František Neuman

Global properties of the nth order linear differential equations

In: Jiří Fábera (ed.): Equadiff IV, Czechoslovak Conference on Differential Equations and Their Applications. Proceedings, Prague, August 22-26, 1977. Springer-Verlag, Berlin, 1979. Lecture Notes in Mathematics, 703. pp. [309]--317.

Persistent URL: http://dml.cz/dmlcz/702231

GLOBAL PROPERTIES OF THE nTH ORDER LINEAR DIFFERENTIAL EQUATIONS

F. Neuman, Brno

In my lecture I should like to describe an approach to problems concerning global properties of linear homogeneous differential equations (LDEs) of the nth order, $n \geqq 2$, and some basic facts of the theory.

Let me start with a few historical remarks. Investigations concerning LDEs of the nth order began in the middle of the last century and were connected with the names of E. E. Kummer [5], E. Laguerre [7], F. Brioschi, G. H. Halphen, A. R. Forsyth, P. Stäckel [19], S. Lie, E. J. Wilczynski [20], and others. Between the main objects of their study were transformations, canonical forms and invariants of LDEs. Their investigations were of local character, which was already noticed by George D. Birkhoff [1] in 1910. He pointed out that not every 3rd order LDE can be reduced to its Laguerre-Forsyth canonical form on its whole interval of definition.

Of course, the local character of results is not suitable for global problems, like questions concerning boundedness of solutions, solutions of the classes L^2 and L^p, periodic solutions, solutions converging to zero, oscillatory behavior of solutions: conjugate points, disconjugate equations etc.

Except that G.Birkhoff [1] introduced a geometrical interpretation of solutions of the 3rd order LDEs using curves in the projective plane, and except for some isolated results of a global character, there was no theory describing global properties of LDEs, not even in the simplest cases $n=2$ and 3.

As a simple illustration that any question of a global character was difficult to solve let me mention the following one. There was a conviction that some properties of LDEs with variable coefficients might be modifications of properties of LDEs with constant coefficients. E.g., the 3rd order LDE with real constant coefficients has always at least one nonvanishing solution; one might expect that in the case of variable coefficients at least one solution of any LDE of the 3rd order would have only finite number of zeros. That this is not the case was discovered by G. Sansone [18] in 1948.

In the last twenty years O. Borůvka [2] developed the theory of global properties of LDEs of the 2nd order as you have heard in his plenar lecture at the conference.

For the nth order LDEs there are now results of N. V. Azbelev and Z. B. Caljuk, J. H. Barrett, T. A. Burton and W. T. Patula, W. A. Coppel, W. N. Everitt, M. Greguš, H. Guggenheimer, G. B. Gustafson, M. Hanan, M. K. Kwong, V. A. Kondrat'jev, A. C. Lazer, A. Ju. Levin, M. Ráb, G. Sansone, C. A. Swanson, M. Švec and others having global character but mainly devoted to oscillatory behavior of solutions, conjugate points and disconjugacy. However there was still no theory of global properties of LDEs of the nth order enabling us to foretell the possible behavior of solutions, to exclude the impossible cases, to enable us to see globally the whole situation.

Global structure of linear differential equations

All our considerations will be in the real domain. Consider a LDE of the nth order, $n \geqq 2$:

$$y^{(n)} + p_{n-1}(x)y^{(n-1)} + \dots + p_0(x)y = 0$$

on an open (bounded or unbounded) interval $I \subset \mathbb{R}$ that will be shortly denoted by $\underline{P}$ (together with its interval of definition, that is important when studying situation globally). The coefficients are supposed to be real and continuous. Let $\underline{Q}$ be another LDE of the same order, say

$$z^{(n)} + q_{n-1}(t)z^{(n-1)} + \dots + q_0(t)z = 0 \quad \text{on } J \subset \mathbb{R}.$$

Let $\underline{y} = (y_1, \dots, y_n)^T$ be a column vector of n linearly independent solutions of $\underline{P}$ considered again on the whole interval I; similarly $\underline{z}$ is defined for $\underline{Q}$.

