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ON LINEAR PROBLEMS IN THE SPACE BV

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BV denotes the Banach space of column n-vector valued functions $x : [0,1] \longrightarrow R_n$ endowed with the norm $x \in BV \longrightarrow \|x\|_{BV} = |x(0)| + \mathrm{var}_0^1 x$. A great variety of linear equations in BV is connected with linear operators of the form

$$(1) \qquad K : x \in BV \longrightarrow Kx = \int_0^1 d_s[K(t,s)]\, x(s)$$

where $K : J = [0,1] \times [0,1] \longrightarrow L(R_n)$ is an $n \times n$-matrix valued function defined on the square J and the integral is taken in the Perron-Stieltjes sense. The main assumption is

$$(2) \qquad v_J(K) + \mathrm{var}_0^1 K(0,.) < \infty$$

with $v_J(K)$ the twodimensional Vitali variation of K on J and $\mathrm{var}_0^1 K(0,.)$ the usual variation with respect to the second variable of $K : J \longrightarrow L(R_n)$ on $[0,1]$. Without any loss of generality we may assume in addition to (2)

$$(3) \qquad K(t,1) = 0,\ K(t,s+) = K(t,s) \quad \text{for any} \quad t \in [0,1],\ s \in [0,1)$$

because any kernel satisfying (2) can be replaced by a new one which satisfies also (3) and for which the operator (1) remains unchanged. Set

$NBV = \{y : [0,1] \longrightarrow R_n;\ y \in BV,\ y(s+) = y(s) \ \text{for}\ s \in (0,1),\ y(1) = 0\}$

(R_n is the space of row n-vectors, the star indicates the transposition of a matrix) and let

$$(4) \qquad K^* : y \in NBV \longrightarrow \int_0^1 d[y^*(t)]\, K(t,s)\ .$$

The following facts are known (see [2]): If (2) holds then the linear operator $K : BV \longrightarrow BV$ given by (1) is compact. If (2), (3) are satisfied then $K : NBV \longrightarrow NBV$ given by (4) is also compact. The spaces BV and NBV form a dual pair with respect to the bilinear form

$$(x,y^*)\in BV\times NBV \longrightarrow \langle x,y^*\rangle = \int_0^1 d[y^*(t)]\,x(t)$$

and we have

(5) $\langle Kx,y^*\rangle = \langle x,K^*y^*\rangle$ for all $x\in BV$, $y\in NBV$.

For the linear equation

(6) $x - Kx = f$, $f\in BV$

the Fredholm theory works in our case provided the operator K^* is used instead of the usual adjoint to K (see [1]). In this situation this is useful because neither the analytic description of the elements of the dual space to BV nor the analytic form of the adjoint operator to K are available. Here we are interested in the resolvent formula for the equation (6).

Assume in addition to (2) that we have $N(I-K) = \{0\}$ for the kernel of the operator $I-K$, i.e. for its range we have $R(I-K) = BV$ (I is the identity operator on BV). Then we obtain easily that

(7) $$x(t) = f(t) + \int_0^1 d_s[R(t,s)]\,f(s)$$

(or shortly $x = f + Rf$) is for any $f\in BV$ a solution to (6) if and only if $x = x - Kx + R(x - Kx)$ for every $x\in BV$, i.e.

$$\int_0^1 d_s\Big[R(t,s) - K(t,s) - \int_0^1 d_r[R(t,r)]\,K(r,s)\Big]\,x(s)$$

for all $x\in BV$, $t\in[0,1]$. Hence we get

Proposition 1. Let $K : J \longrightarrow L(R_n)$ fulfil (2) and let $N(I-K) = \{0\}$. Then (7) yields a solution to (6) for any $f\in BV$ if and only if

$$R(t,s) - K(t,s) - \int_0^1 d_r[R(t,r)]\,K(r,s)\in S$$

for all $t\in[0,1]$ where S is the linear space of $n\times n$-matrix valued functions $A : [0,1]\longrightarrow L(R_n)$ such that $A\in BV$, $A(0) = A(0+) = A(t-) = A(t+) = A(1-) = A(1)$ for all $t\in(0,1)$.

