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A PARTIALLY ORDERED SPACE CONNECTED WITH THE DE LA VALLÉE POUSSIN PROBLEM

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In the paper the m-point BVP

$$(1)\qquad (L(x) \equiv)\sum_{j=0}^{n} p_j(t)x^{(n-j)} = f(t,x,x',\dots,x^{(n-1)}),$$

$$(2)\qquad x^{(i-1)}(t_k) = A_{i,k}(i=1,\dots,r_k,\ k=1,2,\dots,m)$$

is considered where $n \geq 2$, $2 \leq m \leq n$, $1 \leq r_k$ $(k=1,2,\dots,m)$ are natural numbers such that $r_1 + r_2 + \dots + r_m = n$, and $-\infty < a < t_1 < \dots < t_m < b < \infty$, $A_{i,k}(i=1,\dots,r_k,\ k=1,2,\dots,m)$ are real numbers. First the properties of the Green function corresponding to the homogeneous BVP

$$(1')\qquad L(x) = 0,$$

$$(2')\qquad x^{(i-1)}(t_k) = 0 \quad (i=1,\dots,r_k,\ k=1,2,\dots,m)$$

and their consequences are studied. The regularity properties determine continuity and compactness of the integrodifferential operator associated to (1), (2). The sign of the Green function decides the monotonicity properties of this operator in a partially ordered space which is described here. Using the fixed point theory in partially ordered spaces we obtain new results for the mentioned BVP connected with the notion of the lower (upper) solution of (1).

Throughout the paper the following two assumptions are satisfied:

(A_1) $p_j\colon (a,b) \to R$, $p_j \in L_{loc}(a,b)$ $(j=0,1,\dots,n)$, $p_0(t) = 1$ in (a,b) a.e. and $f\colon (a,b) \times R^n \to R$ satisfies Carathéodory conditions, i.e. $f(\cdot,x_0,x_1,\dots,x_{n-1})$ is measurable in (a,b) for each fixed $(x_0,x_1,\dots,x_{n-1}) \in R^n$, for almost all $t_0 \in (a,b)$ $f(t_0,\cdot,\dots,\cdot)$ is continuous in R^n and $\sup\{|f(t,x_0,x_1,\dots,x_{n-1})| : (x_0^2 + x_1^2 + \dots + x_{n-1}^2)^{1/2} \leq r\} \in L_{loc}(a,b)$ for each $r > 0$.

(A_2) The differential equation (1') is disconjugate in (a,b).

With respect to Assumption (A_1), by a solution (1) on an interval $j \subset (a,b)$ each function x with $x^{(n)} \in L_{loc}(j)$ which satisfies (1) a.e. in j is understood.

Under the assumptions (A_1), (A_2) for any m, $2 \leq m \leq n$, any r_k $(k = 1,2,\dots,m)$ with $1 \leq r_k$, $r_1 + r_2 + \dots + r_m = n$, and any $a < t_1 < t_2 < \dots < t_m < b$ the problem (1'), (2') has only the trivial solution and there exists a unique function $G = G(t,s)$, the so-called Green function of (1'), (2') defined on $[t_1,t_m] \times \bigcup_{\ell=1}^{m-1} [t_\ell, t_{\ell+1}]$

and such that if $r \in L(t_1,t_m)$ and w is the unique solution of the problem (1'), (2), then the unique solution y of the problem $L(y) = r(t)$, (2) is given by $y(t) = w(t) + \int_{t_1}^{t} G(t,s)r(s)\,ds$ ([4], p. 137). Hence the BVP (1), (2) is equivalent to the integrodifferential equation

$$(3)\quad x(t)=w(t)+\int_{t_1}^{t_m} G(t,s)f\left[s,x(s),x'(s),\dots,x^{(n-1)}(s)\right] ds \quad (t\in[t_1,t_m])$$

in the sense that each solution x of (3) with $x^{(n-1)} \in L(t_1,t_m)$ possesses $x^{(n)} \in L(t_1,t_m)$ and is a solution of (1), (2) and conversely.