We say that $\underline{P}$ is <u>globally transformable</u> into $\underline{Q}$ if there exist
1. a bijection h of J onto I of the class C^n with $dh(t)/dt \neq 0$ on J,
2. a nonvanishing scalar function $f: J \to \mathbb{R}$ of the class C^n, and
3. an n by n regular constant matrix A such that

$$(\alpha) \qquad \underline{z}(t) = A.f(t).\underline{y}(h(t)) \quad \text{on } J$$

for some (then every) $\underline{y}$ and $\underline{z}$ of $\underline{P}$ and $\underline{Q}$, resp.
Due to Stäckel, (α) is the most general pointwise transformation that for $n \geqq 2$ keeps the kind of our differential equations (i.e., the order and the linearity) unchanged.

The h and f in (α) will be called <u>transformator</u> and <u>multiplicator</u> of the transformation (α), resp. We shall also simply write

$$\alpha \underline{P} = \underline{Q}$$

to express the fact that $\underline{P}$ is globally transformed into $\underline{Q}$ by α.

The relation of global transformability is an equivalence and we often call the $\underline{P}$ and $\underline{Q}$ globally equivalent equations. We come to a decomposition of all LDEs of all orders $n \geqq 2$ into classes of globally equivalent equations.

Let D be one of the classes, $\underline{P} \in$ D, $\underline{Q} \in$ D, and $\alpha \underline{P} = \underline{Q}$. For $\underline{R} \in$ D and $\beta \underline{Q} = \underline{R}$ we may define $(\beta\alpha)\underline{P} := \beta(\alpha \underline{P}) = \underline{R}$. It is easy to check that we have introduced a structure of Brandt groupoid into each class of globally equivalent equations. From the theory of categories it is known that each Brandt groupoid essentially depends on the stationary group of its arbitrary element, e.g. on the group $B(\underline{P})$ of all morphisms (or transformations) of the equation $\underline{P}$ into itself. This stationary group $B(\underline{P})$ for $n = 2$ coincides with the group of dispersions of $\underline{P}$ introduced by O. Borůvka.

If we consider transformations that not only globally transform $\underline{P}$ into itself but, moreover, that transform each solution of the equation $\underline{P}$ into itself (i.e. A is the unit matrix in (α)), then we get a subgroup $C(\underline{P})$ of $B(\underline{P})$.

When studying global properties of solutions then transformations with increasing transformators h, $h' > 0$, are extremely important. Let $B^+(\underline{P})$ and $C^+(\underline{P})$ be the subgroups of $B(\underline{P})$ and $C(\underline{P})$ with increasing transformators.

The fundamental results of that part of the theory are the following ones.

Theorem 1. $B^+(\underline{P})$ is not trivial if and only if D contains an equation with periodic coefficients.

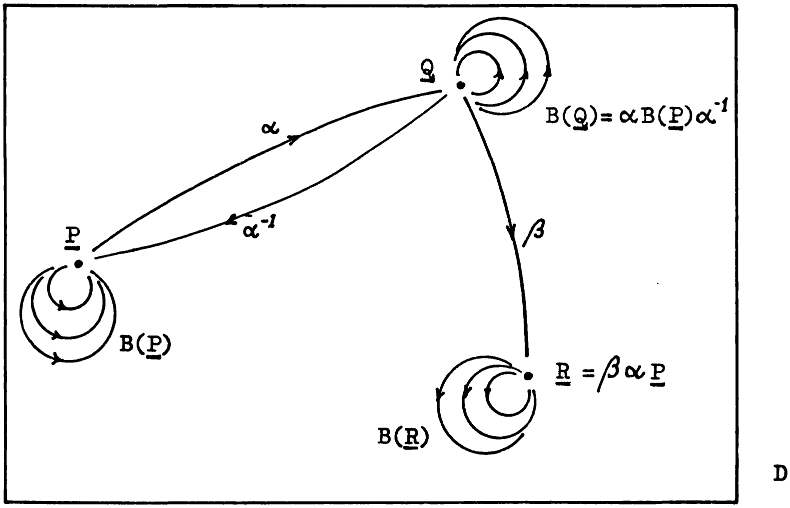
Theorem 2. $C^+(\underline{P})$ is not trivial if and only if there is an equation in D having only periodic solutions with the same period.

