) \qquad \tau_{\mu\nu} = \frac{\partial a}{\partial \gamma_{\mu\nu}} \quad ; \quad \gamma_{\mu\nu} = \frac{\partial w_\mu}{\partial x_\nu} + \frac{\partial w_\nu}{\partial x_\mu}$$

$a(w,w)$ is the well known expression of the elastic potential in the linear elasticity theory. For long range deformations one has to modify the elastic constants in $a(w,w)$ to depend on the present or more general past stress distributions to obtain an adequate description by means of (3) or by more general nonlinear transition rules, Let $u(x,t-t_1)$ define the elastic motion $G_x \to G_z : x + u(x,t_2-t_1) = z$, under suitable regularity assumptions we receive for $t_2 \to t_1$, see [4] the limit equations (6):

$$(6) \quad \varrho(x)\frac{\partial^3 \check{u}}{\partial t^3}(x,t) + \alpha\frac{\partial^2 \check{u}}{\partial t^2}(x,t) + L_2\left(\frac{\partial \check{u}}{\partial t}(x,t)\right) = \sigma \text{, for } t = 0.$$

$L_2\psi$ is the Lagrangean of the second variation $Q_2(\psi,\psi)$.
(6) is the starting point for the following approximation for the solution of our initial problem (2),(1). Before, we mention that in the case of damping material ϱ is supposed to be constant, wheras in the case $\alpha = 0$ our method covers exactly the change of ϱ during the motion, [4]. Dividing the time interval $I : [0,T] : 0 < t_1 < t_2 \ldots < t_n = T$ we determine in the first step the vector $u_i^0(x^0,t)$ along $0 \leq t' \leq t_1$ as the solution of the linear wave equation (7) under the condition (1) and boundary condition $u_i^0(\sigma,t) = 0, \sigma \subset \partial G_0$:

$$(7) \qquad \varrho(x^0)\frac{\partial^2 \check{u}^0}{\partial t^2} + \alpha\frac{\partial \check{u}^0}{\partial t} + L_2^0(\sigma^0)\check{u}^0(x,t) = \sigma$$

$L_2^0(\sigma^0)$ is the Lagrangean of the second variation $Q_2(u,u)$ over G_0 corresponding to the given stress distribution $\sigma_{\mu\nu}^0(x^0)$. (7) is a consequence of the dynamic equations (2) and the equilibrium conditions over G_0. According to (4),(5) the transition $G_0 \to G_1$: $x_i^0 + u_i^0(x^0,t_1) = x_i^1$ defines at time t_1 the new stress distribution

$$(8) \qquad \sigma_{\mu\nu}^1(x^1) = \sigma_{\mu\nu}^0(x^0) + \tau_{\mu\nu}^0(x^0).$$

The starting point for the further approximation must be (6).

Second step: Along the interval $t_1 \leq t' \leq t_2$ the velocity of the points $x^1 \subset G_1$: $u^1_t(x^1, t'-t_1) = \psi^1(x^1, t'-t_1)$ is to be calculated by solving the initial value problem for the linear wave equation :

$$(9)\ \rho \frac{\partial^2 \psi^1}{\partial t^2}(x^1, t'-t_1) + \alpha \frac{\partial \psi^1}{\partial t} = -L^1_2(\sigma^1)\, \psi^1(x^1, t'-t_1)$$

under the conditions:

$$(10)\quad u^1_t(x^1,0) = \psi^1(x^1,0) = u^0_t(x^0, t_1)\ ;\ \psi^1(\sigma, t'-t_1) = 0$$

$$u^1_{tt}(x^1,0) = \psi^1_t(x^1,0) = u^0_{tt}(x^0, t_1)\ ,\ \sigma \subset \partial G_1 .$$

The transition $G_1 \to G_2$ is defined by

$$(11)\quad x^1 + u^1(x^1, t_2 - t_1) = x^1 + \int_{t_1}^{t_2} \psi^1(x^1, t'-t_1)\, dt' = x^2$$

with the new stress configuration $\sigma^2_{\mu\nu}(x^2)$ according to (4).

