SOME MODIFICATIONS OF SOBOLEV SPACES AND NON-LINEAR BOUNDARY VALUE PROBLEMS

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This paper deals with applications of two types of function spaces of Sobolev type to (generally non-linear) elliptic partial differential equations. Especially, the following spaces are considered:

(i) A n i s o t r o p i c Sobolev spaces denoted by

$$W^{E,p}(\Omega) \; ;$$

(ii) Sobolev w e i g h t spaces denoted by

$$W^{k,p}(\Omega;\sigma) \; .$$

Here Ω is a bounded domain in the Euclidean space R^N with a boundary $\partial\Omega$ which can be locally described by Lipschitzian functions and p is a real number, $p > 1$. The spaces under consideration are defined as follows:

(i) The space $W^{E,p}(\Omega)$ with E an arbitrary but fixed set of N-dimensional multiindices is defined as the set of all functions $u \in L_{1,loc}(\Omega)$ such that their distributional derivatives $D^\alpha u$ with $\alpha \in E$ belong to the space $L_p(\Omega)$. The space $W^{E,p}(\Omega)$ is a separable reflexive Banach space under the norm

$$\|u\|_{E,p} = \Big(\sum_{\alpha \in E} \int_\Omega |D^\alpha u(x)|^p \, dx \Big)^{1/p}$$

provided the multiindex set E contains the zero multiindex $\theta = (0,0,\ldots,0)$.

(ii) The space $W^{k,p}(\Omega;\sigma)$ with k a positive integer and $\sigma = \sigma(x)$ an almost everywhere positive function defined on Ω (and called the w e i g h t function) is defined as the set of all functions $u \in L_{1,loc}(\Omega)$ such that their distributional derivatives $D^\alpha u$ of order $|\alpha| \leq k$ (α denotes the usual N-dimensional multiindex, $|\alpha|$ its length) have the property

$$\int_\Omega |D^\alpha u(x)|^p \, \sigma(x) \, dx < \infty$$

for all α with $|\alpha| \leq k$. The space $W^{k,p}(\Omega;\sigma)$ is a separable reflexive Banach space under the norm

$$\|u\|_{k,p,\sigma} = \Big(\sum_{|\alpha| \leq k} \int_{\Omega} |D^{\alpha}u(x)|^p \, \sigma(x) \, dx \Big)^{1/p} .$$

The aim of this paper is to extend some results concerning the solvability of linear partial differential equations in the above mentioned spaces and in some cases the uniqueness of the solution (which is treated in the w e a k sense here) also to the case of n o n - l i n e a r partial differential equations.

1. Anisotropic Sobolev spaces

1.1. In [1], S. M. NIKOL'SKIĬ investigated linear partial differential equations on Ω of the form

$$(1) \qquad \sum_{\alpha,\beta \in E} (-1)^{|\alpha|} D^{\alpha}(a_{\alpha\beta}(x)\, D^{\beta}u(x)\,) = f(x) \; , \quad x \in \Omega \, ,$$

where E is a given set of N-dimensional multiindices and $a_{\alpha\beta}(x)$ are given functions from $L_{\infty}(\Omega)$. Provided that the differential operator on the left-hand side in (1) is conditionally elliptic which means that a constant $c_0 > 0$ exists such that

$$\sum_{\alpha,\beta \in E} a_{\alpha\beta}(x) \xi_{\alpha} \xi_{\beta} \geq c_0 \sum_{\gamma \in E} |\xi_{\gamma}|^2$$

for all $x \in \Omega$ and $\xi_{\alpha} \in R^1$, the concept of the weak solution of a boundary value problem for (1) can be introduced to be a certain function $u \in W^{E,2}(\Omega)$, and existence theorems can be proved.

1.2. Remark. Equations of the form (1) occur in applications. E. g., the fourth order equations appear in elasticity, namely, in the theory of plates. Analogous problems are often formulated also in terms of non-linear equations.

1.3. Here, we shall deal with n o n - l i n e a r analoga of the equation (1), i.e. with equations of the form

$$(2) \qquad \sum_{\alpha \in E} (-1)^{|\alpha|} D^{\alpha} a_{\alpha}(x;\, \delta_E u(x)\,) = f(x) \; , \quad x \in \Omega \, .$$

The symbol $\delta_E u$ denotes the vector function

$$\{ D^{\beta}u \; ; \; \beta \in E \} \; ;$$

let us denote by $\varkappa(E)$ the number of components of this vector and assume that the "coefficients" $a_\alpha(x;\xi)$ in the equation (2) are defined for $x \in \Omega$ and $\xi = \{\xi_\beta\} \in R^{\varkappa(E)}$, satisfy the Carathéodory condition and the growth conditions

$$(3) \qquad |a_\alpha(x;\xi)| \leqq g_\alpha(x) + c_\alpha \sum_{\beta \in E} |\xi_\beta|^{p-1}$$

with $p > 1$ for $\alpha \in E$, $x \in \Omega$, $\xi \in R^{\varkappa(E)}$ and with constants $c_\alpha > 0$ and functions $g_\alpha \in L_q(\Omega)$, $q = p/(p-1)$.