Lemma 1. The function G has the following properties:

1. $\frac{\partial^i G}{\partial t^i}$, $i = 0,\dots,n-2$, is continuous in $[t_1,t_m] \times [t_1,t_m]$.
2. $\frac{\partial^{n-1} G}{\partial t^{n-1}}$ is continuous in the domains $t_1 \leqq t \leqq s \leqq t_m$ and $t_1 \leqq s \leqq t \leqq t_m$. Further,

$$(4)\qquad \lim_{t \to s+} \frac{\partial^{n-1}G(t,s)}{\partial t^{n-1}} - \lim_{t \to s-} \frac{\partial^{n-1}G(t,s)}{\partial t^{n-1}} = 1 \quad (t_1 < s < t_m).$$

3. $G(\cdot,s)$ satisfies (1') in $[t_1,t_m]$ a.e. and the boundary conditions (2').
4. The sign of G is determined by the inequality

   $$G(t,s)(t-t_1)^{r_1}(t-t_2)^{r_2}\dots(t-t_m)^{r_m} \geqq 0,\quad t_1 \leqq t \leqq t_m,\ t_1 \leqq s \leqq t_m$$

   and

   $$G(t,s) \neq 0 \ \text{ for } \ t_k < t < t_{k+1},\ t_1 < s < t_m,\ k = 1,\dots,m-1.$$

5. $$\lim_{s \to t_1+} \frac{\partial^i G(t,s)}{\partial t^i} = \lim_{s \to t_m-} \frac{\partial^i G(t,s)}{\partial t^i} = 0 \ \text{ for } \ t_1 < t < t_m,\quad 0 \leqq i \leqq n-1$$

   and

   $$\lim_{s \to t_1+} \frac{\partial^i G(t,s)}{\partial t^i} = \begin{cases} 0, & 0 \leqq i \leqq n-2 \\ -1, & i = n-1, \end{cases}$$

   $$\lim_{s \to t_m-} \frac{\partial^i G(t,s)}{\partial t^i} = \begin{cases} 0, & 0 \leqq i \leqq n-2 \\ 1, & i = n-1. \end{cases}$$

Proof. Let $u = u(t,s)$ be the Cauchy function for (1'), i.e. $u(\cdot,s)$ is the solution of (1') determined by the conditions $x^{(i-1)}(s) = 0$ $(i = 1,\dots,n-1)$, $x^{(n-1)}(s) = 1$. Let $v_{j\ell}$, $\ell = 1,2,\dots,m$, $j = 1,\dots,r_\ell$, be the solution of (1') satisfying the conditions

$v_{j\ell}^{(i-1)}(t_k) = 0$ $(k \neq l,\ i = 1,\dots,r_k,\ k = 1,2,\dots,m)$

$v_{j\ell}^{(i-1)}(t_\ell) = \delta_{ji}$ $(i = 1,\dots,r_\ell)$. δ_{ji} is the delta Kronecker symbol.

Then, by [4], p. 137,

$$(5)\quad G(t,s) = \begin{cases} \displaystyle\sum_{\ell=1}^{k} \sum_{j=1}^{r_\ell} v_{j\ell}(t) \frac{\partial^{j-1} u(t_\ell,s)}{\partial t^{j-1}}, & t \geqq s \\ \displaystyle -\sum_{\ell=k+1}^{m} \sum_{j=1}^{r_\ell} v_{j\ell}(t) \frac{\partial^{j-1} u(t_\ell,s)}{\partial t^{j-1}}, & t \leqq s, \end{cases}$$

$$t_k \leqq s \leqq t_{k+1} \quad (k = 1,\dots,m-1)$$

and the equality

$$(6)\quad \sum_{\ell=1}^{m} \sum_{j=1}^{r_\ell} v_{j\ell}(t) \frac{\partial^{j-1} u(t_\ell,s)}{\partial t^{j-1}} = u(t,s) \quad \text{for each} \quad (t,s) \in [t_1,t_m] \times [t_1,t_m]$$