Theorem 3. For $\alpha\underline{P} = \underline{Q}$ it holds $B(\underline{Q}) = \alpha B(\underline{P})\alpha^{-1}$, and similarly for $B^+(\underline{Q})$, $C(\underline{Q})$ and $C^+(\underline{Q})$.

Theorem 4. All transformations of $\underline{P}$ into $\underline{Q}$ form the set

$$\alpha B(\underline{P}) = B(\underline{Q})\alpha = B(\underline{Q})\alpha B(\underline{P}).$$

Proofs of the theorems are essentially based on methods of the theory of categories and can be seen from the following picture.



To each class D of globally equivalent LDEs we may assign a (canonical) equation $\underline{E}(D)$. Then for each equation $\underline{P} \in D$ there exists a transformation α (not necessarily unique, it depends on $B(\underline{P})$) that transforms $\underline{E}(D)$ into $\underline{P}$. The transformator h and the multiplicator f of the α are called phase and amplitude of $\underline{P}$ (with respect to the canonical $\underline{E}(D)$). Hence we have introduced "polar coordinates" in each class of globally equivalent LDEs.

The just mentioned categorial description of global structure of LDEs of arbitrary order n, $n \geqq 2$, has its geometrical aspects that enable us to understand the sense of analytic constructions in the theory of global transformations, to solve open problems, and, sometimes, to find occasional inaccuracies in the mathematical literature occurring in complicated and lengthy analytic processes without necessity of a tiresome calculation.

The essence of our geometrical approach is the following observation first introduced in [10] and [11].

Theorem 5. Consider LDE $\underline{P}$ and its n linearly independent solutions $y_1,\dots,y_n$ forming the coordinates of the vector function $\underline{y}$, now considered as a curve in n-dimensional vector space. There is a 1-1 correspondence between all solutions of equation P and all hyperplanes passing through origin, in which parameters of intersections of the curve $\underline{y}$ with a particular hyperplane are zeros of the correspond-

<u>ing solution and vice versa, counting multiplicities that occur as the order of contacts.</u>

This result is essentially used in recent literature, see e.g. [4] .

Moreover, see again [11], if $\underline{y}$ is considered in n-dimensional euclidean space, the central projection of the curve $\underline{y}$ onto the unit sphere S_{n-1} has the same property. But now all intersections are on the unit sphere, and if instead of hyperplanes main circles are under consideration we have all the situation in a compact space, where strong tools of topology are to our disposal. Some open problems were already solved by the method ([13]).

Furthermore, having the central projection of the curve $\underline{y}$ on the unit sphere in n-dimensional euclidean space we introduce a new parametrization as the length of the projection. We could see that, firstly, by the projection the multiplicator was eliminated, and secondly, by specifying the parametrization we unify the transformator. Hence we get a special curve $\underline{u}$ on the sphere. LDEs which conversely correspond to these special curves are called <u>canonical</u>. The explicit forms of the canonical equations are obtained using Frenet formulae of the special curves.

I should like to stress that these special equations are canonical in the global sense, that means, each LDE can be transformed on its whole interval of definition into its canonical form without any restrictions on the smoothness of its coefficients.

E.g.

$$y'' + y = 0 \qquad \text{on } I$$

are all canonical differential equations for n = 2 (there are still several equivalent classes depending on the length of I);

$$y''' - \frac{a'}{a}y'' + (1 + a^2)y' - \frac{a'}{a}y = 0 \qquad \text{on } I,$$

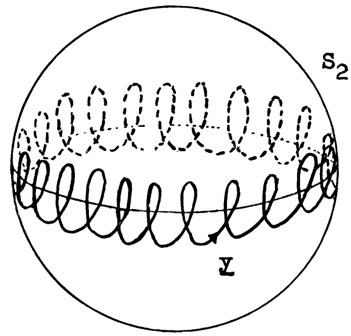
$a \in C^1$, $a > 0$, are all canonical forms for n = 3 (they depend on a function a and an interval I), etc.

