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ON FORMULATION AND SOLVABILITY OF BOUNDARY VALUE PROBLEMS FOR VISCOUS INCOMPRESSIBLE FLUIDS IN DOMAINS WITH NON-COMPACT BOUNDARIES

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Studying boundary value problems for viscous incompressible fluids I have introduced two function spaces, namely $\overset{\circ}{J}(\Omega)$ and $H(\Omega)$. The former is the closure in the norm $L_2(\Omega)$ of the family $\dot{J}^{\infty}(\Omega)$ of all infinitely differentiable solenoidal vector functions $\vec{u}(x)$ with compact supports which belong to the domain Ω of the Euclidean space R^n $(n=2,3)$. The latter is the closure of the same family $\dot{J}^{\infty}(\Omega)$ in the norm of Dirichlet integral. Let us denote by $\overset{\circ}{D}(\Omega)$ the Hilbert space which is the closure in the norm of Dirichlet integral of the family $\dot{C}^{\infty}(\Omega)$ of all infinitely differentiable vector-functions $\vec{u}(x)$ with compact supports which belong to Ω. The scalar product in $\overset{\circ}{D}(\Omega)$ is defined by

$$(1) \qquad \left[\vec{u},\vec{v}\right] = \int_{\Omega} \vec{u}_x \vec{v}_x dx = \int_{\Omega} \sum_{i,k=1}^{n} u_{ix_k} v_{ix_k} dx \,.$$

We introduce the same scalar product in $H(\Omega)$. $H(\Omega)$ is a proper subspace of the space $\overset{\circ}{D}(\Omega)$. We shall regard $\overset{\circ}{J}(\Omega)$ as a subspace of the Hilbert space $L_2(\Omega)$ and introduce a scalar product in both of them by

$$(2) \qquad (\vec{u},\vec{v}) = \int_{\Omega} \vec{u}\vec{v} dx; \quad (\vec{u},\vec{u})^{1/2} \equiv \|\vec{u}\|_{2,\Omega} \,.$$

Let us give a motivation for introducing these spaces and show why they proved useful and suitable for the study of Navier-Stokes equations. First let us consider the Stokes problem

$$(3) \qquad -\nu\Delta\vec{v} = -\nabla\rho + \vec{f}(x) \,,$$

$$(4) \qquad \operatorname{div}\vec{v} = 0 \,, \quad \vec{v}\big|_{\partial\Omega} = 0 \,,$$

restricting ourselves to the case of homogeneous boundary conditions. If Ω is an unbounded domain in R^3 then $\vec{v}$ has to satisfy an additional condition

$$(5_1) \qquad \vec{v}(x) \rightarrow 0 \quad \text{for} \quad |x| \rightarrow \infty \,,$$

while for $\Omega \subset R^2$ it must satisfy

$$(5) \qquad \vec{v}(x) \rightarrow 0 \quad \text{for} \quad |x| \rightarrow \infty \,.$$

In this most current formulation, ρ is subjected to no boundary conditions. Therefore I wanted to "get rid" of ρ and to obtain such a system of relations for $\vec{v}$ which would enable us to determine uniquely $\vec{v}$ and then to find ρ from $\vec{v}$. At the same time, I did not want to put any restrictions on the behavior or ρ near to $\partial\Omega$ and infinity lest I should have to verify them when determining ρ from $\vec{v}$. To this aim I formed the scalar product of (3) with $\vec{\eta}\in \dot{J}^{\infty}(\Omega)$, integrated over Ω and transformed the resulting equation into the form

(6) $$[\vec{v},\vec{\eta}] = (\vec{f},\vec{\eta})$$

using the integration by parts formula, the equation (4) and the fact that $\rho\in L_{2,loc}$. Provided $\vec{f}$ is not too bad, namely

(7) $$|(\vec{f},\vec{\eta})| \le c_f \|\vec{\eta}\|_H$$

for all $\vec{\eta}\in H(\Omega)$, then $\vec{v}\in H(\Omega)$ is found uniquely from the identity (6) [1 , Chap.II] . With regard to all this I introduced the following definition of a (generalized) solution of the problem (3) - (5):

A function $\vec{v}$ is called a solution of the problem (3)-(5) if it belongs to $H(\Omega)$ and satisfies the equality (6) for all $\vec{\eta}\in \dot{J}^{\infty}(\Omega)$.