is true ([4], p. 136). By (5), $\frac{\partial^i G}{\partial t^i}$, $i = 0,1,\dots,n-1$, is piecewise continuous in $[t_1,t_m] \times [t_1,t_m]$ and its only discontinuities may lie on the lines $s = t$ and $s = t_k$, $k = 2,\dots,m-1$ (if $m > 2$). Since $\frac{\partial^{j-1} u}{\partial t^{j-1}}$ is continuous in s and $\frac{\partial^{j-1} u(t_k,t_k)}{\partial t^{j-1}} = 0$ for $j = 1,\dots,r_k$, by (5) we get that $\lim_{s\to t_k+} \frac{\partial^i G(t,s)}{\partial t^i} = \lim_{s\to t_k-} \frac{\partial^i G(t,s)}{\partial t^i}$ $(t \in [t_1,t_m],\ t \neq t_k,\ k = 2,\dots,m-1,\ 0 \leqq i \leqq n-1)$. Further,

$$\lim_{s\to t-} \frac{\partial^i G(t,s)}{\partial t^i} - \lim_{s\to t+} \frac{\partial^i G(t,s)}{\partial t^i} = \begin{cases} 0, & 0 \leqq i \leqq n-2 \\ 1, & i = n-1 \end{cases} \quad (t_1 < t < t_m)$$

follows from (5), (6). This implies the statements 1 and 2 as well as (4). 3 is a direct consequence of (5). The sign of G has been determined in [5], pp. 80-81. (5) and (6) also imply the statement 5.

Denote now the set of all functions x with $x^{(n-1)} \in L(t_1,t_m)$ (with $x^{(n)} \in L(t_1,t_m)$) by $L^{(n-1)}(t_1,t_m)$ $(L^{(n)}(t_1,t_m))$. Further, let $C_{n-1} = C_{n-1}([t_1,t_m])$ be provided with the norm $\|x\| =$

$= \max_{k=0,1,\dots,n-1} \{\max_{t\in[t_1,t_m]} |x^{(k)}(t)|\}$. Similarly, the norm $\|x\|_0 = \max_{t\in[t_1,t_m]} |x(t)|$ will be used in $C = C([t_1,t_m])$.

The right-hand side of (3) defines in $L^{(n-1)}(t_1,t_m)$ an operator T which is given by

$$(7)\qquad T(x)(t) = w(t) + \int_{t_1}^{t_m} G(t,s)f[s,x(s),x'(s),\dots,x^{(n-1)}(s)]\,ds \qquad (t\in[t_1,t_m]).$$

By (A_1) and Lemma 1, $T\colon L^{(n-1)}(t_1,t_m) \to L^{(n)}(t_1,t_m)$. Since $L^{(n)}(t_1,t_m)\subset C_{n-1}\subset L^{(n-1)}(t_1,t_m)$, T maps C_{n-1} into itself.

<u>Lemma 2.</u> T as a mapping from C_{n-1} into itself is continuous and compact.

Proof. Suppose that $S\subset C_{n-1}$ is bounded. Then, by (A_1), there exists a function $m\in L(t_1,t_m)$ such that for any $x\in S$, $|f(t,x(t),\dots,x^{(n-1)}(t)| \leqq m(t)$ $(t\in[t_1,t_m])$. On the basis of (7), this implies that there exist constants A_j, $j = 0,1,\dots,n-1$, such that $T(x) = y$ satisfies the inequalities $|y^{(j)}(t)| \leqq A_j$ $(j = 0,1,\dots,n-1,$ $t\in[t_1,t_m])$ and, since y is the solution of $L(y) = f[t,x(t),x'(t),\dots,x^{(n-1)}(t)]$, (2), $|y^{(n)}(t)| \leqq \sum_{j=1}^{n} |p_j(t)|A_{n-j} + m(t)$. Thus the sets $\{y^{(j)}\}$, $j = 0,1,\dots,n-1$, are equicontinuous and uniformly bounded on $[t_1,t_m]$. From this the relative compactness of $T(S)$ in C_{n-1} follows. Thus T is compact.

If $x_n \to x$ in C_{n-1}, then $[T(x_n)]^{(j)}(t) \to [T(x)]^{(j)}(t)$ $(j = 0,1,\dots,n-1)$ pointwise in $[t_1,t_m]$ and the sequences $\{[T(x_n)]^{(j)}\}$ are equicontinuous on that interval. Hence $[T(x_n)]^{(j)}$ converge uniformly to $[T(x)]^{(j)}$ $(j = 0,1,\dots,n-1)$ on $[t_1,t_m]$ and T is continuous.