In the same way we proceed further and calculate $\psi^2(x^2, t'-t_2)$ ... $\psi^{i-1}(x^{i-1}, t'-t_{i-1})$ ..

$$(12)\quad G_{i-1} \to G_i\ :\ x^i = x^{i-1} + \int_{t_{i-1}}^{t_i} \psi^{i-1}(x^{i-1}, t'-t_{i-1})\, dt'.$$

If, as we suppose, the initial state over G_0 is static stable, the second variation (3) is positiv definit there. Thus, the corresponding eigenvalue problem has strictly positiv eigenvalues, the set of eigenfunctions is complete in $H_{0,2}(G_0)$. The solutions $\psi^i(x^i, t'-t_i)$ can be constructed by the well known Fourier method. As long as $\mathrm{Min}\,\lambda^i_k > \frac{\alpha^2}{4}$ we get with $\sqrt{\beta^i_k} = \sqrt{\lambda^i_k - \frac{\alpha^2}{4}}$ the expansions

$$(13)\quad \psi^i(x^i, t'-t_i) = \sum e^{-\alpha/2(t'-t_i)} B^i_k \cos\sqrt{\beta^i_k}\,(t'-t_i)\, v^i_k(x^i) + \sum e^{-\alpha/2(t'-t_i)} D^i_k \sin\sqrt{\beta^i_k}\,(t'-t_i)\, v^i_k(x^i)$$

$$\psi^i_t(x^i, t'-t_i) = \sum -e^{-\alpha/2(t'-t_i)} \sqrt{\beta^i_k}\, B^i_k \sin\sqrt{\beta^i_k}(t'-t_i)\, v^i_k(x^i) + \sum e^{-\alpha/2(t'-t_i)} \sqrt{\beta^i_k}\, D^i_k \cos\sqrt{\beta^i_k}\,(t'-t_i)\, v^i_k(x^i) - \alpha/2\, \psi^i(x^i, t'-t_i)$$

in the i'th step with the eigenfunctions and eigenvalues $v^i_k(x^i)$, λ^i_k. Summing up the transitions (12) we receive for $G_0 \to G_r$:

$$(14)\quad f^r_k(x^0) = x^r_k\ ,\ k = 1,2,3\ .$$

In[4] we have proved: For $t \in I_s$: $0 \leq t \leq s \leq t^+$ the curves (14) converge uniformly in $C^{4+\mu}(G_0) \times C^2(I_s)$ to curves $x = f(x^0, t)$ which define the motion of the continuum G_0 under the initial conditions (1), further on, the stress components are converging too in $C^{4+\mu}$. The corresponding limit equations (6) are fulfilled in each moment. Thus, because of (7), we conclude that the dynamic equations (2) are fulfilled along this motion.

If the motion, constructed, reaches the boundary, discontinuities caused by reflection will travel inside. To overcome these difficulties we suppose that within a small boundary layer $\hat{G} \subset G_0$ the movement of the particles dies down by high damping. Introducing a C^∞-function $\gamma^i(x^1, t)$ on G_1 being equal to one on the inner domain $G_1 - \hat{G}$ with supp $\gamma^i(x^1, t) \subset \hat{G}$, we define the transitions (12) now by:

$$(15) \quad x^i = x^{i-1} + \int_{t_{i-1}}^{t_i} \gamma^i(x^1, t') \; \gamma^{i-1}(x^{i-1}, t' - t_{i-1}) dt'.$$

As a small strip around the boundary shall remain fixed, the stress cannot change there. Thus, we can perform our existence proof in exactly the same way under the new conditions for times $t \geq t^+$.

Literature

Beckert H. [1] Zur ersten Randwertaufgabe in der nichtlinearen Elastizitätstheorie, ZAMM 55 (1975), S. 47 - 58.

[2] Zur zweiten und dritten Randwertaufgabe in der nichtlinearen Hyperelastizitätstheorie ZAMM 57 (1977) S. 153 - 164.

[3] Variationsrechnung und Stabilitätstheorie, Math. Nachr. 75 (1976) S. 47 - 59.

[4] The initial value problem for the general dynamic equations in nonlinear elasticity theory, ZAMM, in press.

Trefftz E. [5] Zur Theorie der Stabilität des elastischen Gleichgewichts, ZAMM 13 (1933) S. 160 - 165.