If $\vec{f}\in L_{2,loc}$ then it is relatively easy to prove that $\vec{v}\in W^2_{2,loc}$ and satisfies the system (3) with a certain function $\rho\in W^1_{2,loc}$. The function ρ is determined uniquely provided it is normed, say, by the condition $\int_{\Omega_1}\rho\,dx = 0$, $\bar{\Omega}'\subset\Omega$.

Such an approach to the problem (3)-(5) is attractive for its simplicity and generality: it permits to include simultaneously arbitrary domains from R^2 and R^3 not only for the Stokes system but for the complete nonlinear Navier-Stokes system as well (see [1 , Chap.IV]). It accounts also for the Stokes paradox: for unbounded domains $\Omega\subset R^3$ the solution from $H(\Omega)$ converges for $|x|\to\infty$ to zero while for $\Omega\subset R^2$ it converges to a constant, generally non--zero. Thus the suggested re-formulation of the problem (3)-(5) proved to be successful from the mathematical point of view: we have satisfied all the requirements of the problem (3)-(5) proving at the same time its unique solvability for a wide class of right hand side terms $\vec{f}$. Nevertheless, to obtain uniqueness I had to consider $\vec{v}$ in the space $H(\Omega)$. This assumption has not been included in the classical formulation of the problem (3)-(5) and the question whether the suggested specification of the problem (3)-(5) is the only possible is essential. First of all, $\vec{v} \in H(\Omega)$ implies finiteness of the Dirichlet integral for $\vec{v}$. We know quite a number of problems in which the solution, interesting from the physical point of view, does not possess this property. However, to omit it (in the case of nonlinear Navier-Stokes equations and general type of domains) does not seem possible at the moment, and therefore we restrict ourselves by considering only such $\vec{v}$'s for which $\|\vec{v}_x\|_{2,\Omega} < \infty$. This assumption together with zero boundary conditions means that $\vec{v}$ has to be an element of $\overset{o}{D}(\Omega)$. Moreover, taking into account, the equality $\operatorname{div} \vec{v} = 0$ we conclude that $\vec{v}$ belongs to the space $\hat{H}(\Omega)$ which consists of all elements of $\overset{o}{D}(\Omega)$ which have zero divergence.

It is clear that

$$H(\Omega) \subset \hat{H}(\Omega) \subset \overset{o}{D}(\Omega),$$

which raises a question about the dimension of the quotient space $\hat{H}(\Omega)|_{H(\Omega)}$. Its investigation was initiated by J.Heywood [2]. He proved that $\hat{H}(\Omega) = H(\Omega)$ for domains Ω (bounded or not) with compact smooth boundaries of the class C^2. Moreover, he indicated domains for which $\hat{H}(\Omega)$ is wider than $H(\Omega)$. In the three-dimensional case this holds for the whole space R^3 divided by the plane $\{x : x_1 = 0\}$ with "holes" cut in it. For such Ω we have $\dim \hat{H}|_H = 1$ and the elements of $\hat{H}|_H$ may be characterized either by the quantity α ($\alpha \in R^1$) of the total flow through all the holes (their number is assumed finite and they must be bounded two-dimensional domains with smooth boundaries) or by the difference of the limit values of ρ for $x_1 \to \pm\infty$.

In accordance with this, for such domains the system (3)-(5) has a unique solution $\vec{v}$ from $H(\Omega)$ which has a prescribed total flow through the holes. The solution $\vec{v}$ determined above (i.e. $\vec{v}$ from $H(\Omega)$) corresponds to the value of α equal to zero.

Together with V.A.Solonnikov we have carried out a more detailed analysis of the cases $\hat{H} = H$ and $\dim \hat{H}|_H \geq 1$. Furthermore, we have investigated problems of formulation and solvability of boundary value problems for general nonlinear Navier-Stokes equations in the space $\hat{H}$ when $\hat{H}$ is wider than H. The results obtained have been published in [3], [4]. They have been continued in the thesis of K.Pileckas and in a joint paper [5] by V.A.Solonnikov and K.Pileckas. Let us mention the results of [3], [4] without presenting the precise formulations. First, we proved that $\hat{H}$ coincides with H for domains (bounded or not) with compact "not too bad" boundaries (e.g. Lipschitzian). To this aim we had to consider two auxiliary problems:

(8) $\qquad \operatorname{div} \vec{u} = \varphi \;, \quad \vec{u} \in \mathring{D}(\Omega)$

with $\varphi \in L_2(\Omega)$, $\int_\Omega \varphi dx = 0$, and

(9) $$\rho_{x_i} = \sum_{k=1}^{n} (R_{ik})_{x_k} + f_i \;, \quad \int_{\Omega_1} \rho dx = 0 \;, \quad i=1,\dots,n \;,$$

with $\rho \in W^1_{2,loc}$ and $R_{ik} \in L_2(\Omega)\ W^1_{2,loc}$, $f_i \in L_2(\Omega)$.
For (8) we found a solution $\vec{u}$ which satisfies an inequality $\|\vec{u}_x\|_{2,\Omega} \le C_\Omega \|\varphi\|_{2,\Omega}$ with a constant C_Ω which is invariant with respect to similarity mapping of the domain Ω . For ρ satisfying (9) we proved an estimate

$$\|\rho\|_{2,\Omega} \le C_{\Omega,\Omega'}\Big(\sum_{i,k=1}^{n} \|R_{ik}\|_{2,\Omega} + \sum_{i=1}^{n} \|f_i\|_{2,\Omega}\Big) \;.$$

Non-smoothness of the boundary precluded us from using the theory of hydrodynamic potentials. And it is this type of boundaries that we have to deal with even if the boundary $\partial\Omega$ of the original domain Ω is smooth but not compact.

The above presented auxiliary results are useful not only for the problems just considered. They have been applied to deal with problems with free surfaces which meet non-smoothly a rigid wall [6] . They can be used also in the case of the problem (3), (4) on a bounded domain Ω to prove $\rho \in L_2(\Omega)$ for all $\vec{f}$ satisfying the condition (7).

However, let us come back to the problem whether $\hat{H}$ and H coincide or not. We have proved that $\hat{H} = H$ provided Ω has one exit to infinity. If Ω has m exits to infinity, $m>1$ and each of them includes a circular cone (an angle in the case $\Omega \subset R^2$) then $\dim \hat{H}|_H = m-1$. The elements $\vec{v}$ of the quotient space $\hat{H}|_H$ can be characterized by choices of numbers α_k , $k=1,\dots,m-1$ which indicate the flows $\vec{v}$ through $m-1$ exits (as $\vec{v} \in \hat{H}$, the flow through the last exit equals $\alpha_m = -\sum_{k=1}^{m-1} \alpha_k$). For elements $\vec{v}$ from H all α_k are equal to zero. In accordance with this, the problem (3)-(5) for such Ω allows the following more precise formulation:

to find a vector function $\vec{v}$ from $\hat{H}$ for which the flows through $m-1$ exits are equal to α_k , $k=1,\dots,m-1$ and which satisfies the identity (6) for all $\vec{\eta} \in \dot{J}^{\infty}(\Omega)$ (or, which is the same, for all $\vec{\eta} \in H$).

Its unique solvability follows from the above proved solvability of the problem (3)-(5) in the space H. Indeed, let $\vec{a}$ be an element of the space $\hat{H}$ with given flows α_k, $k=1,\dots,m-1$ and let us seek $\vec{v}$ in the form $\vec{u}+\vec{a}$, $\vec{u} \in H$. For $\vec{u}$ we obtain the problem

$$-\nu\Delta\vec{u} = -\nabla\rho + \nu\Delta\vec{a} + \vec{f}, \quad \vec{u} \in H$$

whose unique solvability was proved in [1, Chap.II].

The nonlinear problem

$$(10) \qquad -\nu\Delta\vec{v} + \sum_{k=1}^{n} v_k\vec{v}_{x_k} = -\nabla\rho + \vec{f}(x),$$

$$\operatorname{div}\vec{v} = 0, \quad \vec{v}\big|_{\partial\Omega} = 0,$$

on domains Ω with m exits to infinity which extend "sufficiently quickly" (e.g. they may contain cones (angles)) allows an analogous formulation:

to find $\vec{v}$ from $\hat{H}$ with prescribed flows α_k, $k=1,\dots,m-1$ through $m-1$ exits and satisfying the identity

$$\nu[\vec{v},\vec{\eta}] - (v_k\vec{v}, \vec{\eta}_{x_k}) = (\vec{f},\vec{\eta})$$

for all $\vec{\eta} \in \mathring{J}^{\infty}(\Omega)$.

