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A NEW DEFINITION AND SOME MODIFICATIONS

OF FILIPPOV CONE

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Let G be a region in an $n+1$-dimensional Euclidean space R_{n+1} and let $f(t,x)$ be a measurable function $f : G \to R_n$. Denote by $F(t,x)$ the set

$$F(t,x) = \bigcap_{d>0} \bigcap_{N, m(N)=0} \overline{\text{Conv}}\, f(t, U(x,d)-N) \tag{1}$$

where $U(x,d) = \{y : \|y-x\|<d\} \subset R_n$, $N \subset U(x,d)$, m is the Lebesgue measure in R_n, $\overline{\text{Conv}}\ A$ is the closed convex hull of the set A. The set $\{[q,qy] : q \geqq 0 , y \in F(t,x)\}$ is called the Filippov cone at the point $[t,x] \in G$ and mapping $[t,x] \to F(t,x)$ the Filippov mapping.

A vector function $x(t)$ is a generalized solution in the Filippov sense of $\dot{x} = f(t,x)$ if $x(t)$ is defined on a nondegenerate interval I, $[t,x(t)] \in G$ for $t \in I$, $x(t)$ is absolutely continuous on I and $\dot{x}(t) \in F(t,x(t))$ for almost all $t \in I$. The notion of Filippov's generalized solutions depends essentially on formula (1). J.Kurzweil established a certain minimum property of the mapping F in his book about ordinary differential equations not yet published. We shall mention the property in more detail later. The purpose of the lecture is to show that the Filippov mapping can be constructed on the basis of this property and to present some modifications offered by this approach.

The autonomous case

First some definitions and notation. Let $\mathcal{A}_o$ be the class of all subsets of R_n and let h be a mapping $h : G \to \mathcal{A}_o$ where G is a region in R_n. The mapping h is called locally essentially bounded if to every $x_o \in G$ there exist numbers $d>0$ and $c \geqq 0$ such that $m\{x : x \in U(x_o,d),\ h(x) \not\subset U(0,c)\} = 0$.

The mapping h is called upper semi-continuous if to every $d>0$ and $x\in G$ there exists $r>0$ such that $h(y)\subset U(h(x),d)$ for $y\in \in U(x,r)$ where $U(A,d)$ is the d-neighbourhood of the set A with $U(\emptyset,d)=\emptyset$.

Denote by $\mathcal{C}_0$ the class of all compact subsets of R_n and by $\mathcal{E}_0$ a class fulfilling

a) to every $A\in\mathcal{C}_0$ there exists a set $B\in\mathcal{E}_0$, $A\subset B$;

b) if $B_p\in\mathcal{E}_0$ then $\cap B_p\in\mathcal{E}_0$;

c) if $A\in\mathcal{E}_0$ then $A\in\mathcal{C}_0$.

Let $\mathcal{C}$, $\mathcal{E}$ and $\mathcal{A}$ be classes originating respectively from $\mathcal{C}_0$, $\mathcal{E}_0$ and $\mathcal{A}_0$ by excluding the empty set.

Further, let $\{h_z, z\in Z\}$, $Z\neq\emptyset$ be a family of mappings $h_z: G\to\mathcal{A}_0$. The greatest lower bound $h: G\to\mathcal{A}_0$ of the family is the mapping h defined by $h(x)=\bigcap_{z\in Z} h_z(x)$. We shall write $h=\bigwedge_{z\in Z} h_z$. The mapping $h_1(x)$ is before $h_2(x)$ $(h_1\leqq h_2)$ if $h_1(x)\subset h_2(x)$ for all $x\in G$.

Definition. Let f be a mapping $f: G\to\mathcal{A}$ and $\mathcal{E}$ a class such that $\mathcal{E}_0$ fulfils a) to c). Denote by $R(f,\mathcal{E})$ the family of all mappings h fulfilling

i) $h(x)\in\mathcal{E}$ for all $x\in G$;

ii) h is upper semi-continuous on G;

iii) $f(x)\subset h(x)$ for almost all $x\in G$.

The condition under which the set $R(f,\mathcal{E})$ is nonempty is given in

Theorem 1. Let a class $\mathcal{E}_0$ fulfil a) to c). The set $R(f,\mathcal{E})$ is nonempty if and only if the mapping f is locally essentially bounded.

