Miroslav Sova
Abstract Cauchy problem

In: Jiří Fábera (ed.): Equadiff IV, Czechoslovak Conference on Differential Equations and Their Applications. Proceedings, Prague, August 22-26, 1977. Springer-Verlag, Berlin, 1979. Lecture Notes in Mathematics, 703. pp. [384]--396.

Persistent URL: http://dml.cz/dmlcz/702239

ABSTRACT CAUCHY PROBLEM

M. Sova, Praha

The aim of this lecture is to present some results from the theory of the Cauchy problem for linear differential equations in Banach spaces with unbounded coefficients.

I. PRELIMINARIES

We denote by E an arbitrary Banach space over the complex number field C. By an operator we always mean a linear operator acting on a linear subspace of E into E.

For a function $f:(0,\infty)\to E$ we introduce the notion of the r-th integral of f by the following definition: if f is integrable on bounded subintervals of $(0,\infty)$, then we put for $t>0$ and $r\in$ $\in\{1,2,\dots\}$,

$$\underline{r|}\int_0^t f(\tau)d\tau = \int_0^t\int_0^{\tau_1}\dots\int_0^{\tau_{r-1}} f(\tau_r)d\tau_1 d\tau_2\dots d\tau_r \ .$$

Moreover, for $t>0$ we shall write $\underline{0|}\int_0^t f(\tau)d\tau = f(t)$.

II. SETTING OF THE CAUCHY PROBLEM

Indispensable definitions and properties connected with the basic notion of correctness of the abstract Cauchy problem are shortly summarized in this section.

Let $A_1,A_2,\dots,A_n$ $(n\in\{1,2,\dots\})$ be an arbitrary n-tuple of linear operators in E which will be fixed throughout the whole lecture.

A function $u:(0,\infty)\to E$ will be called a <u>solution of the Cauchy problem for</u> $A_1,A_2,\dots,A_n$ if

(S_1) u is n-times differentiable on $(0,\infty)$,

(S_2) $u^{(n-i)}(t)\in D(A_i)$ for $t>0$ and $i\in\{1,2,\dots,n\}$,

(S_3) the functions $A_i u^{(n-i)}$ are continuous on $(0,\infty)$ and bounded on $(0,1)$ for every $i\in\{1,2,\dots,n\}$,

(S_4) $u^{(n)}(t)+A_1u^{(n-1)}(t)+\dots+A_nu(t)=0$ for $t>0$,

(S_5) $u(0_+)=u'(0_+)=\dots=u^{(n-2)}(0_+)=0$, $u^{(n-1)}(0_+)$ exists.

The notion of a solution in our sense is at the first sight restrictive because we consider only special initial values. However, it turned out that it is sufficiently general for the study of the

Cauchy problem since the solutions with arbitrary initial values can be calculated by means of these special solutions.

We shall say that *the Cauchy problem for* $A_1, A_2, \ldots, A_n$ is *determined* if every solution u satisfying $u^{(n-1)}(0_+) = 0$ is identically zero on $(0, \infty)$.

Further, we shall say that *the Cauchy problem for* $A_1, A_2, \ldots, A_n$ is *extensive* if there exists a dense subset $D \subseteq E$ such that for every $x \in D$ there is a solution u with $u^{(n-1)}(0_+) = x$.

Now let m be a nonnegative integer. Then *the Cauchy problem for* $A_1, A_2, \ldots, A_n$ will be called *correct of class* m if

(I) it is extensive,

(II) there exist nonnegative constants M, ω so that for every solution u, for every $t > 0$ and $i \in \{1, 2, \ldots, n\}$,

$$\left\| \underline{|m+1} \int_0^t A_i u^{(n-i)}(\tau)\, d\tau \right\| \leqq M\, e^{\omega t} \| u^{(n-1)}(0_+) \| .$$

Finally, *the Cauchy problem for* $A_1, A_2, \ldots, A_n$ will be called *correct* if it is correct of class m for some nonnegative integer m.

The operators $A_1, A_2, \ldots, A_n$ will be sometimes called *the generating operators* (*of the Cauchy problem associated with them*).

It follows from the condition (II) that

(II′) $$\left\| \underline{|m} \int_0^t u^{(n-1)}(\tau)\, d\tau \right\| \leqq (1 + nM) e^{\omega t} \| u^{(n-1)}(0_+) \| \text{ for every } t > 0.$$

If $A_1 = A_2 = \ldots = A_{n-1} = 0$, then the conditions (II) and (II′) are equivalent, but in the general case, it is necessary to introduce the condition (II) in a more complicated form to get a satisfactory theory.

