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THE METHOD OF LEAST SQUARES ON THE BOUNDARY AND VERY WEAK SOLUTIONS OF THE FIRST BIHARMONIC PROBLEM

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In this paper, the so-calledMethod of Least Squares on the Boundary is presented and its application to an approximate solution of the first biharmonic problem is shown. This method is applicable even if the boundary conditions are so general that the existence of a weak solution is not ensured, so that current variational methods (the Ritz method, the finite element method, etc.) cannot be applied. Moreover, it enables to solve the first problem of plane elasticity by reducing it into the first biharmonic problem also in the case of multiply connected regions,where other current methods meet with well-known difficulties even in the case of smooth boundary conditions.

Because the origin of this method lies in solving problems of the theory of plane elasticity, let us recall, in brief, basic concepts and results of this theory.

Throughout this paper, G is a bounded region in E_2 with a Lipschitzian boundary Γ.

Under the first problem of plane elasticity we understand a problem to find three functions

(1) $$\sigma_x, \ \sigma_y, \ \tau_{xy},$$

the so-called components of the stress-tensor, sufficiently smooth in G (to be made more precise later), fulfilling in G the equations of equilibrium

(2) $$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} = 0,$$

(3) $$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} = 0$$

and the equation of compatibility

(4) $$\Delta (\sigma_x + \sigma_y) = 0$$

(where Δ is the Laplace operator), and on Γ the boundary conditions

(5) $$\sigma_x \nu_x + \tau_{xy} \nu_y = X(s),$$

(6) $$\tau_{xy} \nu_x + \sigma_y \nu_y = Y(s).$$

Here ν_x, ν_y are components of the unit outward normal to Γ (existing almost everywhere on Γ, because Γ is Lipschitzian), X and Y are components of the outward loading which acts on the boundary, s is the length of arc on Γ. If G is multiply connected, it is required, moreover, that the vector of displacement corresponding to the stress-tensor (1) is a single-valued function in G.
In what follows, we assume that the loading is in the static equilibrium (both in forces and moments).

I. Simply connected regions

In this case, the first problem of plane elasticity can be easily transformed into the first biharmonic problem

$$\Delta^2 u = 0 \text{ in } G, \tag{7}$$

$$u = g_0(s),\ \frac{\partial u}{\partial \nu} = g_1(s) \text{ on } \Gamma. \tag{8}$$

The functions g_0, g_1 are derived, in a simple way, from the functions X,Y (for details see [5]). In this paper, we assume

$$g_0 \in W_2^{(1)}(\Gamma),\quad g_1 \in L_2(\Gamma) \tag{9}$$

only. This assumption is sufficiently general from the point of view of the theory of elasticity and sufficiently interesting from the mathematical point of view. Indeed, (9) does not ensure existence of a weak solution of (7), (8). But (see the Nečas monography [3]) it ensures existence of the so-called wery weak solution: In fact, traces (in the sense of (9)) of functions from the space $W_2^{(2)}(G)$ are dense in $W_2^{(1)}(\Gamma) \times L_2(\Gamma)$. Consequently, a sequence of functions $v_n \in W_2^{(2)}(G)$ exists such that

$$\left(v_n, \frac{\partial v_n}{\partial \nu}\right) \to (g_0, g_1) \text{ in } W_2^{(1)}(\Gamma) \times L_2(\Gamma). \tag{10}$$

Then (see [3] again) the sequence of weak solutions $\tilde{v}_n$ of the problem (7), (8) with g_0, g_1 replaced by v_n, $\partial v_n/\partial \nu$ converges, in $L_2(G)$, to a function $u \in L_2(G)$. This function is uniquely determined by the functions g_0, g_1 and is called the <u>very weak solution</u> of the problem (7), (8). The function u can be shown to be a classical solution of (7) inside of G. The components of the desired stress-tensor are then given by the relations

$$\sigma_x = \frac{\partial^2 u}{\partial y^2},\quad \sigma_y = \frac{\partial^2 u}{\partial x^2},\quad \tau_{xy} = -\frac{\partial^2 u}{\partial x \partial y}. \tag{11}$$

Because a very weak solution of (7), (8) need not be a weak solution and, consequently, need not belong to $W_2^{(2)}(G)$, usual variational methods are not applicable, in general, to get an approximate solution of the problem (7), (8). Therefore, in [1] the above mentioned method of least squares on the boundary has been developed by K. Rektorys and V. Zahradník:

Let

(12) $$z_1(x,y),\ z_2(x,y),\ \dots\ ,\ z_n(x,y)$$

be the system of basic biharmonic polynomials. (For details see [1] ; note that for $n \geqq 2$ there are precisely $4n - 2$ basic biharmonic polynomials of degree $\leqq n$.) Let $n \geqq 2$ be fixed. Denote by M the set of all functions of the form