Many consequences follow from the sign of the Green function. In order to show them, similarly as in [7], a partial ordering in C_{n-1}, and more generally, in C, will be introduced by the rule: If x, $y\in C_{n-1}$ or in C, then $x \leqq y$ iff $(-1)^{r_{k+1}+\dots+r_m}[y(t) - x(t)] \geqq 0$ for all $t\in[t_k,t_{k+1}]$ and all $k = 1,2,\dots,m-1$. By Lemma 1, $x \leqq y$ means that $x(t) \leqq y(t)$ $(x(t) \geqq y(t))$ on each interval $[t_k,t_{k+1}]$ where $G(t,s) \geqq 0$ $(G(t,s) \leqq 0)$. Denote $H_1 = \{k\in\{1,\dots,m-1\} : r_{k+1}+\dots+r_m \text{ is even}\}$, $H_2 = \{k\in\{1,\dots,m-1\} : r_{k+1}+\dots+r_m \text{ is odd}\}$.

Let $P = \{x \in C_{n-1}: x \geqq 0\}$ ($P_0 = \{x \in C: x \geqq 0\}$). Clearly P, P_0 are cones in the corresponding spaces and $(C_{n-1}, P), (C, P_0)$, respectively, ordered Banach spaces with positive cones P and P_0, respectively. While P is not normal, P_0 already is. If $x_0 \leqq y_0$ in C_{n-1} and in C, respectively, then by the interval $[x_0, y_0]$ the set $\{z \in C_{n-1}: x_0 \leqq z \leqq y_0\}$ and $\{z \in C: x_0 \leqq z \leqq y_0\}$, respectively, will be understood.

Further, $x \in L^{(n)}(t_1, t_m)$ will be called a <u>lower solution</u> ($y \in L^{(n)}(t_1, t_m)$ an <u>upper solution</u>) <u>of the differential equation (1)</u> if it satisfies the differential inequality $L(x) \leqq f(t, x, x', \ldots, x^{(n-1)})$ ($L(y) \geqq f(t, y, y', \ldots, y^{(n-1)})$) a.e. in $[t_1, t_m]$.

Finally for each $x \in L^{(n)}(t_1, t_m)$ the solution v_x of the problem

$$L(v) = 0,$$

$$v^{(i-1)}(t_k) = x^{(i-1)}(t_k) \quad (i = 1, \ldots, r_k, \quad k = 1, 2, \ldots, m)$$

will be said to be the <u>solution of (1') associated with the function x.</u>

With help of these notions the following condition will be expressed which plays an important role in subsequent considerations.

(A_3) There exist a lower and an upper solution x_0, y_0, respectively, of the equation (1) such that

$$x_0 \leqq y_0 \tag{8}$$

and the solutions v_{x_0}, v_{y_0} of (1') associated with the function x_0, y_0, respectively, satisfy

$$v_{x_0} \leqq w \leqq v_{y_0} \tag{9}$$

where w is the solution of the problem (1'), (2).

The inequalities (8), (9) are considered in the partially ordered space C_{n-1}.

<u>Theorem 1.</u> Let the differential equation (1) satisfy the conditions $(A_1 - (A_3)$ and let

$$L(x_0)(t) - f(t, x, x_1, \ldots, x_{n-1}) \leqq 0 \leqq L(y_0(t) - f(t, x, x_1, \ldots, x_{n-1}) \tag{10}$$

be true for all $(t, x, x_1, \ldots, x_{n-1}) \in W$ where W is given by

$$W = \{(t, x, x_1, \ldots, x_{n-1}): x_0(t) \leqq x \leqq y_0(t), -\infty < x_i < \infty, \ i = 1, \ldots, n-1\} \cup \{(t, x, x_1, \ldots, x_{n-1}): y_0(t) \leqq x \leqq x_0(t), -\infty < x_i < \infty, \ i = 1, \ldots, n-1\}.$$

Then the BVP (1), (2) has a solution x which lies in the interval $[x_0, y_0]$.