Theorem 1. If $K : J\longrightarrow L(R_n)$ satisfies (2), (3) and $N(I-K) = \{0\}$, then there exists a uniquely determined $n\times n$-matrix valued function

$R : J \longrightarrow L(R_n)$ such that

$$(8) \qquad R(t,s) = K(t,s) + \int_0^1 d_r[K(t,r)]\,R(r,s), \quad t,s \in [0,1]$$

$$(9) \qquad R(t,s) = K(t,s) + \int_0^1 d_r[R(t,r)]\,K(r,s), \quad t,s \in [0,1]$$

and $\mathrm{var}_0^1\, R(.,s) < \infty$ for each $s \in [0,1]$. Furthermore $v_J(R) + \mathrm{var}_0^1\, R(0,.) < +\infty$, $R(t,1) = 0$ for all $t \in [0,1]$ and, given $f \in BV$, (7) gives the unique solution to $x - Ky = f$ in BV.

P r o o f. According to the Bounded Inverse Theorem there is a unique $R : J \longrightarrow L(R_n)$ fulfilling (8) and such that $R(.,s) \in BV$ for all $s \in [0,1]$. Moreover,

$$\|R(.,s)\|_{BV} \le c\|K(.,s)\|_{BV} \le m < \infty \quad (c < \infty)$$

for all $s \in [0,1]$ and it may be shown that $v_J(R) < \infty$. Let us put

$$\tilde{R}(t,s) = \begin{cases} R(t,s) & \text{if } s = 0 \text{ or } 1,\ t \in [0,1] \\ R(t,s+) & \text{if } s \in (0,1),\ t \in [0,1] \end{cases}$$

and $\hat{R}(t,s) = R(t,s) - \tilde{R}(t,s)$ on J. Then

$$\hat{R}(t,s) - \int_0^1 d_r[K(t,r)]\,\hat{R}(r,s) = 0$$

for all $t,s \in [0,1]$ and thus $\hat{R}(t,s) \equiv 0$ on J. This means that the rows of $R(t,.)$ belong for every $t \in [0,1]$ to the class NBV. Moreover, it is easy to verify that $x = f + Rf$ is for any $f \in BV$ a solution to (6). According to Proposition 1 we have $Q(t,.) \in S$ for all $t \in [0,1]$, where

$$Q(t,s) = R(t,s) - K(t,s) - \int_0^1 d_r[R(t,r)]\,K(r,s) \quad \text{on } J.$$

Since $R(t,.) \in NBV$ and $K(t,.) \in NBV$ for all $t \in [0,1]$, $Q(t,.) \in NBV$ for all $t \in [0,1]$, i.e. $Q(t,s) \equiv 0$ on J. This yields (9).

Remark. If $K : J \longrightarrow L(R_n)$ satisfies (2), (3) but $\dim N(I-K) = k > 0$, then a "pseudoresolvent technique" can be used for showing that there exists $R : J \longrightarrow L(R_n)$ with $v_J(R) < \infty$, $\mathrm{var}_0^1\, R(0,.) < \infty$, $R(t,.) \in NBV$ for all $t \in [0,1]$ such that if for a given $f \in BV$ the equation (6) possesses a solution, then $x = f + Rf$ is also its solution. All the solutions to (6) may be then written in the form $x(t) = x(t) + \sum_{i=1}^{k} d_i x_i(t)$, where $x_1, x_2, \ldots, x_k$ is a basis in $N(I-K)$ and d_i, $i=1,\ldots,k$ are real numbers.