(13) $$v(x,y) = \sum_{i=1}^{4n-2} b_{ni}\, z_i(x,y)$$

with b_{ni} arbitrary (real) and let

(14) $$Fv = \int_\Gamma (v-g_0)^2\, ds + \int_\Gamma \left(\frac{\partial v}{\partial s} - \frac{dg_0}{ds}\right)^2 ds + \int_\Gamma \left(\frac{\partial v}{\partial \nu} - g_1\right)^2 ds$$

be a functional on M. (Because of (10) and of the Lipschitzian boundary, all integrals in (14) have a sense.) Let us look for an approximate solution in the form

(15) $$u_n = \sum_{i=1}^{4n-2} a_{ni} z_i(x,y),$$

where the coefficients a_{ni} are determined from the condition

(16) $$Fu_n = \text{min. on } M.$$

The functional F being quadratic, the condition (16) leads to the solution of a system of $4n - 2$ linear equations for $4n - 2$ coefficients a_{ni}.

Theorem 1. The above mentioned system is uniquely solvable.

The proof is relatively simple: On the set M of all functions (13) one defines the scalar product $(u,v)_\Gamma$ by

$$(u,v)_\Gamma = \int_\Gamma uv\, ds + \int_\Gamma \frac{\partial u}{\partial s}\frac{\partial v}{\partial s}\, ds + \int_\Gamma \frac{\partial u}{\partial \nu}\frac{\partial v}{\partial \nu}\, ds.$$

It turns out that the determinant of the above mentioned system is the Gram determinant of the linearly independent functions (13), and, consequently, it is different from zero.

Theorem 2. For $n \to \infty$ we have

$$u_n \to u \text{ in } L_2(G),$$

where u(x,y) is the very weak solution of (7), (8). Moreover, on every subregion $G' \subset G$ the convergence is uniform. The same holds for the convergence of the derivatives $D^i u_n$ to $D^i u$ in G'.

The proof is not simple (see [1] , pp. 119-130). It is based on the following two lemmas:

Lemma 1. Let u_0 be a weak solution of a first biharmonic problem in G. Then to every $\varepsilon > 0$ there exist a region $\tilde{G} \supset G$ and a function $\tilde{u}$ biharmonic in $\tilde{G}$ such that for its restriction on G we have

$$\|\tilde{u} - u_0\|_{W_2^{(2)}(G)} < \varepsilon .$$

(Because u is biharmonic in $\tilde{G}$, it has continuous derivatives of all orders in $\bar{G}$; thus, Lemma 1 says that every weak biharmonic function in G can be approximated, in $W_2^{(2)}(G)$, with an arbitrary accuracy, by a very smooth biharmonic function in $\bar{G}$.)

For the proof of this lemma, one constructs a sequence of bounded regions G_j,

$$\bar{G} \subset G_j, \quad G_{j+1} \subset G_j \text{ for every } j = 1,2, \ldots ,$$
$$\lim_{j\to\infty} \operatorname{mes} (G_j - \bar{G}) = 0$$

(thus G_j converge for $j \to \infty$ to G "from outside"), extends the function u_0 to G_1 so that this extension - let us denote it by U_0 - belongs to $W_2^{(2)}(G_1)$ (and, consequently, to every $W_2^{(2)}(G_j)$, as the restriction on G_j, $j = 2,3, \ldots$). This is possible (cf. [3] , p. 80). On every G_j one constructs the (uniquely determined) weak solution u_j of the first biharmonic problem with boundary conditions given by the function U_0 and proves for the restrictions $\tilde{u}_j$ of u_j on G that

$$\lim_{j\to\infty} \|u_0 - \tilde{u}_j\|_{W_2^{(2)}(G)} = 0.$$

For the function $\tilde{u}$ it is then sufficient to take the restriction $\tilde{u}_j$ of a function u_j with a sufficiently high index j. (For details see [1] , pp. 122 - 128.)

Lemma 2 (on density). The traces of biharmonic polynomials are dense in $W_2^{(1)}(\Gamma) \times L_2(\Gamma)$. In detail : To every pair of functions $g_0 \in W_2^{(1)}(\Gamma)$, $g_1 \in L_2(\Gamma)$ and to every $\varepsilon > 0$ there exists a biharmonic polynomial p satisfying

$$\|p - g_0\|_{W_2^{(1)}(\Gamma)} < \varepsilon , \quad \left\|\frac{\partial p}{\partial \nu} - g_1\right\|_{L_2(\Gamma)} < \varepsilon .$$

The proof is relatively simple and is based on Lemma 1, on the well--known representation of biharmonic functions by holomorphic functions (see [5]) and application of the Walsh theorem on approximation of holomorphic functions by polynomials. For details see [1] .