The solvability of this problem follows also from the results which I proved about the solvability of the system (10) in H provided at least one of the representants $\vec{a}$ of the element of $\hat{H}|_H$ which corresponds to the prescribed values α_k, $k=1,\dots,m-1$ possesses the following property:

$$(11) \qquad \int_\Omega \vec{a}^2\vec{\eta}^2 dx \le \nu_1 \int_\Omega \vec{\eta}_x^2 dx, \qquad \nu_1 \in (0,\nu)$$

for all $\vec{\eta} \in H(\Omega)$. In the paper [4] such $\vec{a}$'s will be constructed for "almost" all the class of domains Ω for which we proved $\dim \hat{H}|_H = m-1$ in [3]. Here α_k, ν^{-1} as well as the other data of the problem are subjected to no smallness requirements.

If the domain Ω has m "sufficiently quickly" extending exits to infinity and r "insufficiently quickly" extending ones then the prescribed values of α_k, $k=1,\dots,m-1$ of flows through the exits of the first kind are added to the equations (3)-(5) and (10) provided $m>1$. The dimension $\dim \hat{H}|_H$ is then equal to $m-1$.

The words "insufficiently quickly" extending exit indicate the fact that for any element $\vec{v}$ from $\hat{H}(\Omega)$ the flow through this exit is equal to zero. It is not difficult to obtain sufficient conditions guaranteeing this property of an exit. For example, let it have the form $B = \{x : x_1 > a , (x_2,x_3) \in S(x_1)\}$, where $S(x_1)$ is a family of two-dimensional domains with meas $S(x_1) > 0$. If $\vec{v} \in \overset{o}{D}(\Omega)$ then it is well known that for almost all x_1 the following inequalities hold:

$$|j(x_1)|^2 \equiv \Big(\int_{S(x_1)} |\vec{v}(x_1,x_2,x_3)| dx_2 dx_3 \Big)^2 \le \text{meas } S(x_1) ,$$

$$\int_{S(x_1)} \vec{v}^2(x) dx_2 dx_3 \le c \text{ meas}^2 S(x_1) \int_{S(x_1)} \vec{v}_x^2(x) dx_2 dx_3 .$$

This together with the finiteness of the Dirichlet integral implies that $j(x_1^k) \to 0$ for a certain sequence x_1^k , $k=1,2,\ldots$ of values of x_1 tending to infinity, provided

$$\int_a^{\infty} \lambda_1(x_1) \text{ meas}^{-1} S(x_1) dx_1 = \infty . \tag{12}$$

On the other hand, the flow $\int_{S(x_1)} v_3(x_1,x_2,x_3) dx_2 dx_3$ is independent of $x_1 > a$. Consequently, it is equal to zero.

It turned out that the convergence of the integral (12) together with the Lipschitz condition on ∂B already guarantee the property called by myself "sufficiently quick" extension of the exit B . In the paper [5] we have weakened our original assumption for an exit to contain a cone in the sense just described.

Let us pass now to non-stationary problems

$$\vec{v}_t - \nu \triangle \vec{v} + \sum_{k=1}^{n} v_k \vec{v}_{x_k} = -\nabla \rho + \vec{f}(x,t) , \tag{13}$$

$$\text{div } \vec{v} = 0 , \quad \vec{v}|_{\partial\Omega} = 0 , \quad \vec{v}|_{t=0} = \vec{\varphi}(x) .$$

In my papers, and actually in papers of other authors as well it is assumed that $v \in L_2((0,T) , \overset{o}{J}{}_2^1(\Omega))$. The space $\overset{o}{J}{}_2^1(\Omega)$ is the closure of the space $\vec{J}^{\infty}(\Omega)$ in the norm of $W_2^1(\Omega)$. The scalar product in both $\overset{o}{J}{}_2^1(\Omega)$ and $\overset{o}{W}{}_2^1(\Omega)$ is given by

$$(\vec{u},\vec{v})^{(1)} = \int_{\Omega} (\vec{u}\vec{v} + \vec{u}_x v_x) dx . \tag{14}$$

All elements $\vec{v}$ of the space $\overset{o}{\vec{W}}{}_2^1(\Omega)$ which satisfy the equality $\text{div } \vec{v} = 0$ form a subspace which will be denoted by $\hat{\overset{o}{J}}{}_2^1(\Omega)$. It is easily seen that

$$\overset{o}{J}{}_2^1(\Omega) \subset \hat{\overset{o}{J}}{}_2^1(\Omega) \subset \hat{\overset{o}{W}}{}_2^1(\Omega) .$$