Given a class $\mathcal{E}$ and a mapping f, Theorem 1 enables us to construct the greatest lower bound $S=\bigwedge_{h\in R(f,\mathcal{E})} h$. Basic properties of S are given in

Theorem 2. Let a class $\mathcal{E}_0$ fulfil a) to c). If the mapping $f:G\to\mathcal{A}$

is locally essentially bounded then $S \in R(f, \mathcal{E})$.

The mapping S depends on the class $\mathcal{E}$. The most important classes are subclasses of $\mathcal{K}$ where $\mathcal{K}$ is the class of all compact, convex and nonempty subsets of R_n. In these cases the existence theorem [2] can be applied due to Theorem 2 to a differential relation $\dot{x} \in S(x)$. Thus if a locally essentially bounded mapping f and a class $\mathcal{E} \subset \mathcal{K}$ are given, then the mapping S exists and its properties guarantee that the set of all solutions of the differential relation $\dot{x} \in S(x)$ is nonempty. These solutions can be called $\mathcal{E}$-generalized solutions of the differential relation $\dot{x} \in f(x)$. If $\mathcal{E} = \mathcal{K}$ then the $\mathcal{E}$-generalized solutions of $\dot{x} \in f(x)$ can be called the generalized solutions of $\dot{x} \in f(x)$ in the Filippov sense. This definition is justified by the following theorem.

Theorem 3. Let f be a measurable and locally essentially bounded function. If $\mathcal{E} = \mathcal{K}$ then $S = F$.

This theorem directly implies that the $\mathcal{K}$-generalized solutions of $\dot{x} = f(x)$ are exactly Filippov's generalized solutions of the equation. Theorem 3 together with the definition of S yield $F \leqq h$ for $h \in R(f, \mathcal{K})$. This means that F is the minimum mapping from those fulfilling i) to iii) and this is the minimum property mentioned in the introduction.

Let $\mathcal{Q}$ be the class of all Cartesian products $\prod_{i=1}^{n} J_i$ of compact, nonempty intervals and put $\mathcal{Q}_0 = \mathcal{Q} \cup \{\emptyset\}$. Certainly $\mathcal{Q}_0$ fulfils conditions a) to c). Another interesting choice of $\mathcal{E}$ is $\mathcal{E} = \mathcal{Q}$.

Theorem 4. Let f be a measurable and locally essentially bounded function. Assume $\mathcal{E} = \mathcal{Q}$. Then a vector function $x(t)$ is an $\mathcal{E}$-generalized solution of $\dot{x} = f(x)$ if and only if $x(t)$ is a generalized solution of $\dot{x} = f(x)$ in the sense of Viktorovskii.

The generalized solutions in the sense of Viktorovskii are defined in [3] : A vector function $x(t)$ is a generalized solution of $\dot{x} = f(x)$ in the sense of Viktorovskii if $x(t)$ is defined on a non-degenerate interval I, $x(t)$ is absolutely continuous on I, if to every $d>0$ and to every subset N of $I \times G$ with zero $n+1$-dimensional Lebesgue measure exist vector functions $z^{(i)}(t)$, $i=1,\dots,n$, defined on I such that $z^{(i)}(t) \in G$ for $t \in I$, $f_i(z^{(i)}(t))$ are integrable on I, $\|x(t) - z^{(i)}(t)\| < d$ on I, $\left| x_i(t) - x_i(t_o) - \int_{t_o}^{t} f_i(z^{(i)}(s))ds \right| < d$ for $t, t_o \in I$ and $[t, z^{(i)}(t)] \notin N$ for almost all $t \in I$ and $i=1,\dots,n$. Theorem 4 is a consequence of theorems from [4].

Let us sketch the proofs of the previous theorems. If $R(f,\mathcal{E})$ is nonempty, i.e. there exists $h \in R(f,\mathcal{E})$, then items ii) and iii) immediately imply the local essential boundedness of f. On the other hand, let f be locally essentially bounded. Choose $x_o \in G$. By definition there exists $d>0$, $c>0$ such that

$$m\,\{x : x \in U(x_o,d),\ h(x) \not\subset U(0,c)\} = 0 .$$

Due to a) there exists $B \in \mathcal{E}$, $\overline{U(0,c)} \subset B$. Denote by h_{x_o} the mapping $h_{x_o}(x) = B$ for $x \in U(x_o,d)$ and $h_{x_o}(x) = R_n$ for $x \notin U(x_o,d)$. The mapping h_{x_o} fulfils ii) and iii). We can easily construct a set X_o such that the greatest lower bound $h = \bigwedge_{x_o \in X_o} h_{x_o}$ fulfils all conditions of the definition. Theorem 1 is proved.