Theorem 1. *Every correct Cauchy problem is determined as well.*

Proof. Immediately from the property (II′). :::

Our definition of correctness was motivated by the classical Hadamard's considerations on the correctness of the initial value problem and by certain well-known special cases which have been examined lately. The first and second order Cauchy problems were studied in a different setting as generation problems for operator semigroups, distribution operator semigroups and cosine and sine operator functions (see [1] - [6]). Further, the Timoshenko equation of transverse vibrations of a beam or a plate led to the study of the fourth order Cauchy problem and its correctness (cf. [7]).

Let $u, h:(0,\infty) \to E$. The function u will be called a *response to the excitation* h *for the Cauchy problem for* $A_1, A_2, \ldots, A_n$ if

(R_1) u is n-times differentiable on $(0,\infty)$,

(R_2) $u^{(n-i)}(t) \in D(A_i)$ for every $t > 0$ and $i \in \{1,2,\ldots,n\}$,

(R_3) the functions $A_i u^{(n-i)}$ are continuous on $(0,\infty)$ and bounded on $(0,1)$ for every $i \in \{1,2,\ldots,n\}$,

(R_4) $u^{(n)}(t)+A_i u^{(n-1)}(t)+\ldots+A_n u(t) = h(t)$ for every $t > 0$,

(R_5) $u(0_+) = u'(0_+) = \ldots = n^{(n-1)}(0_+) = 0$.

Further, we define the space $L_{loc}((0,\infty),E)$ as the space of all functions $f:(0,\infty) \to E$ integrable over bounded subsets of $(0,\infty)$ and topologized by the natural Fréchet topology determined by the seminorms $\int_0^T \|f(\tau)\| d\tau$, $T > 0$.

Theorem 2. *Let* m *be a nonnegative integer. If the operators* $A_1, A_2, \ldots, A_n$ *are closed, then the following two statements are equivalent*:

(A) *the Cauchy problem for* $A_1, A_2, \ldots, A_n$ *is correct of class* m ,

(B) (I) *there is a dense subset* J *of* $L_{loc}((0,\infty),E)$ *so that for every* $h \in J$ *there exists a response* u *to the excitation* h ,

(II) *there exist nonnegative constants* M, ω *so that for every* u, h *such that* u *is a response to the excitation* h *and* $h \in L_{loc}((0,\infty),E)$, *for every* $t > 0$ *and* $i \in \{1,2,\ldots,n\}$,

$$\left\| \underline{|m+1} \int_0^t A_i u^{(n-i)}(\tau) d\tau \right\| \leq M e^{\omega t} \int_0^t \|h(\tau)\| d\tau .$$

Proof. See [8], p. 124 and 128. :::

III. SPECTRAL PROPERTIES OF THE CAUCHY PROBLEM

Our aim in this section will be to clarify the relations between the correctness and the spectral properties of the generating operators.

Under spectral properties of operators $A_1, A_2, \ldots, A_n$ we understand various properties of the operator polynomial $z^n I + z^{n-1} A_1 + \ldots + A_n$, $z \in C$, which will be called the *spectral polynomial of the operators* $A_1, A_2, \ldots, A_n$ and denoted by $P(.;A_1, A_2, \ldots, A_n)$.

We introduce some notions and notation related to the spectral polynomial of the operators $A_1,A_2,\dots,A_n$.

We denote by $\rho(A_1,A_2,\dots,A_n)$ the set of all $z \in C$ for which the operator $P(z;A_1,A_2,\dots,A_n)$ is one-to-one and its inverse is everywhere defined and bounded. This inverse $P(z;A_1,A_2,\dots,A_n)^{-1}$ will be denoted by $R(z;A_1,A_2,\dots,A_n)$. Finally, we put $\sigma(A_1,A_2,\dots,A_n) = C \setminus \rho(A_1,A_2,\dots,A_n)$.

The set $\rho(A_1,A_2,\dots,A_n)$ is called the resolvent set and $\sigma(A_1,A_2,\dots,A_n)$ the spectrum of the operators $A_1,A_2,\dots,A_n$. The function $R(.;A_1,A_2,\dots,A_n)$, defined on $\rho(A_1,A_2,\dots,A_n)$, will be called the resolvent function of these operators.

In this notation, the specification of operators $A_1,A_2,\dots,A_n$ is sometimes omitted in proofs.

It should be still noted that the above introduced notions and notation do not coincide with those usually defined for one operator. They are slightly modified.

