Jiří Taufer; Emil Vitásek

A-stability and numerical solution of abstract differential equations

In: Jiří Fábera (ed.): Equadiff IV, Czechoslovak Conference on Differential Equations and Their Applications. Proceedings, Prague, August 22-26, 1977. Springer-Verlag, Berlin, 1979. Lecture Notes in Mathematics, 703. pp. [415]--423.

Persistent URL: http://dml.cz/dmlcz/702242

A-STABILITY AND NUMERICAL SOLUTION OF ABSTRACT DIFFERENTIAL EQUATIONS

J. Taufer and E. Vitásek, Praha

1. Introduction

In this paper we try to give a survey of some results which have been achieved during a few last years in the Mathematical Institute of the Czechoslovak Academy of Sciences and which are related on the one hand to the problems connected with the numerical solution of stiff differential systems, on the other hand, to the problem of construction of methods for the numerical solution of partial differential equations of parabolic type which are of arbitrarily high order of accuracy with respect to the time integration step. It is well known that these two problems are very closely connected and thus, let us first of all mention this connection. Let us begin with recalling the concept of a stiff differential system. For the simplicity we speak only about the linear system with constant coefficients of the form

$$u' = Au. \tag{1.1}$$

This system is said to be stiff if some eigenvalues of the matrix A have negative real parts the magnitudes of which are great in comparison with the magnitudes of the real parts of the other eigenvalues. The solution of the system (1.1) then contains rapidly decreasing components which are negligible in comparison with the other components and very often, we are not interested in them. But when solving the system (1.1) numerically we are generally in such a situation that the magnitude of the integration step is controlled exactly by these decreasing components. The practical consequences are that the integration step must be chosen unpractically small. Thus, it would be desirable to have at our disposal for solving stiff systems such methods in which the components of the approximate solution that correspond to the rapidly damped components of the exact solution would be negligible in comparison with other components even for relatively large values of the integration step. The A-stability introduced by Dahlquist is a property which guarantees such behaviour. Since it is now clear that this property will play an important role in our investigation let us also recall its definition. To define the A-stability, apply first the given method to one differential equation of the type (1.1) where A is a complex constant with a negative real part. Then we say that the method is

A-stable if any approximate solution obtained with help of any integration step converges at infinity to zero. Now it is clear that the magnitude of the integration step is controlled in the case of an A-stable method only by the accuracy with which we want to approximate those components of the vector of the solution which we are interested in.

Let us now demonstrate on the trivial example of the heat conduction equation

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}, \quad 0 < x < 1,\ 0 < t < T \tag{1.2}$$

with the initial condition

$$u(x,0) = \eta(x) \tag{1.3}$$

and with the boundary conditions

$$u(0,t) = u(1,t) = 0 \tag{1.4}$$

the connection of the problem of solving a stiff differential system with the problem of constructing methods for solving parabolic differential equations.

Let us solve the problem (1.2) to (1.4) in such a way that we first discretize only the space variable, for the simplicity, by the finite-difference method. Putting $h = 1/m$ we obtain

$$\underline{u}_h' = A_h \underline{u}_h \tag{1.5}$$

where $\underline{u}_h = \underline{u}_h(t)$ is (m-1)-dimensional vector the components of which approximate the exact solution at the points $x_i = ih$, $i = 1, \ldots, m-1$ and A_h is the matrix given by

$$A_h = \frac{1}{h^2} \begin{bmatrix} -2 & 1 & 0 & \cdots & 0 \\ 1 & \ddots & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & 1 \\ 0 & \cdots & 0 & 1 & -2 \end{bmatrix} \tag{1.6}$$

It is commonly known that this method is convergent and that its error is of order h^2. If we want to replace the original problem by the finite dimensional one completely we must, moreover, solve the system (1.5). But the eigenvalues λ_ν of the matrix A_h are given by the formula

$$(1.7) \qquad \lambda_\nu = -\frac{4}{h^2}\sin^2\frac{\nu\pi}{2m}, \qquad \nu = 1,\dots,m-1$$

and, consequently, the system (1.5) is stiff since λ_{m-1} behaves for small h as $-4/h^2$. Thus, if we apply to it a general method (not satisfying further special assumptions) the best result which we can expect is a relatively stable method, i.e., a method in which the magnitude of the time integration step is restricted by the magnitude of the space integration step. It may also happen that we get a completely unstable method. On the other hand, the application of an A-stable method will lead to an absolutely stable method and the convergence proof will cause no substantial difficulties.