We pass to the proof of Theorem 2. First we mention that the set $R(f,\mathcal{E})$ is closed with respect to the greatest lower bounds of countably many mappings, i.e. we have

Lemma 1. If $h_p \in R(f,\mathcal{E})$ then $\bigwedge h_p \in R(f,\mathcal{E})$.

The second step consists in approximating S by a sequence of mappings from $R(f,\mathcal{E})$. The approximation of S at one point is given by

Lemma 2. Let $x \in G$ and $d > 0$ be given. Then there exists $h \in R(f, \mathcal{E})$ such that $h(x) \subset U(S(x), d)$.

This lemma yields that S is upper semi-continuous. Since the mapping h is upper semi-continuous there exists $r > 0$ such that $h(y) \subset U(h(x), d)$ for $y \in U(x, r)$. By the definition of S and by Lemma 2 we have $S(y) \subset h(y) \subset U(h(x), d) \subset U(S(x), 2d)$ for $y \in U(x, r)$. The upper semi-continuity of S is proved.

Let now a point $x \in G$ and a nonnegative integer p be given. By Lemma 2 there exists $h \in R(f, \mathcal{E})$ such that $h(x) \subset U(S(x), 1/p)$. Since h is upper semi-continuous there exists $r(x,p) > 0$ such that $h(y) \subset U(h(x), 1/p) \subset U(S(x), 2/p)$ for $y \in U(x, r(x,p))$. For a given p the balls $U(x, r(x,p))$ cover G and we can choose a countable covering. Denote the corresponding points by x_p^i, $i=1,2,\ldots$ and the corresponding mappings by $h^{(i,p)}$. The upper semi-continuity of S and the properties of $h^{(i,p)}$ imply $\bigwedge_{i,p} h^{(i,p)} \leqq S$. Since $S \leqq \bigwedge_{i,p} h^{(i,p)}$ by the definition of S we have $S = \bigwedge_{i,p} h^{(i,p)}$. The statement of Theorem 2 now follows from Lemma 1.

Theorem 3 can be now easily proved.

The inclusion $F \in R(f, \mathcal{K})$ follows directly from formula (1). Let now a mapping $h \in R(f, \mathcal{K})$ be given. Choose a point $x_0 \in G$ and a number $d > 0$. There exists $r > 0$ such that $h(x) \subset U(h(x_0), d)$ for $x \in U(x_0, r)$ and condition iii) implies $f(x) \in h(x) \subset U(h(x_0), d)$ for almost all $x \in U(x_0, r)$, i.e. the set $N_r = \{x : x \in U(x_0, r), f(x) \notin U(h(x_0), d)\}$ has Lebesgue measure zero. Put $\tilde{N} = \bigcup_{r>0} N_r$. Consider the formula (1) with the set $\tilde{N}$ instead of N. We obtain $F(x_0) \subset \bigcap_{d>0} \overline{\mathrm{Conv}}\, f(U(x_0, r) - N_r) \subset \bigcap_{d>0} \overline{U(h(x_0), d)} = h(x_0)$. We proved $F \leqq h$ which completes the proof of Theorem 3.

The nonautonomous case

The construction of S in the nonautonomous case can be reduced to the autonomous case. Denote the points of R_{n+1} by $[t,x]$ where $t \in R_1$ and $x \in R_n$. We shall use the notation $A_t = \{x : [t,x] \in A\}$ for $A \subset R_{n+1}$. Let f be a mapping $f : G \to \mathcal{A}_0$ where G is a region in R_{n+1}. We can define mappings f_t on G_t for every t by $f_t(x) = f(t,x)$ for $x \in G_t$.

Definition. A mapping $f : G \to \mathcal{A}$ where G is a region in R_{n+1} is t-locally essentially bounded if to every point $[t_0,x_0] \in G$ there exist $d > 0$ and a function $c(t)$ defined and integrable on the interval $\langle t_0-d, t_0+d \rangle$ such that $m\{x : x \in U(x_0,d), f(t,x) \not\subset \not\subset U(0,c(t))\} = 0$ for almost all $t \in \langle t_0-d, t_0+d \rangle$.