Theorem 3. If the operators $A_1,A_2,\dots,A_n$ are closed, then

(a) the set $\rho(A_1,A_2,\dots,A_n)$ is open,

(b) the function $R(.;A_1,A_2,\dots,A_n)$ is analytic on the set $\rho(A_1,A_2,\dots,A_n)$,

(c) the operators $A_iR(z;A_1,A_2,\dots,A_n)$ are everywhere defined and bounded for every $z \in \rho(A_1,A_2,\dots,A_n)$ and $i \in \{1,2,\dots,n\}$,

(d) for every $i \in \{1,2,\dots,n\}$, the function $A_iR(.;A_1,A_2,\dots,A_n)$ is analytic on the set $\rho(A_1,A_2,\dots,A_n)$.

Proof. See [9], p. 233-236. :::

Theorem 4 (uniqueness for the Cauchy problem). We assume that there exist nonnegative constants ω, d, D such that

(α) every $\lambda > \omega$ belongs to $\rho(A_1,A_2,\dots,A_n)$,

(β) $\|R(\lambda;A_1,A_2,\dots,A_n)\| \leqq De^{d\lambda}$ for every $\lambda > \omega$.

Under these assumptions, the Cauchy problem for $A_1,A_2,\dots,A_n$ is determined.

Proof. See [10], p. 38. :::

In what follows we need still the notion of higher domains of operators $A_1,A_2,\dots,A_n$ which will be defined by induction in the following way:

$D_0(A_1,A_2,\dots,A_n) = E$,

$D_{k+1}(A_1,A_2,\dots,A_n) = \{x : x \in D(A_1) \cap D(A_2) \cap \dots \cap D(A_n)$ and $A_ix \in D_k(A_1,A_2,\dots,A_n)$ for every $i \in \{1,2,\dots,n\}\}$, $k \in \{0,1,\dots\}$.

Finally, we define $D_\infty(A_1,A_2,\dots,A_n)$ as the intersection $\bigcap_{k=0}^{\infty} D_k(A_1,A_2,\dots,A_n)$.

Theorem 5 (existence for the Cauchy problem). Let m be a nonnegative integer. We assume that

(α) the operators $A_1,A_2,\dots,A_n$ are closed,

(β) the set $D_{m+1}(A_1,A_2,\dots,A_n)$ is dense in E,

(γ) there exist nonnegative constants M, ω so that

(I) $(\omega,\infty) \subseteq \varrho(A_1,A_2,\dots,A_n)$,

(II) $$\left\| \frac{d^p}{d\lambda^p}\lambda^{n-m-i-1} A_i R(\lambda;A_1,A_2,\dots,A_n)\right\| \leqq \frac{Mp!}{(\lambda-\omega)^{p+1}}$$

for every $\lambda > \omega$, $i \in \{1,2,\dots,n\}$ and $p \in \{0,1,\dots\}$.

Under these assumptions, the Cauchy problem for $A_1,A_2,\dots,A_n$ is correct of class m.

Proof. See [10], p. 46. :::

The preceding existence Theorem 5 for the Cauchy problem can be essentially converted in the following way:

Theorem 6 (converse). Let m be a nonnegative integer. We assume that

(α) the operators $A_1,A_2,\dots,A_n$ are closed,

(β) the Cauchy problem for $A_1,A_2,\dots,A_n$ is correct of class m.

Under these assumptions,

(a) the set $D_1(A_1,A_2,\dots,A_n)$ is dense in E,

(b) the condition (γ) of Theorem 4 holds.

Proof. See [10], p. 55. :::

The preceding Theorems 5 and 6 cover the well-known Hille-Yosida theorem on generation of strongly continuous semigroups, the related theorem on generation of cosine and sine operator functions and the Lions theorem on generation of exponential distribution semigroups (cf. [1] - [6]).

The assumption (γ) of Theorem 5 can be replaced by another condition which does not involve the derivatives of the resolvent function on a real halfaxis but makes use of the behavior of this function on a complex halfplane.

Theorem 7. If the operators $A_1,A_2,\dots,A_n$ are closed, then the condition (γ) of Theorem 5 is equivalent with

(γ') there exist nonnegative constants M, ω so that

(I') $\{z:\mathrm{Re}\, z > \omega\} \subseteq \varrho(A_1,A_2,\dots,A_n)$, (II′) $\|z^{n-m-i-1}A_iR(z;A_1,A_2,\dots,A_n)\| \leq \frac{M}{\mathrm{Re}\,z-\omega}$ for every Re z and $i \in \{1,2,\dots,n\}$,

(III′) $\left\| \int_{\alpha-i\infty}^{\alpha+i\infty} \frac{z^{n-m-i-1}A_iR(z;A_1,A_2,\dots,A_n)}{(1-s(z-\alpha))^r}\,dz \right\| \leq M$ for every $\alpha > \omega$, $s > 0$, $i \in \{1,2,\dots,n\}$ and $r \in \{2,3,\dots\}$.

