Felix M. Arscott

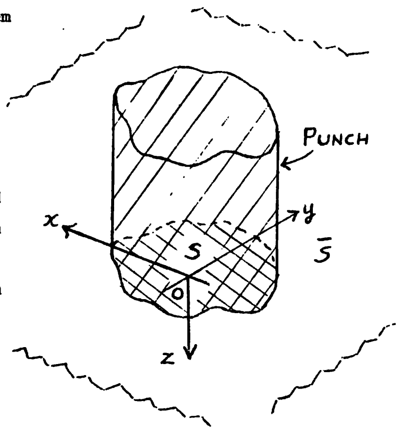
On the solution of indentation and crack problems in linear elasticity by use of higher special functions

In: Michal Greguš (ed.): Equadiff 5, Proceedings of the Fifth Czechoslovak Conference on Differential Equations and Their Applications held in Bratislava, August 24-28, 1981. BSB B.G. Teubner Verlagsgesellschaft, Leipzig, 1982. Teubner-Texte zur Mathematik, Bd. 47. pp. 21--24.

Persistent URL: http://dml.cz/dmlcz/702250

ON THE SOLUTION OF INDENTATION AND CRACK PROBLEMS IN LINEAR ELASTICITY BY USE OF HIGHER SPECIAL FUNCTIONS

F. M. Arscott
University of Manitoba, Winnipeg, Canada



By an <u>indentation</u> (or <u>punch</u>) problem we mean the following: The infinite half-space $z > 0$ is filled with uniform, isotropic elastic material. A rigid punch is pressed into that material, in the plane $z = 0$, over a region S, so that it indents the elastic material completely and there is perfect contact between the punch and the material. The 'profile' of the punch - that is to say, the depth to which it penetrates the elastic material - is a prescribed function $g(x,y)$, this being presumed small enough that the approximations of linear elasticity hold good. The edge of the punch is frictionless, and the remaining part of the surface $z = 0$, denoted by $\bar{S}$, is assumed to be stress-free.

The problem is to determine the displacement and stress in the material.

Mathematical analysis

(a) It is known [3] that the displacement vector $\underset{\sim}{D}$ in the material can be represented in the form

$$\underset{\sim}{D} = (3 - 4\nu)\,\Phi\underset{\sim}{k} - z\nabla\Phi + \nabla\psi \tag{1}$$

with
$$\frac{\partial\psi}{\partial z} = -\,(1 - 2\nu)\Phi, \tag{2}$$

where Φ and ψ are harmonic functions. The quantity ν is Poisson's ratio, a constant for the material

Hence we may obtain two quantities of particular interest:

(b) the displacement on the surface, outside the punch :=

$$w(x,y,0) = 2(1 - \nu)\Phi(x,y,0),\ (x,y) \in \bar{S}, \tag{3}$$

(c) the stress at the surface underneath the punch :=

$$\sigma = \sigma_{zz} = 2\mu \frac{\partial}{\partial z}\Phi(x,y,0),\ (x,y) \in S, \tag{4}$$

where μ is the shear modulus of the material, also a constant.

Formulation as a boundary value problem

Hence the mathematical problem is to determine Φ such that

$$
\begin{array}{rll}
\text{(i)} & \nabla^2\Phi = 0 \text{ for } z > 0, & \\
\text{(ii)} & 2(1-\nu)\Phi(x,y,0) = g(x,y) \text{ for } (x,y)\in S, & \\
\text{(iii)} & \frac{\partial}{\partial z}\Phi(x,y,0) = 0 \qquad\qquad \text{for } (x,y)\in \bar{S}, & \\
\text{(iv)} & \Phi \to 0 \text{ as } r \to \infty \text{ in } z > 0, \qquad r = (x^2+y^2+z^2)^{\frac{1}{2}}. &
\end{array}
\tag{5}
$$

We thus have a mixed boundary value problem for Laplace's equation. A crack problem here means the following: we have an infinite elastic medium, with a plane region of discontinuity S in the plane $z = 0$. Equal normal pressures are applied to the faces of this crack, tending to open it out (so we have symmetry about the plane $z = 0$ and need only consider the region $z > 0$). We have relations (1),(2) again, but now the boundary conditions are
(α) $w(x,y,0) = 0$ for $(x,y) \in \bar{S}$, and (β) $\sigma(x,y,0) = -g(x,y)$, for $(x,y) \in S$, where $g(x,y)$ is the applied pressure. Brief consideration shows that this leads to a similar mixed boundary value problem for Φ, but with Dirichlet conditions on $\bar{S}$ and Neumann conditions on S.