<u>Examples</u>

Let me demonstrate the above few facts from the groundwork of the theory of global properties of LDEs on special problems.

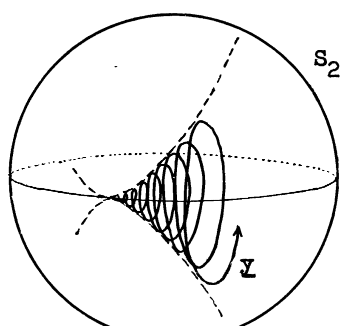
Let us see the following picture of "a prolonged cycloid" $\underline{y}$ infinitely many times surrounding the equator of the unit sphere in

3-dimensional space:



If a curve $\underline{y}$ is three times differentiable and without points of inflexion (that corresponds to nonvanishing Wronskian of its coordinates), then its coordinates may be considered as 3 linearly independent solutions of a LDE of the 3rd order. Since each plane going through the origin intersects $\underline{y}$ infinitely many times, each solution of the LDE has infinitely many zeros. We have Sansone's interesting result using our approach.

Considering again LDEs of the 3rd order with only constant coefficients we can observe that if one oscillatory solution occurs, then necessarily there must be two linearly independent oscillatory solutions. One may ask whether for general LDEs of the 3rd order (with variable coefficients) the same situation holds. Using our method, we want to know whether a curve of the class C^3 without points of inflexion on the unit sphere S_2 of 3-dimensional space exists such that it is intersected infinitely many times just by one plane passing through origin, whereas any other plane passing through origin has only finite number of intersections with our curve.



The above picture gives the answer: there exists such an equation; there is again no analogy to the case of constant coefficients.

One may ask, why is the situation for $n = 3$ different from the case $n = 2$, where there are so many analogies. The answer follows from our results: because for $n = 2$ each class of globally equivalent equations has a global representation (e.g. its canonical equation) with constant coefficients (i.e. $y'' + y = 0$), however this is not the case for $n > 2$.

Let me come to other type of applications of our approach. Many recent problems and results concern LDE of the 2nd order in the form

(1) $$u'' + q(t)u = 0 \quad \text{on } I$$

having all solutions square integrable. There was a problem whether in this case all solutions of (1) are also bounded, see [17] and [6].

Using our method we may proceed as follows.

(2) $$y'' + y = 0 \quad \text{on } J$$

is a canonical form of (1). The curve $\underline{y} = \binom{\sin x}{\cos x}$ corresponds to (2), hence the curve $\underline{u}(t) = \binom{f(t)\ \sin h(t)}{f(t)\ \cos h(t)}$, $f, h \in C^2$, $f.h' \neq 0$ on I, corresponds to LDE of the second order. Since the coefficient by u' in (1) is zero, we have $f(t) = \text{const.}\cdot|h'(t)|^{-1/2}$ (cf. O. Borůvka's lecture). Hence $f \in C^2$ implies $h \in C^3$ and $|h'(t)|^{-1/2}.\sin h(t)$, $|h'(t)|^{-1/2}.\cos h(t)$ are two linearly independent solutions of (1).

It is easy to derive the following succession of implications:

Each solution of (1) is square integrable <u>iff</u>
Two linearly independent solutions of (1) are square integrable <u>iff</u>
$\int_I |h'(t)|^{-1}.\sin^2 h(t)dt < \infty$ and $\int_I |h'(t)|^{-1}.\cos^2 h(t)dt < \infty$ <u>iff</u>
$\int_I |h'(t)|^{-1}dt < \infty$.

Analogously

Each solution of (1) is bounded <u>iff</u>
Two linearly independent solutions of (1) are bounded <u>iff</u>
Both $|h'(t)|^{-1}.\sin^2 h(t)$ and $|h'(t)|^{-1}.\cos^2 h(t)$ are bounded on I <u>iff</u>
$|h'(t)|^{-1}$ is bounded on I,
where $h' \neq 0$ and $h \in C^3$.