Proof. We shall need the following

L e m m a . Let $\alpha \in R$, $J \in \{z:\mathrm{Re}\,z \geq \alpha\} \to E$, $k \in \{0,1,\dots\}$. If the function J is continuous for $\mathrm{Re}\,z \geq \alpha$, analytic for $\mathrm{Re}\,z > \alpha$ and $\|J(z)\| \leq K(1+|z|)^k$ for $\mathrm{Re}\,z \geq \alpha$ with $K \geq 0$, then for every $\lambda > \alpha$ and $p \in \{k+1,k+2,\dots\}$,

$$[*] \quad J^{(p)}(\lambda) = (-1)^p \frac{p!}{2\pi i} \int_{\alpha-i\infty}^{\alpha+i\infty} \frac{J(z)}{(\lambda-z)^p}\,dz .$$

The verification of this Lemma is easy by means of the Cauchy integral formula.

Now to the proof itself. We fix $i \in \{1,2,\dots,n\}$ and write $H(z) = z^{n-m-i-1}A_iR(z)$ for $z \in \varrho$.

$(\gamma') \Rightarrow (\gamma)$ According to Theorem 3, our Lemma is applicable with arbitrary $\alpha > \omega$ to corresponding restriction of the function H and then the identity [*] gives immediately the estimate (II) of (γ) for $p \in \{1,2,\dots\}$ if we finally let $\alpha \to \omega_+$. For $p = 0$, we deduce the desired estimate from the case $p = 1$ because $H(\lambda) \to 0$ $(\lambda \to \infty)$ according to (II′).

$(\gamma) \Rightarrow (\gamma')$ Fix z in the halfplane $\mathrm{Re}\,z > \omega$.

It is easy to see from the inequality (II) in (γ) that, for sufficiently large $\lambda > \omega$, the series $\sum_{k=0}^{\infty} \frac{(z-\lambda)^k}{k!} R^{(p)}(\lambda)$ and $\sum_{k=0}^{\infty} \frac{(z-\lambda)^k}{k!} A_iR^{(p)}(\lambda)$ converge and define $R(z)$ and $A_iR(z)$.

This shows in particular that (I) is true.

On the other hand, after a little calculation, we find that there exists a $\lambda_0 \geq \omega$ such that $\lambda-\omega > |z-\lambda|$ for $\lambda > \lambda_0$. By means of this inequality and of the inequality (II) we can now estimate the above series for $A_iR(z)$ and we easily obtain the desired inequality (II′) letting $\lambda \to \infty$.

The facts proved above together with Theorem 3 enable us to apply again our Lemma with arbitrary $\alpha > \omega$ to corresponding restriction of the function H and now it is a matter of routine to get (III′) from (II) by means of the identity [*]. :::

The following two theorems concern the general notion of correctness.

Theorem 8 (existence for the Cauchy problem). We assume that

(α) the operators $A_1,A_2,\ldots,A_n$ are closed,

(β) the set $D_\infty(A_1,A_2,\ldots,A_n)$ is dense in E ,

(γ) there exist nonnegative constants L, l, $\varkappa$ so that

(I) $\{z:\mathrm{Re}\,z \geq \varkappa\} \subseteq \rho(A_1,A_2,\ldots,A_n)$,

(II) $\|A_i R(z;A_1,A_2,\ldots,A_n)\| \leq L(1+|z|)^l$ for every $\mathrm{Re}\,z > \varkappa$ and $i \in \{1,2,\ldots,n\}$.

Under these assumptions, the Cauchy problem for $A_1,A_2,\ldots,A_n$ is correct.

Proof. Choosing m sufficiently large, for example $m = l+2$, we verify easily that the condition (γ') in Theorem 7 is satisfied. Consequently, by Theorem 7, the assumptions of Theorem 5 hold and this implies the assertion of the present theorem. :::

Theorem 9 (converse). We assume that

(α) the operators $A_1,A_2,\ldots,A_n$ are closed,

(β) the Cauchy problem for $A_1,A_2,\ldots,A_n$ is correct.

Under these assumptions,

(a) the set $D_1(A_1,A_2,\ldots,A_n)$ is dense in E ,

(b) the condition (γ) of Theorem 8 holds.

Proof. Immediate consequence of Theorems 6 and 7 (l may be taken equal to m chosen for Theorem 6). :::

In the following theorem we shall make some a priori restrictions concerning the basic space E and the generating operators $A_1,A_2,\ldots,A_n$. Then the Cauchy problem for these operators is not only always determined and extensive but above all, its correctness is fully specified merely by the location of the spectrum of generating operators. All the new notions used (normal operator, abelian system, spectral measure, spectral integral) can be found in [11] (see in particular Chap. VII, VIII and X).