Assume that f is t-locally essentially bounded. Then there exists a set T, $T \subset R_1$ with Lebesgue measure zero such that f_t are locally essentially bounded for $t \in R_1 - T$. We can construct tne corresponding S_t for $t \in R_1 - T$. We put $S(t,x) = S_t(x)$. The mapping S is defined on $G - T \times R_n$, i.e. almost everywhere on G. As in the autonomous case the solutions of $\dot{x} \in S(t,x)$ will be called the $\mathcal{E}$--generalized solutions of $\dot{x} \in f(t,x)$.

Formula (1) implies also $F_t(x) = F(t,x)$ for almost all t and this allows us to generalize Theorem 3.

Theorem 5. Let f be a measurable and t-locally essentially bounded function, where G is a region in R_{n+1}. If $\mathcal{E} = \mathcal{K}$ then $S(t,x) = F(t,x)$ for almost all t.

Also Theorem 4 can be generalized.

Theorem 6. Let f fulfil the conditions of Theorem 5. If $\mathcal{E} = \mathcal{Q}$ then an n-dimensional function $x(t)$ is an $\mathcal{E}$-generalized solution of $\dot{x} = f(t,x)$ if and only if $x(t)$ is a generalized solution of $\dot{x} = f(t,x)$ in the sense of Viktorovskii.

Nevertheless, the nonautonomous case is more complicated than the autonomous one since a problem of measurability of S may arise.

Definition. Let h be a mapping $h : G \to \mathcal{A}_0$, $G \subset R_{n+1}$. The mapping h is measurable if the sets $\{[t,x] : h(t,x) \cap A \neq \emptyset\}$ are Lebesgue measurable for all closed sets A, $A \subset R_n$. The mapping h is t-measurable if the sets $\{t : h(t,x) \cap A \neq \emptyset\}$ are Lebesgue measurable for all closed sets A and all $x \in R_n$.

Generally, both measurability and t-measurability of f imply neither measurability nor t-measurability of S but there is a wide family of classes $\mathcal{E}$ for which the problem has an affirmative answer.

Let a class $\mathcal{E}$ be given. If A is a set in R_n, then $\mathcal{E}(A) = \bigcap_{B \supset A, B \in \mathcal{E}} B$ is called the $\mathcal{E}$-closure of A. The $\mathcal{E}$-closure exists if and only if A is bounded, and $\mathcal{E}(A) \in \mathcal{E}$ if and only if $\mathcal{E}(A) \neq \emptyset$.

Definition. The $\mathcal{E}$-closure is called continuous if $\bigcap_n \mathcal{E}(A_n) = \mathcal{E}(\bigcap_n A_n)$ for every sequence of nonempty, compact sets $A_1 \supset A_2 \supset \dots$.

Theorem 7. Let f be a measurable and t-locally essentially bounded mapping $f : G \to \mathcal{A}$, where G is a region in R_{n+1}. If a class $\mathcal{E}_0$ fulfils a) to c) and the $\mathcal{E}$-closure is continuous then the corresponding mapping S is both measurable and t-measurable.

This theorem can be applied e.g. for the classes $\mathcal{E} = \mathcal{C}$, $\mathcal{E} = \mathcal{K}$, $\mathcal{E} = Q$ etc.

Sketch of the proof of Theorem 7. First we shall investigate the case $\mathcal{E} = \mathcal{C}$. Let $x \in G_t$ denote

$$B_t(x) = \{z : m_e[f_t^{-1}(U(z,d_2)) \cap U(x,d_1)] > 0 \text{ for all } d_1 > 0,\ d_2 > 0\}$$

where $f_t^{-1}(A) = \{x : f(t,x) \cap A \neq \emptyset\}$ and m_e is the Lebesgue outer measure.

Lemma 3. Let f be a t-locally essentially bounded mapping $f : G \to \mathcal{A}$ and $\mathcal{E} = \mathcal{C}$. Then $S(t,x) = B(t,x) = B_t(x)$ almost everywhere in G .