The use of curvilinear coordinates

If the shape of the region S is simple, we may be able to embed it in a suitable coordinate system, transform the Laplace equation to the corresponding variables, and solve by separation of variables. This possibility arises notably in four cases.

(i) S is circular (ii) S is elliptic (iii) S is an infinite strip (iv) S is a parabola.

It should be noted that although this line of attack requires that S have such a simple shape, there is no inherent restriction on the profile of the punch, which is essentially arbitrary.

The analysis when S is circular has long been known; that for an elliptic punch was given recently by Shail [4] and prompted the investigation (by the author and Mr. A. Darai) of the infinite strip punch. We are now working on the parabolic punch.

For the student of differential equations, the problem is challenging because, when the separation is performed, the ordinary differential equations which result are quite complicated (Lamé's equation in case (ii), Mathieu's in (iii) and (iv)) and the demands of the original problem highlight the need for deeper study of these equations.

It is also worth remarking that in cases (ii),(iii) and (iv) we obtain a two-parameter eigenvalue problem. In (ii) we have a discrete spectrum in both parameters, but in (iii) and (iv) the spectrum is discrete in one parameter and continuous in the other - an unusual feature.

The infinite strip punch

Let S, the region of the z plane occupied by the punch, be an infinite strip of width 2f, its sides being parallel to the x axis. Introduce elliptic cylinder coordinates (x,ξ,η), where

$$y = f \cosh \xi \cos \eta, \quad z = f \sinh \xi \sin \eta, \quad \xi \geq 0, \ -\pi < \eta \leq \pi. \tag{6}$$

Then the strip is given by $\xi = 0$, $0 < \eta < \pi$ and the elastic material by $\xi > 0$, $0 < \eta < \pi$, so the boundary value problem is to determine $\Phi(x,\xi,\eta)$ such that

(i) $\nabla^2\Phi = 0$, $\xi > 0$, $0 < \eta < \pi$, $x \in \mathbb{R}$,

(ii) $\Phi(x,0,\eta) = H(x,\eta), 0 < \eta < \pi$, $x \in \mathbb{R}$,

(where $2(1-\nu)\ H(x,\eta) = g(x,y)$, the punch profile) (7)

(iii) $\frac{\partial}{\partial\eta}\Phi(x,\xi,0) = \frac{\partial}{\partial\eta}\Phi(x,\xi,\pi) = 0$, $\xi > 0$, $x \in \mathbb{R}$,

(iv) $\Phi \to 0$ as $\xi \to \infty$, $0 < \eta < \pi$, $x \in \mathbb{R}$.

Now the punch profile $g(x,y)$ can be split into the sum of 4 functions, each of which is either symmetric or antisymmetric about each axis Ox,Oy, and the corresponding boundary value problem solved for each of the four functions separately; hence to illustrate the analysis we shall assume $g(x,y)$ is symmetric about both Ox and Oy, hence $H(x,\eta)$ is even in x and even about $\eta = \frac{1}{2}\pi$.

On transforming to elliptic cylinder coordinates and separating we obtain the three separated equations (θ,λ are the separation constants)

$$\text{(a)}\quad X''(x) = \theta X(x), \qquad \text{(b)}\quad F''(\xi) + (\tfrac{1}{2}\theta f^2 \cosh 2\xi-\lambda)F(\xi) = 0, \tag{8a,b}$$

$$\text{(c)}\quad G''(\eta) + (\lambda - \tfrac{1}{2}\theta f^2 \cos 2\eta)G(\eta) = 0. \tag{8c}$$

Of these, 8(c) is <u>Mathieu's equation</u> and 8(b) is the <u>modified Mathieu equation.</u>

In order to have (as is clearly necessary) $|\Phi| < \infty$ as $|x| \to \infty$, we must take $\theta < 0$, say $\theta = -\alpha^2$, $\alpha \geq 0$, and $X = A \cos \alpha x$.