Theorem 10. If the operators $A_1,A_2,\ldots,A_n$ are normal and form an abelian system in a Hilbert space E , then the Cauchy problem for $A_1,A_2,\ldots,A_n$ is always determined and extensive.

Moreover, it is correct if and only if there exists a constant ω such that $\sigma(A_1,A_2,\ldots,A_n) \subseteq \{z:\mathrm{Re}\,z \leq \omega\}$.

Proof. We denote by $\mathcal{B}$ the family of all Borel subsets of C .

According to [11] (in particular Chap. X), we can find a spectral measure $\mathcal{E}$ on $\mathcal{B}$ and Borel measurable functions $a_1, a_2, \dots, a_n : C \to C$ so that $A_i x = \int_C a_i(\sigma)\, \mathcal{E}(d\sigma)x$ for $x \in D(A_i)$ in the sense of spectral integration described in [11], Chap. VII. In the rest of this proof we shall frequently use this integration without special reference to [11].

We begin with proving that our Cauchy problem is determined and extensive. Let $\mathcal{B}_0$ be the family of bounded sets from $\mathcal{B}$. It is easy to prove that the set $Q = \{x : \mathcal{E}(X)x = x$ for some $X \in \mathcal{B}_0\}$ is dense in E. Moreover, $Q \subseteq D_\infty(A_1, A_2, \dots, A_n)$. On the other hand, the operators $A_i \mathcal{E}(X)$ are bounded and $A_i \mathcal{E}(X) \supseteq \mathcal{E}(X) A_i$ for $X \in \mathcal{B}_0$. These facts enable us to construct easily a solution for every $x \in Q$ and so to prove that the problem is extensive. To prove that it is determined we use Theorem 4 for bounded operators $A_i \mathcal{E}(X)$, $X \in \mathcal{B}_0$, (since in this case it is clearly valid) and the fact that there is a sequence $X_k \in \mathcal{B}_0$ such that $\mathcal{E}(X_k)x \to x$ for any $x \in E$.

Now to the proof of the last assertion of our theorem.

We shall write $p(z,s) = z^n + a_1(s) z^{n-1} + \dots + a_n(s)$ for $z, s \in C$. Further for $X \in \mathcal{B}$ we put $K(X) = \{z : p(z,s) = 0$ for some $s \in C \setminus X\}$.

We first need to prove that

[*] there exists $N \in \mathcal{B}$ such that $\mathcal{E}(X) = 0$ and $K(N) \subseteq \subseteq \sigma(A_1, A_2, \dots, A_n)$.

To this aim, let us denote $N_z = \{s : p(z,s) < \|R(z)\|^{-1}\}$ for $z \in \rho = \rho(A_1, A_2, \dots, A_n)$.

It is clear that $N_z \in \mathcal{B}$ for any $z \in \rho$. Now we shall show that $\mathcal{E}(N_z) = 0$ for $z \in \rho$. Proceeding indirectly, we fix $z \in \rho$ such that $\mathcal{E}(N_z) \neq 0$. We have shown above that $D_\infty(A_1, A_2, \dots, A_n)$ is dense in E and hence we can find an $x \neq 0$, $x \in D_\infty(A_1, A_2, \dots, A_n)$ and $\mathcal{E}(N_z)x = x$. Since then $C \setminus N_z$ is a null set for the measure $\|\mathcal{E}(.)x\|^2$ we obtain $\|P(z)x\| =$

$$= \left\|\int_C p(z,\sigma)\, \mathcal{E}(d\sigma)x\right\| = \sqrt{\int_C |p(z,\sigma)|^2 \, \|\mathcal{E}(d\sigma)x\|^2} =$$

$$= \sqrt{\int_{N_z} |p(z,\sigma)|^2 \, \|\mathcal{E}(d\sigma)x\|^2} < \sqrt{\int_{N_z} \|R(z)\|^{-2} \, \|\mathcal{E}(d\sigma)x\|^2} =$$

$= \|R(z)\|^{-1} \, \|\mathcal{E}(N_z)x\| = \|R(z)\|^{-1} \|x\|$. But this inequality is contradictory since it implies that $\|R(z)\|^{-1}\|x\| = \|R(z)\|^{-1} \|R(z)P(z)x\| \leq$

$\|R(z)\|^{-1} \|R(z)\| \, \|P(z)x\| = \|P(z)x\| < \|R(z)\|^{-1} \|x\|$.