Proof. The interval $[x_0, y_0]$ is a closed and convex subset of C_{n-1}. If $x \in [x_0, y_0]$, then $(t, x(t), x'(t), \dots, x^{(n-1)}(t)) \in W$ for all $t \in [t_1, t_m]$ and by (9), (10), for all $t \in [t_k, t_{k+1}]$, $k \in H_1$, $T(x)(t) = w(t) + \int_{t_1}^{t_m} G(t,s)\, f[s, x(s), x'(s), \dots, x^{(n-1)}(s)]\, ds \leqq v_{y_0}(t) + \int_{t_1}^{t_m} G(t,s)\, L(y_0)(s)\, ds = y_0(t)$. For $k \in H_2$ we get $T(x)(t) \geqq y_0(t)$, $t \in [t_k, t_{k+1}]$. In a similar way, $T(x)(t) \geqq x_0(t)$ $(T(x)(t) \leqq x_0(t))$ for $t \in [t_k, t_{k+1}]$, $k \in H_1 (k \in H_2)$. Thus we see that (8) - (10) imply that $T([x_0, y_0]) \subset [x_0, y_0]$. By Lemma 2, T is continuous.

Now we prove that $T([x_0, y_0])$ is relatively compact in C_{n-1}. Let $1 \leqq j \leqq n-1$ be a natural number. By the continuity properties of G there exists a constant $G_j > 0$ such that $\left| \frac{\partial^j G(t,s)}{\partial t^j} \right| \leqq G_j$ in $[t_1, t_m] \times [t_1, t_m]$, and, on the basis of (10), the inequality

$$(11) \qquad \left| [T(x)]^{(j)}(t) \right| \leqq \left| w^{(j)}(t) \right| + G_j \int_{t_1}^{t_m} \left(|L(x_0)(s)| + |L(y_0)(s)| \right) ds = H_j$$

is true for $x_0 \leqq x \leqq y_0$. Let $H_0 = \max\left(\max_{t \in [t_1, t_m]} |x_0(t)|, \max_{t \in [t_1, t_m]} |y_0(t)| \right)$. Since $y = T(x)$ satisfies the equation $L(y) = f[t, x(t), x'(t), \dots, x^{(n-1)}(t)]$, (11) with (A_1) give that $\left| [T(x)]^{(n)}(t) \right| \leqq \sum_{j=1}^{n} |p_j(t)| H_{n-j} + m(t)$ where $m \in L(t_1, t_m)$. This proves that the functions $\{[T(x)]^{(j)}\}$, $j = 0, 1, \dots, n-1$, are uniformly bounded and equicontinuous in $[t_1, t_m]$ which means that $T([x_0, y_0])$ is relatively compact. The Schauder fixed point theorem ensures a fixed point of T in $[x_0, y_0]$.

In [7] two sufficient conditions are given for x_0 and y_0 to satisfy (9). Here they will be stated as

<u>Lemma 3.</u> Any of the following conditions is sufficient for the lower and the upper solutions x_0, y_0, respectively, of (1) to satisfy the inequalities (9).

1. x_0 and y_0 satisfy the boundary conditions

$x_0^{(i-1)}(t_k) = A_{i,k} = y_0^{(i-1)}(t_k)$, $i = 1,\ldots,r_k$, $k = 2,\ldots,m-1$ (if such points exist),

$x_0^{(i-1)}(t_j) = A_{i,j} = y_0^{(i-1)}(t_j)$, $i = 1,\ldots,r_j-1$, $j = 1,m$ (if $r_1, r_m \geqq 2$), $(-1)^{n+r_1-1}\left[x_0^{(r_1-1)}(t_1) - A_{r_1,1}\right] \geqq 0 \geqq (-1)^{n+r_1-1}\left[y_0^{(r_1-1)}(t_1) - A_{r_1,1}\right]$,

$x_0^{(r_m-1)}(t_m) \geqq A_{r_m,m} \geqq y_0^{(r_m-1)}(t_m)$.

2. $r_1 = n-1$, $r_2 = 1$, $m = 2$, $L(x) = x^{(n)}$ and the lower and the upper solutions x_0, y_0, respectively, of

$$x^{(n)} = 0 \tag{12'}$$

satisfy the boundary conditions

$$y_0^{(i-1)}(t_1) \leqq A_{i,1} \leqq x_0^{(i-1)}(t_1), \quad i = 1,\ldots,n-1,$$

$$y_0(t_2) \leqq A_{1,2} \leqq x_0(t_2).$$

On the basis of Lemma 3, Theorem 1 generalizes Theorem 3.1 from [6], pp. 672-673.