If f is a measurable function then the asymptotical continuity of f yields the property iii) (i.e. $f_t(x) \in B_t(x)$). Let $x_p \to x_o$, $y_p \in B_t(x_p)$, $y_p \to y_o$. If numbers $d_1 > 0$, $d_2 > 0$ are given we can find an index q such that $\|x_p - x_o\| < d_1/2$, $\|y_p - y_o\| < d_2/2$ for $p \geqq q$. Since $f_t^{-1}(U(y_o,d_2)) \cap U(x_o,d_1) \supset f_t^{-1}(U(y_p,d_2/2)) \cap \cap U(x_p,d_1/2)$ and the measure of the latter set is positive we obtain $y_o \in B_t(x_o)$ as a consequence of $y_p \in B_t(x_p)$. The last assertion implies that B_t are upper semi-continuous. Certainly $B_t(x) \in \mathcal{C}$ so that $B_t \in R(f_t, \mathcal{C})$ and $S \leqq B$. The proof of $B \leqq S$ is similar as in the proof of Theorem 3. In the general case (f is a mapping) the proof is much more complicated.

In the case $\mathcal{E} = \mathcal{C}$ we can prove Theorem 7 using a lemma from measure theory.

Lemma 4. Let f be a measurable and t-locally essentially bounded mapping $f : G \to \mathcal{A}$. Then the sets

$$\left\{[t,x] : m_e\left[f_t^{-1}(\mathcal{O}) \cap U(x,d)\right] > 0\right\}$$ are Lebesgue measurable in G for every open set $\mathcal{O}$ and $d > 0$.

The sets $\left\{t : m_e\left[f_t^{-1}(\mathcal{O}) \cap U(x,d)\right] > 0\right\}$ are Lebesgue measurable on R_1 for every open set $\mathcal{O}$, $d > 0$ and $x \in G_t$.

Now we shall use the notation $S^{(\mathcal{E})}$ to stress the dependence of S on $\mathcal{E}$. Since $\mathcal{E} \subset \mathcal{C}$ we have $S^{(\mathcal{C})} \subset S^{(\mathcal{E})}$.

Lemma 5. Let f be a t-essentially bounded mapping and let the $\mathcal{E}$ -closure be continuous. Then $S_t^{(\mathcal{E})}(x) = \mathcal{E}(S_t^{(\mathcal{C})}(x))$ for almost all t .

Since $S_t^{(\mathcal{C})}(x) \subset S_t^{(\mathcal{E})}(x)$ we obtain $\mathcal{E}(S_t^{(\mathcal{C})}(x)) \subset \mathcal{E}(S_t^{(\mathcal{E})}(x)) = = S_t^{(\mathcal{E})}(x)$. On the other hand, to every $d_1 > 0$, $x_o \in G_t$ there exists

$d_2 > 0$ so that $\overline{\mathcal{E}(U(S_t^{(C)}(x_o),d_2))} \subset U(\mathcal{E}(S_t^{(C)}(x_o),d_1))$ since the $\mathcal{E}$-closure is continuous and the set $S_t^{(C)}(x_o)$ is compact. The upper semi-continuity of $S_t^{(C)}$ implies that there exists $d_3 > 0$ so that $S_t^{(C)}(x) \subset U(S_t^{(C)}(x_o),d_2)$ for $x \in U(x,d_3)$. These inclusions yield that the mappings $\mathcal{E}(S_t^{(C)}(.))$ are upper semi-continuous. Condition iii) follows from $\mathcal{E}(S_t^{(C)}(x)) \supset S_t^{(C)}(x)$ so that $\mathcal{E}(S_t^{(C)}(.)) \in R(f_t, \mathcal{E})$ and finally $S_t^{(\mathcal{E})}(x) = \mathcal{E}(S_t^{(C)}(x))$.

Theorem 7 in the general case now follows from

Lemma 6. Let h be a measurable mapping $h : \langle 0,1\rangle \longrightarrow C$. Then $\mathcal{E}(h(.))$ is a measurable mapping $\langle 0,1\rangle \longrightarrow \mathcal{E}$.

The case of discontinuous closure was also investigated. Generally there exists a class $\mathcal{E}$ such that the $\mathcal{E}$-closure is discontinuous in only one set A but the corresponding problem of measurability has no positive answer. Nevertheless, if some regularity conditions are fulfilled then the problem has an affirmative answer.

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