And we ask whether (1) with all square integrable solutions has only bounded solutions. In our model it is equivalent to the question, whether

$$\int_I |h'(t)|^{-1} dt \quad \Rightarrow \quad |h'|^{-1} \text{ is bounded on } I$$

for $h \in C^3$, $h' \neq 0$; see [8] and [9].

Of course, the implication is not true. Taking suitable h' we can explicitly construct an example of such an equation if it is necessary. Similarly we may construct explicitly examples of LDEs with certain properties using coordinates of the corresponding curves and making some boring computation.

I should like to conclude my lecture by the following remark. The above sketched method and results are suitable for reviewing globally the whole situation, to see what can and what cannot happen, they are applicable in cases when problems concern behaviour of solutions, distribution of their zeros and other properties of this kind. On the other hand, within the reach of our approach there are only few results for both second and higher order equations which make use of conditions on coefficients.

References

[1] G. D. Birkhoff, On the solutions of ordinary linear homogeneous differential equations of the third order. Annals of Math. 12 (1910/11), 103-127.

[2] O. Borůvka, Linear differential transformations of the second order. The English Univ. Press, London 1971.

[3] O. Borůvka, Teorija global'nych svojstv obyknovennych linejnych differencial'nych uravnenij vtorogo porjadka. Differencial'nyje uravnenija 12 (1976), 1347-1383.

[4] H. Guggenheimer, Distribution of zeros and limit behavior of solutions of differential equations. Proc. AMS 61 (1976), 275-279.

[5] E. E. Kummer, De generali quadam aequatione differentiali tertii ordinis. Progr. Evang. Royal & State Gymnasium Liegnitz 1834, reprinted in J. Reine Angew. Math. (Crelle Journal) 100 (1887), 1-10.

[6] M. K. Kwong, On boundedness of solutions of second order differential equations in the limit circle case. Proc. AMS 52 (1975), 242-246.

[7] M. Laguerre, Sur les équations différentielles linéaires du troisième ordre. Comptes rendus 88 (1879), 116-119.

[8] F. Neuman, Relation between the distribution of the zeros of the solutions of a 2nd order linear differential equation and the boundedness of these solutions. Acta Math. (Hung.) 19 (1968), 1-6.

[9] F. Neuman, L^2-solutions of $y'' = q(t)y$ and a functional equation. Aequationes Math. 6 (1971), 66-70.

[10] F. Neuman, Some results on geometrical approach to linear differential equations of the n-th order, Comm. Math. Univ. Carol. 12 (1971), 307-315.

[11] F. Neuman, Geometrical approach to linear differential equations of the n-th order. Rend. Mat. 5 (1972), 579-602.

[12] F. Neuman, On n-dimensional closed curves and periodic solutions of linear differential equations of the n-th order. Demonstratio Math. 6 (1973), 329-337.

[13] F. Neuman, On two problems about oscillation of linear differential equations of the third order, J. Diff. Equations 15 (1974), 589-596.

[14] F. Neuman, Global transformations of linear differential equations of the n-th order. Knižnice odb. a věd. spisů VUT Brno, B-56 (1975), 165-171.

[15] F. Neuman, On solutions of the vector functional equation $\underline{y}(\xi(x)) = f(x).A.\underline{y}(x)$. To appear in Aequationes Math. in 1977.

[16] F. Neuman, Categorial approach to global transformations of the n-th order linear differential equations, Časopis Pěst. Mat. 102 (1977), 350-355.

[17] W. T. Patula, J. S. W. Wong, An L^p-analogue of the Weyl alternative. Math. Ann. 197 (1972), 9-28.

[18] G. Sansone, Studi sulle equazioni differenziali lineari omogenee di terzo ordine nel campo reale. Revista Mat. Fis. Teor. Tucuman 6 (1948), 195-253.

[19] P. Stäckel, Über Transformationen von Differentialgleichungen. J. Reine Angew. Math. (Crelle Journal) 111 (1893), 290-302.

[20] E. J. Wilczynski, Projective differential geometry of curves and ruled surfaces. Teubner - Leipzig 1906.

Author's address: Mathematical Institute of the Czechoslovak Academy of Sciences, Branch Brno,

662 95 Brno, Janáčkovo nám. 2a,
Czechoslovakia