Thus, we see that the problem of constructing absolutely stable methods the orders of which are arbitrarily high is very closely connected with the problem of constructing A-stable methods of arbitrarily high orders. Since it is known that the order of a classical method which is A-stable is at most 2 we must begin with introducing a class of methods which contain A-stable methods of arbitrarily high orders. There are various possibilities; we introduce the so called block onestep methods, especially for that reason that they are very simple and easily applicable to parabolic equations.

2. BO methods and A-stability

The method will be formulated for one differential equation

$$(2.1) \qquad u' = f(t,u), \qquad t \in \langle 0,T \rangle$$

with the initial condition

$$(2.2) \qquad u(0) = \eta .$$

The right-hand term of this differential equation is assumed to be defined, continuous and satisfying the Lipschitz condition with respect to u in the strip $0 < t < T$, $-\infty < u < \infty$ so that the solution of the problem (2.1),(2.2) exists and is unique in the whole interval $\langle 0,T \rangle$.

Let an integer $k \geqq 1$, a matrix C of order k and a k-dimensional vector $\underline{d}$ be given. Further, put $t_i = ih$, $i = 0,1,\dots$ where $h > 0$ is the integration step and denote by u_j the approximate solution at the point t_j. Then the block onestep method (BO method) is given by the formula

$$(2.3) \qquad \begin{bmatrix} u_{n+1} \\ \\ u_{n+k} \end{bmatrix} = \begin{bmatrix} u_n \\ \\ u_n \end{bmatrix} + hC \begin{bmatrix} f_{n+1} \\ \\ f_{n+k} \end{bmatrix} + hf_n\underline{d}, \quad n = 0,k,2k,\ldots$$

($f_j = f(t_j,u_j)$). One step of the BO method consists therefore in computing k values of the approximate solution simultaneously from the generally nonlinear system of equations and the following step is started with the last one of these k values. The Lipschitz property of f guarantees that the method is practicable at least for sufficiently small h.

Defining now in the more or less usual way the local truncation error of the method and with its help the order it can be proved without substantial difficulties that the method of order at least 1 is convergent and that the method of order p leads to the accuracy of order h^p (supposing that the exact solution is sufficiently smooth).

If we now want to study the A-stability of a BO method we must apply it to the equation (1.1). If we eliminate unnecessary values of the approximate solution we get

$$(2.4) \qquad u_{(r+1)k} = \frac{P(z)}{Q(z)} u_{rk}, \quad r = 0,1,\ldots$$

where

$$(2.5) \qquad z = hA,$$

$$(2.6) \qquad Q(z) = \det(I - zC)$$

and $P(z)$ is the determinant of the matrix which is obtained from the matrix $I - zC$ by replacing its last column by the vector $\underline{e} + z\underline{d}$ where $\underline{e} = (1,\ldots,1)^T$. Thus, the BO method leads in this special situation to the rational approximation of the exponential function exp(kz) and the fulfilment of the inequality

$$(2.7) \qquad \left|\frac{P(z)}{Q(z)}\right| < 1$$

for any z with a negative real part forms obviously the necessary and sufficient condition for the A-stability of the BO method.

Further, the class of BO methods has such a property that to any rational approximation of the exponential there exists a BO method such that the ratio in (2.4) is exactly this approximation. This fact is very important and it implies among other that in the class of BO methods there exist A-stable methods of arbitrarily high orders.

3. Approximate solution of abstract differential equations

Let us pass now to the numerical solution of parabolic differential equations. As we have mentioned above the problem we are mostly interested in is the problem of the order of accuracy with respect to the time mesh-size. In order to emphasize this fact we will not deal in what follows with the partial differential equations of parabolic type but we will be interested in the abstract ordinary differential equation

$$\frac{du(t)}{dt} = Au(t) + f(t), \quad t \in (0,T) \tag{3.1}$$

with the initial condition

$$u(0) = \eta \tag{3.2}$$

where the unknown function $u(t)$ is a function of the real variable t with values in a Banach space B, the given function $f(t)$ has also its values in B and is assumed to be continuous while A is generally an unbounded operator in B. We will suppose about it that its domain $\mathcal{D}(A)$ is dense in B, that A is closed and that it is the generator of a strongly continuous semigroup of operators, i.e., that there exist (real) constants M and ω such that