In some cases the uniqueness of the solution of the BVP (1), (2) can be ensured by the Kellog uniqueness theorem [3]. Since its proof can be based on the theory of the Leray-Schauder degree in locally convex spaces where the degree is defined also for unbounded sets ([2], p. 101), a slightly modified version of that theorem is given here as

Lemma 4. Let X be a real Banach space with a convex open (not necessarily bounded) subset D and let $F: \overline{D} \to \overline{D}$ be a continuous mapping which is continuously Fréchet differentiable on D and for which $\overline{F(\overline{D})}$ is compact. Suppose that

(a) for each $x \in D$, 1 is not an eigenvalue of $F'(x)$, and

(b) for each $x \in \partial D$, $x \neq F(x)$.

Then F has a unique fixed point.

Theorem 2. Suppose that

1. there exist two functions $x_0, y_0 \in C_n([t_1,t_2])$, $a < t_1 < t_2 < b$, such that $y_0(t) < x_0(t)$ $(t \in [t_1,t_2])$ and $y_0^{(i-1)}(t_1) < x_0^{(i-1)}(t_1)$, $i = 1,\ldots,n-1$,

2. f, together with $\frac{\partial f}{\partial x_i}$, $i = 0,1,\ldots,n-1$, is continuous on the set $W_1 = \{(t,x,x_1,\ldots,x_{n-1}) : y_0(t) \leqq x \leqq x_0(t),\ -\infty < x_i < \infty,$

$i = 1,\dots,n-1, \quad t_1 \leqq t \leqq t_2\}$,

3.

(13) $x_0^{(n)}(t) - f(t,x,x_1,\dots,x_{n-1}) < 0 < y_0^{(n)}(t) - f(t,x,x_1,\dots,x_{n-1})$

on W_1,

4. the BVP

$$y^{(n)} = \sum_{i=0}^{n-1} \frac{\partial f[t,x(t),x'(t),\dots,x^{(n-1)}(t)]}{\partial x_i} y^{(i)},$$

(14') $\quad y^{(i-1)}(t_1) = 0, \quad i = 1,\dots,n-1, \quad y(t_2) = 0$

has only the trivial solution for all $x \in C_{n-1}([t_1,t_2])$ such that $y_0(t) < x(t) < x_0(t), \quad t \in [t_1,t_2]$.

Then for all numbers $A_{i,1}$, $i = 1,\dots,n-1$, $A_{1,2}$ such that $y_0^{(i-1)}(t_1) < A_{i,1} < x_0^{(i-1)}(t_1)$, $i = 1,\dots,n-1$, $y_0(t_2) < A_{1,2} < x_0(t_2)$ there exists a unique solution of the problem

(12) $$x^{(n)} = f(t,x,x',\dots,x^{(n-1)})$$

(14) $$x^{(i-1)}(t_1) = A_{i,1}, \quad i = 1,\dots,n-1, \quad x(t_2) = A_{1,2}.$$

Proof. Consider the ordered Banach space $D_{n-1} = C_{n-1}([t_1,t_2])$ with the norm and ordering similar to those in C_{n-1}. Let $D = \{x \in D_{n-1}: y_0(t) < x(t) < x_0(t), \quad t_1 \leqq t \leqq t_2\}$. D is open and convex in D_{n-1}. Then $\bar{D} = [x_0, y_0]$ (we remind that $x \leqq y$ in D_{n-1} iff $x(t) \leqq y(t)$ in $[t_1,t_2]$). Lemma 3 gives that the assumptions 1 and 3 of the theorem imply (A_3) and (10). (A_1) is ensured by the assumption 2. (A_2) is clearly satisfied. Hence, from the proof of Theorem 1 it follows that the mapping T_1 which corresponds to the problem (12), (14) maps $\bar{D}$ into itself, is continuous and $\overline{T_1(\bar{D})}$ is compact. The Fréchet derivative of T_1 at $x \in D$ is