Having proved Lemmas 1 and 2, the proof of the first assertion of Theorem 2 is only a technical matter. (One applies a procedure similar to that described in the text following (10) and some almost obvious properties of the method of least squares.) For details see [1] pp. 129-130.
The second assertion of this theorem is an easy consequence of Theorem 4.1.3 from [3] , p. 200 (on the behaviour, in the interior of G, of solutions of equations with sufficiently smooth coefficients).

Remark 1. In [1] also a numerical example can be found. Note that the second integral on the right-hand side of (14) plays an essential role in the proof of convergence as well as in the numerical process (as a "stabilizator").

II. Multiply connected regions

Let G be a bounded (k+1)-tuply connected region in E_2 with a Lipschitzian boundary

(17) $$\Gamma = \Gamma_0 \cup \Gamma_1 \cup \ldots \cup \Gamma_k,$$

$\Gamma_1, \ldots , \Gamma_k$ being inner boundary curves. Let a loading be acting on each of the boundary curves. As well as in the case of the simply connected region, the functions g_{i0}, g_{i1} $(i = 0,1, \ldots ,k)$ can be constructed and the problem

(18) $$\Delta^2 u = 0 \text{ in } G,$$

(19) $$u = g_{i0}, \quad \frac{\partial u}{\partial \nu} = g_{i1} \text{ on } \Gamma_i, \ i = 0,1, \ldots ,k$$

can be solved. Assuming that

(20) $$g_{i0} \in W_2^{(1)}(\Gamma_i), \quad g_{i1} \in L_2(\Gamma_i), \ i = 0,1, \ldots ,k,$$

it can be shown, in a quite similar way as in the case of the simply connected region, that a (unique) very weak solution of (18), (19) exists. It is a classical solution inside of G again. But in contrast to the case of a simply connected region, the functions (11) need not be components of a stress-tensor, because the corresponding vector of displacement need not be a single-valued function in G. (For details and for an example see [2] , Part I.)
Definition 1. A(very weak) biharmonic function to which there corresponds - through the functions (11) - a single-valued displacement is called an Airy function. In the opposite case we speak of a singular biharmonic function.
In a simply connected region, every biharmonic function is an Airy function. In a multiply connected region it need not be the case. From the point of view of the theory of elasticity, we are interested

in Airy functions only.
From the construction of the functions g_{i0}, g_{i1} it follows (see [2], Part I) that the functions

(21) $$\bar{g}_{i0} = g_{i0} + \ell_i(x,y), \quad \bar{g}_{i1} = g_{i1} + \frac{\partial \ell_i}{\partial \nu},$$

where

(22) $\ell_i(x,y) = a_i x + b_i y + c_i$ (a_i, b_i, c_i real constants), correspond to the same loading on Γ_i. A question arises if it is possible to find, on Γ_i ($i = 1,\dots,k$), the constants a_i, b_i, c_i in such a way that the very weak solution of the problem

(23) $$\Delta^2 u = 0,$$

(24) $$u = g_{i0} + \ell_i, \quad \frac{\partial u}{\partial \nu} = g_{i1} + \frac{\partial \ell_i}{\partial \nu}, \quad i = 0,1,\dots,k,$$

be an Airy function. Here

$$\ell_i = 0 \text{ for } i = 0, \quad \ell_i = a_i x + b_i y + c_i \text{ for } i = 1,\dots,k.$$

Definition 2. An Airy function which is the (very weak) solution of (23), (24), is called an Airy function corresponding to the given loading (given by the functions g_{i0}, g_{i1}).

Formulation of the problem: The functions g being given, to find an Airy function corresponding to the given loading. In detail: To find the constants a_i, b_i, c_i ($i = 1,\dots,k$) in such a way that the solution of (23), (24) be an Airy function, and to find this function.

Theorem 3. Let the functions g_{i0}, g_{i1} satisfy (20). Then there exists precisely one (very weak) Airy function corresponding to the given loading.