Now we put $N = \bigcup N_z$ where z runs through $z \in \rho$ with rational real and imaginary parts. It is immediate from the preceding result that $N \in \mathcal{B}$ and $\mathcal{E}(N) = 0$. With regard to the continuity of $p(.,s)$ we obtain further that $|p(z,s)| \geqq \|R(z)\|^{-1}$, i.e. in particular $p(z,s) \neq 0$, for $z \in \rho$ and $s \in C \setminus N$. But this implies that $K(N) \cap \rho = \emptyset$, i.e. $K(N) \subseteq \sigma$ which proves [*].

Let our Cauchy problem be correct. Then by Theorem 9, there is an ω such that $\rho(A_1,A_2,\ldots,A_n) \subseteq \{z:\mathrm{Re}z \leqq \omega\}$.

Conversely, let ω be such that $\rho(A_1,A_2,\ldots,A_n) \subseteq \{z:\mathrm{Re}z \leqq \omega\}$. According to [*] we can find an $N \in \mathcal{B}$ so that $\mathcal{E}(N) = 0$ and $K(N) \subseteq \{z:\mathrm{Re}z \leqq \omega\}$. On the other hand, we can write $p(z,s) = (z-z_1(s))(z-z_2(s))\ldots(z-z_n(s))$ for $z, s \in C$. For $s \in C \setminus N$ we obtain that $z_i(s) \in K(N)$, i.e. $\mathrm{Re}z_i \leqq \omega$. Consequently $\left|\frac{1}{z-z_i(s)}\right| \leqq \frac{1}{\mathrm{Re}z-\omega}$ for every $s \in C \setminus N$ and $\mathrm{Re}z > \omega$. This yields $\left|\frac{z_i(s)}{z-z_i(s)}\right| \leqq 1 + \frac{|z|}{\mathrm{Re}z-\omega}$ for $s \in C \setminus N$ and $\mathrm{Re}z > \omega$. Using Vièta's formulas expressing $a_i(s)$ in terms of $z_i(s)$ we obtain that there are constants $L \geqq 0$ and $l \in \{0,1,\ldots\}$ such that $\left|\frac{a_i(s)}{p(z,s)}\right| \leqq L(1+|z|)^l$ for every $s \in C \setminus N$ and $\mathrm{Re}z > \omega+1$. From this estimate we deduce easily that $\|A_iR(z)x\| = \left\|\int_{C \setminus N} \frac{a_i(\sigma)}{p(z,\sigma)}\,\mathcal{E}(d\sigma)x\right\| \leqq L(1+|z|)^l\|x\|$ for $x \in E$ and $\mathrm{Re}z > \omega+1$. This fact together with the above proved density of $D_\infty(A_1,A_2,\ldots,A_n)$ shows that Theorem 8 is applicable and hence our Cauchy problem is correct.

Finally, let us remark that a more accurate result can be proved, namely that our problem is in fact correct of class n-1, but this requires a little lengthy estimates. :::

The a priori restrictions in the preceding theorem as to the basic space E and the generating operators $A_1,A_2,\ldots,A_n$ can be weakened on the ground of the theory of scalar-type operators in Banach spaces, extensively studied in [12].

Theorem 11. *The preceding Theorem 10 is valid also in an arbitrary Banach space E if we replace normal operators by scalar-type ones.*

Proof. Similar argument as in Theorem 10 can be used. :::

IV. ASYMPTOTIC PROPERTIES OF THE CAUCHY PROBLEM

In this section, we shall deal with examining the growth of solutions of the Cauchy problem on the whole time halfaxis in dependence upon their boundedness on finite time intervals.

It is known that in certain special cases, the condition of exponential growth of solutions, required in our definition of correctness of the Cauchy problem, can be replaced by a formally weaker condition of local boundedness. This concerns the cases $n=1$, $m=0$ and $n=2$, $m=0$ with $A_1=0$ where it is possible to use the functional equations either for operator semigroups or for cosine operator functions to obtain exponential estimates of growth from local boundedness. However, in general such an approach is not available because we do not know any functional equation for solutions of the general Cauchy problem. Despite of this, we have succeeded recently to prove a general theorem of this type for the class of correctness $m = 0$.

Theorem 12. *We assume that*

(α) *the operators* $A_1, A_2, \dots, A_n$ *are closed,*

(β) *the Cauchy problem for* $A_1, A_2, \dots, A_n$ *is extensive,*

(γ) *for every* $T > 0$, *there is a nonnegative constant* K *so that for every solution* u , *every* $0 < t \leqq T$ *and* $i \in \{1,2,\dots,n\}$,

$$\| \int_0^t A_i u^{(n-i)}(\tau)d\tau \| \leqq K \| u^{(n-1)}(0_+) \| .$$

Under these assumptions, there exist nonnegative constants M, ω *so that for every solution* u , *every* $t > 0$ *and* $i \in \{1,2,\dots,n\}$,

$$\| \int_0^t A_i u^{(n-i)}(\tau)d\tau \| \leqq M e^{\omega t} \| u^{(n-1)}(0_+) \| .$$

In other words, we can say that the Cauchy problem for $A_1, A_2, \dots, A_n$ *is correct of class* $m = 0$.