J.Heywood proved that $\hat{\mathring{J}}{}^1_2(\Omega)$ coincides with $\hat{\mathring{J}}{}^1_2(\Omega)$ for domains Ω with compact boundaries of the class C^2 as well as for the domain from R^3 which was described above in connection with his results. For an analogous domain in the plane (i.e. the whole R^2 except a straight line with some open segments cut in it) he raised the problem of finding $\dim \hat{\mathring{J}}{}^1_2|\mathring{J}{}^1_2$. Together with V.A.Solonnikov we have shown that for all three-dimensional domains of the above considered types we have $\dim \hat{\mathring{J}}{}^1_2|\mathring{J}{}^1_2 = \dim \hat{H}|_H$ while in the planar case $\hat{\mathring{J}}{}^1_2|\mathring{J}{}^1_2$. According to this result we can formulate the problem (13) for three-dimensional domains with $\dim \hat{\mathring{J}}{}^1_2|\mathring{J}{}^1_2 = m-1>0$ in the following way:

to find a vector function $\vec{v}$ from the space $L_2((0,T);\ \hat{\mathring{J}}{}^1_2(\Omega))$ with the prescribed flows $\alpha_k(t)$, $k=1,\dots,m-1$ through the "sufficiently quickly" extending exits and satisfying the integral identity corresponding to the Navier-Stokes system.

We omit here the integral identity which replaces the Navier--Stokes system. It is known that it can be written in various, nonetheless equivalent forms. The analogue of the condition (12) for the space $\hat{\mathring{J}}{}^1_2$ is $\int_a^{\infty} \text{meas}^{-1}\, S(x_1)dx_1 = \infty$ (this is clear from the above argument for elements of $\hat{H}$ and from the fact that the elements of $\hat{\mathring{J}}{}^1_2$ have a finite $L_2(\Omega)$-norm). It guarantees that the flow through an exit B for any element from $\hat{\mathring{J}}{}^1_2(\Omega)$ is equal to zero. The divergence of the integral implies a "sufficiently quick" extension of the exit B for the couple of spaces $\hat{\mathring{J}}{}^1_2$ and $\mathring{J}{}^1_2$. The question of solvability of the problem (13) in the given formulation reduces by means of a substitution $\vec{v}(x,t) = \vec{u}(x,t) + \vec{a}(x,t)$ to the problem of finding the function $\vec{u}$ from the space $L_2((0,T);\ \mathring{J}{}^1_2(\Omega))$. For the function $\vec{a}(x,t)$ we can take any element of $\hat{\mathring{J}}{}^1_2(\Omega)$ with the prescribed flows α_k , $k=1,\dots,m-1$ which are constructed in [3] . If $\alpha_k(t)$ possesses some smoothness properties in t (e.g. $\alpha_k \in C^1([0,T])$) then the existence of $\vec{u}$ is established in essentially the same way as for the problem (13) in the space $L_2((0,T);\ \mathring{J}{}^1_2(\Omega))$. In general, the problem (13) in the new formulation allows an application of the results by E.Hopf as well as their modifications and a majority of results proved by the author for the problem (13) in the original formulation (i.e. for $\alpha_k(t) \equiv 0$) . They are given together with their proofs in [1 , Chap.VI] .

References

(All references except [2] in Russian)

[1] Ladyženskaja O.A.: Mathematical problems of the dynamics of viscous incompressible fluid. 1st edition Fizmatgiz 1961, 2nd edition Nauka 1970

[2] Heywood J.G.: On uniqueness questions in the theory of viscous flow. Acta Mathematica 136 (1976), 61-102

[3] Ladyženskaja O.A., Solonnikov V.A.: On some problems of vector analysis and generalized formulations of boundary value problems for Navier-Stokes equations. Zapiski nauč.sem. Leningr.otd.Matem.Inst.Akad.Nauk SSSR 59 (1976), 81-116

[4] Ladyženskaja O.A., Solonnikov V.A.: On the solvability of boundary and initial-boundary value problems for the Navier--Stokes equations on domains with non-compact boundaries. Vestnik Leningr.gosud.univ. 13 (1977), 39-47

[5] Pileckas K., Solonnikov V.A.: On some spaces of solenoidal vectors and on solvability of the boundary value problem for the system of Navier-Stokes equations on domains with non--compact boundaries. Zapiski nauč.sem.Leningr.otd.Matem. inst.Akad.nauk SSSR 1978

[6] Solonnikov V.A.: Solvability of the problem of planar motion of a viscous incompressible capillary fluid in a non-closed vessel. Preprint LOMI, R-5-77, Leningrad 1977

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