Let an $n \times n$-matrix valued function $A : [0,1] \longrightarrow L(R_n)$ be of bounded variation on $[0,1]$ and $f \in BV$. We consider the integral equation

$$(10) \qquad x(t) - x(0) - \int_0^t d[A(s)]x(s) = f(t) - f(0), \quad t \in [0,1]$$

called the generalized linear ordinary differential equation. It is known (see [3]) that if $x : [0,1] \longrightarrow R_n$ satisfies (10) then $x \in BV$. Furthermore if

$$(11) \qquad \det[I - \Delta^{-}A(t)] \neq 0 \quad \text{on} \quad (0,1]$$

($I \in L(R_n)$ is the identity matrix, $\Delta^{-}A(t) = A(t) - A(t-)$) then for any $f \in BV$ and $c \in R_n$ the equation (10) possesses a unique solution $x(t)$ on $[0,1]$ such that $x(0) = c$. If we assume, in addition,

$$(12) \qquad \det\,[I + \Delta^{+}A(t)] \neq 0 \quad \text{on} \quad [0,1] \quad (\Delta^{+}A(t) = A(t+) - A(t))$$

then there exists an $n \times n$-regular matrix valued function $X : [0,1] \longrightarrow L(R_n)$ of bounded variation on $[0,1]$ such that for any $t,s \in [0,1]$

$$(13) \qquad X(t) = X(s) + \int_s^t d[A(r)]X(r) .$$

$X^{-1}(t) : [0,1] \longrightarrow L(R_n)$ is also of bounded variation and satisfies

$$(14) \qquad X^{-1}(t) = X^{-1}(s) - X^{-1}(t)A(t) + X^{-1}(s)A(s) + \int_s^t d[X^{-1}(r)]A(r)$$

for all $t,s \in [0,1]$. For given $f \in BV$ and $c \in R_n$ the corresponding solution $x(t)$ of (10) with $x(0) = c$ is given by the variation-of-constants formula

$$(15) \qquad x(t) = X(t)X^{-1}(0)c + f(t) - f(0) - X(t)\int_0^t d[X^{-1}(s)](f(s) - f(0)) .$$

Let us consider the boundary value problem (P) of determining a solution $x : [0,1] \longrightarrow R_n$ of (10) which fulfils the side condition

(16) $\quad Mx(0) + Nx(1) = r$

where M,N are $m{\times}n$-matrices and $r \in R_m$. Inserting the variation-of-constants formula (15) into (16) we get that our problem (P) is solvable if and only if

$$d^*Nf(1) - d^*NX(1)X^{-1}(0)f(0) - d^*NX(1)\int_0^1 d[X^{-1}(s)]f(s) = d^*r$$

for any $d \in R_m$ such that

(17) $\quad d^*[MX(0) + NX(1)] = 0$.

It follows from the properties of the matrix function $X^{-1}(s)$ (see (14)) that, given $d \in R_m$ fulfilling (17), the couple $(y^*(s),d^*)$, $y^*(s) = d^*NX(1)X^{-1}(s)$, satisfies the system

(18) $$y^*(s) - y^*(1) + \int_s^1 d[y^*(t)]A(t) - y^*(1)A(1) + y^*(s)A(s) = 0,$$

$$y^*(0) + d^*M = 0, \quad y^*(1) - d^*N = 0 .$$

Hence if

(19) $$y^*(1)f(1) - y^*(0)f(0) - \int_0^1 d[y^*(t)]f(t) = d^*r$$

for any solution $(y(s),d) \in BV{\times}R_m$ of (18), then our problem (P) has a solution. On the other hand, if (P) has a solution x , then

$$y^*(1)f(1) - y^*(0)f(0) - \int_0^1 d[y^*(t)]f(t) =$$

$$= (y^*(1) - d^*N)x(1) - (y^*(0) + d^*M)x(0) +$$

$$+ \int_0^1 d\Big[y^*(s) - \int_0^s d[y^*(t)]A(t) + y^*(s)A(s)\Big]x(s) = 0 .$$

Theorem 2. Under our assumptions the problem (P) has a solution if and only if (19) holds for every couple $(y(s),d) \in BV{\times}R_m$ satisfying (18).

The system (18) is called "the conjugate problem to (P)".