Proof. We denote by A_0 the identity operator.

It is easy to prove from our assumptions that there exists a function $W: (0,\infty) \times E \to E$ such that

(1) W is continuous in both variables, linear in the second,

(2) $$\sum_{j=0}^{n} A_j \underline{|j} \int_0^t W(\tau,x)d\tau = x \quad \text{for } x \in E \text{ and } t > 0 ,$$

(3) $W(0_+,x) = x$ for every $x \in E$,

(4) $$\| A_j \underline{|j} \int_0^t W(\tau,x)d\tau \| \leqq K(T) \|x\| \quad \text{for } x \in E , \; 0 < t \leqq T \text{ and}$$

$j \in \{0,1,\dots,n\}$ with some constant $K(T)$.

Further we choose a function $\vartheta:(0,\infty)\to R$ such that

(5) $\vartheta(t) = 1$ for $0 < t \leqq 1$, $0 \leqq \vartheta(t) \leqq 1$ for $1 < t \leqq 2$ and $\vartheta(t) = 0$ for $t > 2$,

(6) ϑ is n-times continuously differentiable on $(0,\infty)$.

Let us multiply (2) by ϑ and consider the Laplace transform to the identity so obtained. Simple but careful calculations based on current properties of the Laplace transform (see [10], p. 16) and on interchange of closed operators and integrals (see [10], p. 10) enable us to infer, with regard to the properties (α), (1), (3), (5) and (6), that for every $x \in E$ and $\lambda > 0$,

$$(7)\quad \sum_{j=0}^{n} \lambda^{n-j} A_j \Big(\int_0^\infty e^{-\lambda\tau}\, \vartheta(\tau)\big(\underline{\ln}\int_0^\tau W(\sigma,x)d\sigma\big)d\tau\Big) = x -$$

$$- \Big[-\int_0^\infty e^{-\lambda\tau}\, \vartheta'(\tau)d\tau\, x + \sum_{j=0}^{n-1}(-1)^{n-j}A_j\,\cdot$$

$$\cdot\Big(\sum_{k=0}^{n-j-1}\binom{n-j}{k}(-\lambda)^{k+1}\int_0^\infty e^{-\lambda\tau}\, \vartheta^{(n-j-k)}(\tau)\big(\underline{\ln}\int_0^\tau W(\sigma,x)d\sigma\big)d\tau\Big)\Big].$$

Let us denote by $G(\lambda)x$ the left hand side of (7) and by $H(\lambda)x$ the expression in square brachets on the right hand side of (7). Then

$$(8)\quad \sum_{j=0}^{n} \lambda^{n-j} A_j G(\lambda) = I - H(\lambda) \quad \text{for} \quad \lambda > 0.$$

The inequalities (4) together with some assumed or already proved properties enable us to deduce analogous estimates for higher iterated integrals of solutions. Using these estimates and the properties (5) and (6) we obtain, after a little lenghty calculations, that there is a $K \geqq 0$ such that

$$(9)\quad \Big\|\frac{d^p}{d\lambda^p}\lambda^{n-i-1}G(\lambda)\Big\| \leqq \frac{Kp!}{\lambda^{p+1}}, \quad \Big\|\frac{d^p}{d\lambda^p}H(\lambda)\Big\| \leqq \frac{Kp!}{\lambda^{p+1}} \quad \text{for every}$$
$$\lambda > 0, \quad i \in \{1,2,\dots,n\} \ \text{and} \ p \in \{0,1,\dots\}.$$

From the second inequality in (8) we see that there is a constant $\omega_0 \geqq 0$ such that $\|H(\lambda)\| < 1$ for $\lambda > \omega_0$ which implies together with (8) that

$$(10)\quad (\omega_0,\infty) \subseteq \varrho(A_1,A_2,\dots,A_n),$$

$$(11)\quad R(\lambda;A_1,A_2,\dots,A_n) = \Big(\sum_{j=0}^{n}\lambda^{n-j}A_j\Big)^{-1} = G(\lambda)(I-H(\lambda))^{-1} \quad \text{for}$$
$$\lambda > \omega_0.$$

Using the lemma from [13], p. 49, we obtain from (9) and (11) that there are two constants $M \geqq 0$ and $\omega \geqq \omega_0$ such that

(12) $\left\| \dfrac{d^p}{d\lambda^p} R(\lambda; A_1, A_2, \dots, A_n) \right\| \leqq \dfrac{Mp!}{(\lambda-\omega)^{p+1}}$ for $\lambda > \omega$ and $p \in$ $\in \{0, 1, \dots\}$.