The idea of the proof is the following: Let u_0 be the very weak solution of (18), (19). This solution need not be an Airy function. Now, if it is not, the so-called basic singular biharmonic functions r_{ij} ($i = 1,\dots,k$ $j = 1,2,3$) are constructed which are weak solutions of the first biharmonic problem with functions of the form (22) as boundary conditions. It is shown that there exists a linear combination of these functions which "removes" the singularity from the solution u_0, i.e., if added to this solution, an Airy function is obtained. In this way, we get the required Airy function corresponding to the given loading. Uniqueness: It is shown that the difference $U(x,y)$ of two Airy functions corresponding to the given loading is a linear combination of basic singular biharmonic functions. At the same time, $U(x,y)$ - as a difference of two Airy functions - should be an Airy function. But this is possible, as shown in the work, only if all the coefficients of the above-mentioned linear combination are equal to zero. - For a detailed proof of Theorem 3 see [2], Part I.

Also in the case of multiply connected regions, the method of least squares on the boundary can be applied and is shown to be very convenient as a numerical method. The approximate solution cannot be assumed in the simple form (15) only, but in the form

$$(25) \qquad u_{mn}(x,y) = U_{mn}(x,y) + V_{mn}(x,y),$$

where

$$(26) \qquad U_{mn}(x,y) = \sum_{i=1}^{4n-2} a_{mni}\, z_i(x,y) + \sum_{i=1}^{k} \sum_{q=1}^{4m} b_{mniq}\, v_{iq}(x,y) + \sum_{i=1}^{k} c_{mni} \ln \left[(x-x_i)^2 + (y-y_i)^2 \right]$$

and

$$(27) \qquad V_{mn}(x,y) = \sum_{i=1}^{k} \sum_{j=1}^{3} \alpha_{mnij}\, r_{ij}(x,y).$$

Here, $z_i(x,y)$ are basic biharmonic polynomials,

$$v_{i,4\ell+1}(x,y) = \mathrm{Re}\left[\frac{\bar{z}}{(z-z_i)^{\ell+1}}\right], \quad v_{i,4\ell+2}(x,y) = \mathrm{Im}\left[\frac{\bar{z}}{(z-z_i)^{\ell+1}}\right],$$

$$v_{i,4\ell+3}(x,y) = \mathrm{Re}\left[\frac{1}{(z-z_i)^{\ell+1}}\right], \quad v_{i,4\ell+4}(x,y) = \mathrm{Im}\left[\frac{1}{(z-z_i)^{\ell+1}}\right],$$

$z_j = x_j + y_j$ is an (arbitrary) point lying inside of the inner boundary curve Γ_j $(j = 1,\dots,k)$ and $r_{ij}(x,y)$ are the above mentioned basic singular biharmonic functions. (These functions cause no difficulties in the numerical process, because in this process there appear only their values on the boundary curves Γ_i $(i = 1,\dots,k)$, and these are extremely simple.) (26) represents the "Airy part" and (27) the "singular part" of the approximation, respectively.

The coefficients a_{mni}, b_{mniq}, c_{mni}, α_{mnij} are determined from the condition (16) again, M being the set of functions of the form (25) with arbitrary (real) coefficients. The condition (16) leads to the solution of a system of linear equations for the unknowns a_{mni}, b_{mniq}, c_{mni}, α_{mnij} (for details see [2], Part I).

<u>Theorem 4.</u> The above mentioned system is uniquely solvable.

The <u>proof</u> is simple and is an analogue of the proof of Theorem 1.

<u>Theorem 5.</u> For $m,n \longrightarrow \infty$ we have

$$u_{mn}(x,y) \longrightarrow u(x,y) \text{ in } L_2(G),$$

where $u(x,y)$ is the very weak solution of the problem (18), (19). At the same time, the "Airy part" $U_{mn}(x,y)$ converges, in $L_2(G)$, to the Airy function corresponding to the given loading, and thus to the desired solution of the first problem of plane elasticity. The conver-

gence is uniform on every subregion G' of G such that $\overline{G'} \subset G$. The same holds for the convergence of the derivatives $D^i u_{mn}$ or $D^i U_{mn}$.
The proof follows the same idea as the proof of Theorem 2. Only the technique is more pretentious because of the multiple connectivity of the region. Especially, the Walsh theorem on approximation of a holomorphic function by a polynomial should be replaced by a more general theorem on approximation by a rational function (this is also the cause why functions v_{iq} appear in (26)), etc.

Remark 2. Because, solving the problem of plane elasticity, we are interested only in the Airy function corresponding to the given loading, it is not at all necessary to construct, actually, the "singular" functions $r_{ij}(x,y)$.

Remark 3. The method of least squares on the boundary suggested by the authors proved to be a very effective approximate method when solving problems of the theory of elasticity and of related fields. Especially, it has been applied with success to the solution of some rather difficult problems of wall-beams in soil mechanics. In the case of the biharmonic problem, it takes advantage of the form of the biharmonic equation. However, it can be applied as well, when properly modified, to the solution of other problems.

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