$$T'(x)(h(t)) = \int_{t_1}^{t_2} G_1(t,s) \left[\sum_{i=0}^{n-1} \frac{\partial f[s,x(s),x'(s),\dots,x^{(n-1)}(s)]}{\partial x_i} \cdot h^{(i)}(s)\right] ds, \qquad (t \in [t_1,t_2])$$

where $h \in D_{n-1}$ and G_1 is the Green function of (12'), (14'). $T_1'(x)$ continuously depends on x in D. Condition (a) in Lemma 4 means that the linear operator $T_1'(x)$ has no nontrivial fixed point which is equivalent to the assumption 4. The condition (b) follows from $T_1(\bar{D}) \subset D$ which is guaranteed by (13). Thus all assumptions of Lemma 4 being satisfied, by this lemma Theorem 2 is true.

In the sequel a special case of (1) will be considered, namely

(15) $$L(x) = f(t,x),$$

where f shows the following monotonicity property.

(A_4) The function $f = f(t,x)$ is nondecreasing in x for each $t \in [t_k, t_{k+1}]$, $k \in H_1$, and nonincreasing in x for each $t \in [t_k, t_{k+1}]$, $k \in H_2$.

With respect to (15), the operator T defined by (7) which now has the form $T(x)(t) = w(t) + \int_{t_1}^{t_m} G(t,s)f[s,x(s)]\, ds \quad (t \in [t_1, t_m])$ can be considered either as a mapping from C_{n-1} into itself or as a mapping from C into C. In both cases it is continuous and compact. The advantage of using C lies in the fact that in this space an interval is a bounded set. The condition (A_4) implies that T is isotone which means that if $x \leqq y$ are two comparable elements of C or C_{n-1}, then $T(x) \leqq T(y)$. When x and y are a lower and an upper solution of (15), (A_4) gives (10), and by Theorem 1 the following lemma is true.

<u>Lemma 5.</u> Let the differential equation (15) satisfy the conditions (A_1) - (A_4). Then the BVP (15), (2) has a solution $x \in [x_0, y_0]$.

Remarks. 1. By Theorem 2, [7], there exist a least $\overline{x}$ and a greatest solution $\overline{y}$ of (15), (2) in the interval $[x_0, y_0]$. They can be obtained by an iterative procedure starting with x_0 and y_0, respectively.

2. In Theorem 1 and Lemma 5 instead of (A_2) it suffices to assume that (1') is disconjugate in $[t_1, t_m]$. On the basis of Lemma 7, [1], p. 93, the latter assumption implies that (1') is disconjugate in a greater interval $(a,b) \supset [t_1, t_m]$ and thus, the use of (A_2) does not weaken the results.

<u>Theorem 3.</u> Assume that (15) satisfies the conditions (A_1) - (A_3) and

(16) $$f(t,x) - f(t,\overline{x}) \geqq - p(t)(x-\overline{x}) \quad (x \geqq \overline{x},\ t \in [t_k, t_{k+1}],\ k \in H_1)$$
$$f(t,x) - f(t,\overline{x}) \leqq - p(t)(x-\overline{x}) \quad (x \geqq \overline{x},\ t \in [t_k, t_{k+1}],\ k \in H_2)$$

where $p \in L(t_1, t_m)$ and it is such that the differential equation

(17) $$L(x) + p(t)x = 0$$

is disconjugate in $[t_1, t_m]$.

Then the BVP (15), (2) has the least $\overline{x}$ and the greatest solution $\overline{y}$ in $[x_0, y_0]$.

Proof. Defining the operator $L_1(x) = L(x) + p(t)x \quad (x \in$

$\in L^{(n)}(t_1,t_m))$ and $f_1(t,x) = f(t,x) + p(t)x$ $(t \in [t_1,t_m],$ $x \in \in (-\infty,\infty))$ we can write (15) in the form

$$(15_1) \qquad L_1(x) = f_1(t,x).$$

By (17), (16) and Remark 2, (15_1) satisfies (A_1) - (A_4). Hence Lemma 5, together with Remark 1, brings the result.

Remark. By Proposition 9, [1], p. 95, there exists a $\delta > 0$ depending on (1') as well as on $[t_1,t_m]$ such that for $|p(t)| < \delta$ on $[t_1,t_m]$ the equation (17) is disconjugate.

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