$$\|(\lambda I - A)^{-n}\| \leqq \frac{M}{(\mathrm{Re}\,\lambda - \omega)^n} \tag{3.3}$$

for any positive integer n and for any (complex) λ such that $\mathrm{Re}\,\lambda \geqq \omega$. In this situation, it is possible to speak also about the generalized solution of (3.1),(3.2) which is defined by the formula

$$u(t) = U(t)\eta + \int_0^t U(t-\tau)f(\tau)d\tau \tag{3.4}$$

where $U(t)$ is the semigroup generated by A. Consequently, this generalized solution exists for any $\eta \in B$.

Let us apply the BO method to the problem (3.1),(3.2). We get

$$\begin{bmatrix} u_{n+1} \\ \vdots \\ u_{n+k} \end{bmatrix} = \begin{bmatrix} u_n \\ \vdots \\ u_n \end{bmatrix} + hC \otimes A \begin{bmatrix} u_{n+1} \\ \vdots \\ u_{n+k} \end{bmatrix} + hD \otimes A \begin{bmatrix} u_n \\ \vdots \\ u_n \end{bmatrix} + hC \begin{bmatrix} f_{n+1} \\ \vdots \\ f_{n+k} \end{bmatrix} + hD \begin{bmatrix} f_n \\ \vdots \\ f_n \end{bmatrix} \tag{3.5}$$

where D is the diagonal matrix with the components of the vector $\underline{d}$ on the main diagonal and the operator $C \otimes A$ mapping $\mathcal{D}(A) \times \ldots \times \mathcal{D}(A)$ into $B \times \ldots \times B$ is defined by

$$(3.6) \qquad C \otimes A = \begin{bmatrix} c_{11}A & \cdots & c_{1k}A \\ \vdots & & \vdots \\ c_{k1}A & \cdots & c_{kk}A \end{bmatrix}$$

and an analogous definition holds for the operator $D \otimes A$.

Here we cannot conclude as simply as above that (3.5) has a solution since here the operator $(I - hC \otimes A)$ is generally unbounded. Thus, the first question which must be answered is the question of the feasibility of our method. About this problem the following theorem can be easily proved.

<u>Theorem 3.1</u> Let A be the generator of a strongly continuous semigroup of operators and let C have its eigenvalues in the right-hand half plane. Then there exists h_o such that the operator $I - hC \otimes A$ has a bounded inverse for all $h \leqq h_o$ and it holds

$$(3.7) \qquad (I - hC \otimes A)^{-1} = \begin{bmatrix} M_{11} & \cdots & M_{1k} \\ \vdots & & \vdots \\ M_{k1} & \cdots & M_{kk} \end{bmatrix}$$

where M_{ij} are rational functions of hA.

Strictly speaking, this theorem guarantees the feasibility of our method only in the case of the classical problem, i.e., in the case $\eta \in \mathcal{D}(A)$. But the operators M_{ij} from Theorem 3.1 allow to rewrite (3.5) in the form which has sense also in the general case $\eta \in B$. The details will be omitted.

The practicability of the method does not guarantee the convergence. The convergence is controlled, as it can be expected, by the behaviour of the operator $R(hA) = P(hA)Q^{-1}(hA)$ where $P(z)$ and $Q(z)$ are the polynomials defined by (2.6).

<u>Theorem 3.2</u> Let a BO method of order $p \geqq 1$ with a regular matrix C be given and let A be the generator of a strongly continuous semigroup of operators. Then the approximate solution obtained by this method converges at the point t to the generalized solution of the problem (3.1).(3.2) if and only if

$$(3.8) \qquad \|R^n(\tfrac{t}{kn} A)\| \leq M(t)$$

for $n = 0,1,\ldots$ Moreover, supposing that the generalized solution is sufficiently smooth the order of the error is h^p.

The proof is a simple consequence of expressing $R^n((t/kn)A)$ by the Dunford integral.

From this theorem it follows immediately that, e.g., in the case of a Hilbert space and a selfadjoint operator A the A-stability is sufficient for the convergence. In general case, the results are not yet final. Nevertheless, the following theorem solves our problem in a special case.