Further, we deduce easily from (3) that

(13) $D_1(A_1, A_2, \dots, A_n)$ is dense in E.

Now the properties (10), (12) and (13) permit to apply Theorem 5 with $m = 0$ and the assertion of this theorem completes the proof. :::

It would be desirable to have a theorem analogous to Theorem 12 for higher classes of correctness. However, the following Theorem 13 suggests that such a result will hardly hold because the estimates of growth, obtained in this theorem, do not guarantee the correctness in our sense and moreover, as can be easily shown by examples, cannot be generally improved.

Theorem 13. *We assume that*

(α), (β) *as in Theorem* 12,

(γ) *for every* $T > 0$, *there exist a nonnegative constant* K *and a nonnegative integer* r *so that for every solution* u, every $0 < t \leqq T$ *and* $i \in \{1, 2, \dots, n\}$,

$$\left\| \underline{|r}\int_0^t A_i u^{(n-i)}(\tau)\,d\tau \right\| \leqq K\|u^{(n-1)}(0_+)\| .$$

Under these assumptions, there exist nonnegative constants M, ω *and nonnegative integers* χ, m *so that for every solution* u, every $\nu \in \{1, 2, \dots\}$, $0 < t \leqq \nu$ *and* $i \in \{1, 2, \dots, n\}$,

$$\left\| \underline{|\chi\nu + m}\int_0^t A_i u^{(n-i)}(\tau)\,d\tau \right\| \leqq Me^{\omega t}\|u^{(n-1)}(0_+)\| .$$

Roughly speaking, our Theorem guarantees an exponential growth only of certain linearly increasing (*with respect to time*) *iterated integrals of solutions.*

Proof. We begin similarly as in the first part of the proof of Theorem 12. We express the resolvent function R again in the form (11) but for λ from a logarithmic domain $\{\lambda : \mathrm{Re}\,\lambda \geqq \alpha \log(1+|\mathrm{Im}\,\lambda|) + \varkappa\}$ where the constants $\alpha \geqq 0$ and $\varkappa$ depend on $A_1, A_2, \dots, A_n$ only. In this domain, we obtain easily $\|R(\lambda)\| \leqq K(1+|\lambda|)^l$, $K \geqq 0$, $l \geqq 0$ (derivatives of R need not be estimated). On the other hand, the iterated integrals of solutions which we have to estimate can be expressed in terms of the resolvent function R by means of a modified Laplace complex inverse integral whose integral path is the boundary of our logarithmic domain. This formula together with the above proved estimate of the resolvent function R yields the required estimates of solutions almost immediately. :::

References

[1] Hille, E.: Functional analysis and semigroups, 1948.

[2] Yosida, K.: On the differentiability and the representation of one-parameter semi-groups, J. Math. Soc. Japan, 1(1948), 15-21.

[3] Lions, J. L.: Les semi-groupes distributions, Portugal. Math., 19(1960), 141-164.

[4] Sova, M.: Cosine operator functions, Rozprawy Matematyczne, 49(1966).

[5] Sova, M.: Problèmes de Cauchy paraboliques abstraits de classes supérieures et les semi-groupes distributions, Ricerche di Mat., 18(1969), 215-238.

[6] Sova, M.: Encore sur les équations hyperboliques avec petit paramètre dans les espaces de Banach généraux (appendice), Colloquium Math., 25(1972), 155-161.

[7] Sova, M.: On the Timoshenko type equations, Čas. pěst. mat., 100(1975), 217-254.

[8] Sova, M.: Inhomogeneous linear differential equations in Banach spaces, Čas. pěst. mat., 103(1978), 112-135.

[9] Obrecht, E.: Sul problema di Cauchy per le equazioni paraboliche astratte di ordine n, Rend. Sem. Mat. Univ. Padova, 53(1975), 231-256.

[10] Sova, M.: Linear differential equations in Banach spaces, Rozpravy Československé akademie věd, Řada mat. a přír. věd, 85(1975), No 6.

[11] v. Sz.-Nagy, B.: Spektraldarstellung linearer Transformationen des Hilbertschen Raumes, 1942.

[12] Dunford, N., Schwartz, J. T.: Linear operators III, 1971.

[13] Sova, M.: Equations différentielles opérationelles linéaires du second ordre à coefficients constants, Rozpravy Československé akademie věd, Řada mat. a přír. věd, 80(1970), No 7.

Author's address: Matematický ústav ČSAV, Žitná 25, 115 67 Praha 1, Czechoslovakia.