Remarks. 1. If $A(t-) = A(t)$ on $(0,1]$, $A(0+) = A(0)$ and $B(0) = A(0)$, $B(t) = A(t+)$ on $(0,1)$, $B(1-) = A(1)$, then the first equation from (18) reduces to

$$y^*(s) = y^*(1) - \int_s^1 y^*(t)dB(t) .$$

2. Under our assumptions, given $d \in R_n$, the function $y^*(s) = d^*NX(1)X^{-1}(s)$ is a unique solution of the first equation from (18) on $[0,1]$ such that $y^*(1) = d^*$.

3. It may be shown that if the homogeneous problem corresponding to (P)($f(t) \equiv 0$, $r = 0$) has exactly k linearly independent solutions in BV, then its conjugate problem possesses exactly $k^* = k+m-n$ linearly independent solutions in $BV \times R_m$, i.e. the index of the problem is $n - m$.

4. The homogeneous problem corresponding to (P) possesses only the trivial solution if and only if

$$\operatorname{rank} \left[MX(0) + NX(1)\right] = n .$$

Similarly as in the classical case we may show

<u>Theorem 3</u>. Let in addition $m=n$ and $\det D = \det \left[MX(0) + NX(1)\right] \neq 0$. Let us put

$$G(t,s) = \begin{cases} -X(t)D^{-1}MX(0)X^{-1}(s) & \text{for } s<t , \\ X(t)D^{-1}NX(1)X^{-1}(s) & \text{for } s>t \end{cases}$$

($G(0,0) = X(0)D^{-1}NX(1)X^{-1}(0)$, $G(1,1) = -X(1)D^{-1}MX(0)X^{-1}(1)$, the values $G(t,t)$, $t \in (0,1)$ need not be defined at this moment),

$$H(t) = X(t)D^{-1} \quad \text{on } [0,1] .$$

Then for any $f \in BV$ and $r \in R_m$ the function

$$x(t) = H(t)r + G(t,1)f(1) - G(t,0)f(0) - \int_0^1 d_s\left[G(t,s)\right]f(s)$$

is a unique solution of the problem (P).

In virtue of the properties of X^{-1} we have

<u>Theorem 4</u>. Let $P(t,s) = G(t,s)$ if $t,s \in [0,1]$, $t \neq s$, $P(1,1) = G(1,1)$, $P(t,t) = X(t)D^{-1}NX(1)X^{-1}(t)$ if $0 \le t < 1$. Then under the assumptions of Theorem 3 the functions $H(t)$, $P(t,s)$ are such that $v_J(P) + \operatorname{var}_0^1 P(0,.) + \operatorname{var}_0^1 P(.,0) < \infty$, $\operatorname{var}_0^1 H < \infty$ and moreover

$$P(t,s) - P(t,1) + \int_s^1 d_r\left[P(t,r)\right]A(r) + P(t,s)A(s) - P(t,1)A(1) =$$

$= \triangle(t,s)$ for $t \in (0,1)$, $s \in [0,1]$, $P(t,0) = -H(t)M$, $P(t,1) = H(t)N$ for $t \in (0,1)$ where $\triangle(t,s) = 0$ if $t \leq s$ and $\triangle(t,s) = -I$ if $t > s$.

($P(t,.)$, $H(t)$ is for any $t \in [0,1]$ a solution of the conjugate problem with $\triangle(t,s)$ on the right hand side.)

Remark. Theorem 4 describes the behaviour of the functions occuring in the solution formula for the problem (P) given in Theorem 3. The connection of the matrix $G(t,s)$ with the original problem (P) involves the first variable t . The proofs of these relations are straightforward but tedious, they will be given in a separate paper of the authors.

Boundary value problems of the form (P) are of interest since by generalized linear differential equations (10) some special interface problems may be described.

The more general boundary value problem with the side condition

$$Mx(0) + Nx(1) + \int_0^1 d[K(s)]x(s) = r$$

can be also handled in the same manner or it can be transferred to a boundary value problem with a side condition of the type (16) using the Jones transform for the boundary value problem similarly as was done in the paper [4] .

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