Theorem 3.3 Let an A-stable BO method of order $p \geq 1$ be given. Further, let A be an operator with the domain which is dense in B and let its resolvent $(\lambda I - A)^{-1}$ satisfy

$$(3.9) \qquad \|(\lambda I - A)^{-1}\| \leq M(1 + |\lambda|)^q, \qquad q \geq 0,$$

for $\operatorname{Re} \lambda > \omega$. Then it is possible to apply the method to the homogeneous problem (3.1),(3.2) with this operator and the sequence of elements obtained in this way forms for $\eta \in \mathcal{D}(A^{\ell})$ where $\ell > q + 1$ a convergent sequence.

Proof. Let $\ell > q + 1$ and let us fix t and $\eta \in \mathcal{D}(A^{\ell})$. According to the preceding we have to prove that the sequence

$$(3.10) \qquad u_n = R^n(\tfrac{t}{kn} A)\eta$$

converges in B. To prove this fact, let us put first of all $U_n = \{\lambda, \operatorname{Re} \lambda \geq \omega_1, |\lambda| \leq Kn\}$ where $\omega_1 > \omega$ and K is such a constant that the function $R((t/k)\lambda)$ is holomorphic outside the circle $|\lambda| \leq K$. In virtue of this fact it follows immediately that $R((t/kn)\lambda)$ is holomorphic outside U_n for sufficiently large n. Further, since $\eta \in \mathcal{D}(A^{\ell})$, there exists $z_o \in B$ such that

$$(3.11) \qquad \eta = (\lambda_o I - A)^{-\ell} z_o$$

and λ_o is an arbitrary element from U_1. Thus, if we denote by Γ_n the boundary of U_n we can write, for any sufficiently large n and for any $m \geq n$,

$$(3.12) \qquad u_n = R^n(\tfrac{t}{kn} A)(\lambda_o I - A)^{-\ell} z_o =$$

$$\frac{1}{2\pi i} \int_{\Gamma_m} R^n\left(\frac{t}{kn}\lambda\right)(\lambda_0 - \lambda)^{-\ell}(\lambda I - A)^{-1} z_0 \, d\lambda .$$

The assumption (3.9) allows us to pass in (3.12) to the limit for $m \longrightarrow \infty$. We get

$$(3.13) \qquad u_n = \frac{1}{2\pi i} \int_{\omega_1 - i\infty}^{\omega_1 + i\infty} R^n\left(\frac{t}{kn}\lambda\right)(\lambda_0 - \lambda)^{-\ell}(\lambda I - A)^{-1} z_0 \, d\lambda .$$

The property $|R(\lambda)| < 1$ for $\operatorname{Re}\lambda < 0$ (following immediately from the A-stability of the given method) implies the existence of a constant L (independent of n) such that

$$(3.14) \qquad \left| R^n\left(\frac{t}{kn}\lambda\right) \right| \leqq \exp(tL)$$

for $n = 1,2,\ldots$ and for $\operatorname{Re}\lambda = \omega_1$. Thus, the function

$$M \exp(tL)(\lambda_0 - \lambda)^{-\ell}(1 + |\lambda|^q)$$

forms an integrable majorant for the integrand in (3.13) and we can pass in (3.13) to the limit under the integral sign. We obtain

$$(3.15) \qquad \lim_{n \to \infty} u_n = \frac{1}{2\pi i} \int_{\omega_1 - i\infty}^{\omega_1 + i\infty} \exp(\lambda t)(\lambda_0 - \lambda)^{-\ell}(\lambda I - A)^{-1} z_0 \, d\lambda$$

and since the last integral converges absolutely the assertion of the theorem follows immediately.

Since in the case that A is the generator of a strongly continuous semigroup of operators it is possible to choose for q in Theorem 3.3 the value 0, it follows that in our situation the A-stable method is convergent for problems with sufficiently smooth initial data.

References

[1] N. Dunford and J.T. Schwartz: Linear Operators, Interscience Publishers, New-York, London, 1958.

[2] M. Práger, J. Taufer and E. Vitásek: Overimplicit Multistep Methods, Aplikace matematiky 6(1973), 399 - 421.

[3] E. Vitásek and J. Taufer: Numerical Solution of Evolution Problems in Banach Spaces, Topics in Numerical Analysis II, Proceedings of the Royal Irish Academy Conference on Numerical

Analysis, 1974, edited by John J.H. Miller, 243 - 251.

Authors' address: Matematický ústav ČSAV, Opletalova 45,
110 00 Praha 2, Czechoslovakia