To obtain the boundary conditions for equation 8(c), we recall conditions 7(iii) and the symmetry of $g(x,y)$ about $y = 0$; these imply that $G'(0) = G'(\pi) = 0$ and $G(\eta)$ is even about $\eta = \frac{1}{2}\pi$. Then the theory of Mathieu's equation ([1]) shows that

$G(\eta) = ce_{2n}(\eta,-h^2)$ where $h = \frac{1}{2}\alpha f$, n is an arbitrary non-negative integer, and $\lambda = a_{2n}(-h^2)$ in the usual notation.

The only condition now attaching to equation (b) is that $F(\xi) \to 0$ as $\xi \to \infty$; this is sufficient to determine $F(\xi)$ as a constant multiple of $Fek_{2n}(\xi,-h^2)$ so a separated solution of the problem is

$$\Phi_n(x,\xi,\eta,h) = A_n(h) \cos \frac{2hx}{f} Fek_{2n}(\xi,-h^2)ce_{2n}(\eta,-h^2) \tag{9}$$

with $n \geq 0$ integral, $h \geq 0$ arbitrary; $A_n(h)$ is an arbitrary constant.

By the principle of superposition, a more general solution is

$$\Phi(x,\xi,\eta) = \int_0^\infty \sum_{n=0}^\infty A_n(h) \cos \frac{2hx}{f} Fek_{2n}(\xi,-h^2)ce_{2n}(\eta,-h^2)dh. \tag{10}$$

This satisfies (i),(iii) and (iv). In order to satisfy (ii) we must have

$$H(x,\eta) = \int_0^\infty \sum_{n=0}^\infty B_n(h) \cos \frac{2hx}{f} ce_{2n}(\eta,-h^2)dh, \tag{11}$$

where $B_n(h) = A_n(h) \, Fek_{2n}(0,-h^2)$. Applying the inversion theorem for the Fourier cosine transform gives

$$\sum_{n=0}^\infty B_n(h) ce_{2n}(\eta,-h^2) = \frac{4}{\pi f}\int_0^\infty H(x,\eta) \cos \frac{2hx}{f} dx \; ; \tag{12}$$

then the orthogonality property of Mathieu functions yields, finally the formal expression

$$B_n(h) = \frac{8}{\pi^2 f} \int_0^\pi \int_0^\infty H(x,\eta) \cos \frac{2hx}{f} ce_{2n}(\eta,-h^2) dx \, d\eta. \tag{13}$$

Special cases yielding explicit solutions are:-

(I) Flat punch of rectangular cross-section: $H(x,\eta) = \varepsilon \quad |x| < a,$ (14)

$= 0 \quad |x| > a;$

then
$$B_n(h) = \frac{(-1)^n 4}{\pi^2 h} \varepsilon \; A_0^{(2n)}(-h^2) \sin \frac{2ha}{f}, \tag{15}$$

(II)
$$H(x,\eta) = \frac{\varepsilon \ell^2}{\ell^2+x^2} \sin^2\eta \, . \tag{16}$$

Then

$$B_n(h) = \frac{\varepsilon\ell}{f} e^{-2h\ell/f}(-1)^n \left[2A_0^{(2n)}(-h^2) + A_2^{(2n)}(-h^2)\right] \tag{17}$$

where $A_{2r}^{(2n)}$ is the coefficient of $\cos 2r\eta$ in the Fourier expansion of $ce_{2n}(\eta,-h^2)$.

REFERENCES

[1] F. M. Arscott, Periodic Differential Equations, Pergamon, 1964.

[2] F. M. Arscott and A. Darai, Curvilinear coordinate systems in which the Helmholtz equation separates, IMA Jour. of Appl. Math. 27, 33-70, 1981.

[3] W. D. Collins, Some axially symmetric stress distributions in elastic solids containing penny-shaped cracks, Proc. Roy. Soc. A 266, 359-386, 1962.

[4] R. Shail, Lamé polynomial solutions to some elliptic crack and punch problems, Int. Jour. Engng. Sci., 16, 551-563, 1978.