1.4. The multiindex set E and the parameter p in (3) allow now to introduce the (anisotropic) space $W^{E,p}(\Omega)$. Before defining the weak solution of a boundary value problem for the equation (3), let us summarize various assumptions and notions:

1.5. Assumptions. (I) Let us assume that the set E fulfils the following conditions:

(i) E is a convex set;

(ii) if $\beta \in E$ and γ is such a multiindex that $\gamma \leqq \beta$ (i.e., $\gamma_i \leqq \beta_i$ for $i = 1,\dots,N$) then $\gamma \in E$, too.

(II) Let us introduce the space

$$W_0^{E,p}(\Omega)$$

as the closure of the space $C_0^\infty(\Omega)$ of infinitely differentiable functions with compact support in Ω with respect to the norm $\|\cdot\|_{E,p}$, and a space V such that

$$W_0^{E,p}(\Omega) \subset V \subset W^{E,p}(\Omega) ;$$

both space $W_0^{E,p}(\Omega)$ and V are again normed by the expression $\|\cdot\|_{E,p}$.

(III) Let us introduce a Banach space Q such that $V \subset Q$, a constant c_1 exists with

$$\|v\|_Q \leqq c_1 \|v\|_{E,p} \qquad \text{for every } v \in V ,$$

and $C_0^\infty(\Omega)$ is dense in V.

(IV) Let a function $\varphi \in W^{E,p}(\Omega)$ and functionals $f \in Q^*$ and $g \in V^*$ be given such that

$$(4) \qquad \langle g, v \rangle = 0 \quad \text{for } v \in W_0^{E,p}(\Omega) .$$

1.6. Definition. Let the coefficients a_α from (2) fulfil the growth conditions (3) and let the assumptions (I) - (IV) from 1.5 be satisfied. A function $u \in W^{E,p}(\Omega)$ is said to be a weak solution of the boundary value problem for the equation (2) - we shall denote this boundary value problem by $(\{a_\alpha\}; W^{E,p}(\Omega); V,Q)$ - if the following conditions are satisfied:

(i) $u - \varphi \in V$,

(ii) the identity

$$\sum_{\alpha \in E} \int_\Omega a_\alpha(x; \delta_E u(x)) \, D^\alpha v(x) \, dx = \langle f,v \rangle + \langle g,v \rangle \tag{5}$$

holds for every $v \in V$.

1.7. Notation. Let B be a subset of E such that

(i) E is the convex hull of $B \cup \theta$;

(ii) for every $\beta \in E - B$ there exist multiindices $\alpha^{(1)}, \dots, \alpha^{(N)} \in B$ such that

$$\beta \leq \alpha^{(i)}, \quad \beta_i < \alpha_i^{(i)} \quad \text{for } i = 1,\dots,N .$$

Such a set B is called a complete basis of E.

Let us write

$$a_\alpha(x; \xi) = a_\alpha(x; \omega, \eta)$$

where $\omega \in R^{\varkappa(E-B)}$ corresponds to the elements of $E - B$ and $\eta \in R^{\varkappa(B)}$ corresponds to the elements of B, i.e.

$$a_\alpha(x; \delta_E u(x)) = a_\alpha(x; \delta_{E-B} u(x), \delta_B u(x)) .$$

Now, we are able to formulate the following existence theorem:

1.8. Theorem. Let the assumptions of Definition 1.6 be fulfilled. Further, let the coefficients a_α fulfil the monotonicity condition

$$\sum_{\alpha \in B} (a_\alpha(x; \omega, \xi) - a\ (x; \omega, \eta))(\xi_\alpha - \eta_\alpha) \geq 0 \tag{6}$$

for every $\omega \in R^{\varkappa(E-B)}$ and $\xi, \eta \in R^{\varkappa(B)}$, the condition

$$\frac{\sum_{\alpha \in B} \xi_\alpha a_\alpha(x; \xi)}{\sum_{\alpha \in E} |\xi_\alpha| + \sum_{\alpha \in E} |\xi_\alpha|^{p-1}} \to \infty \quad \text{if} \quad \sum_{\alpha \in E} |\xi_\alpha| \to \infty \tag{7}$$

and the coerciveness condition

$$(8) \qquad \sum_{\alpha \in E} \xi_\alpha a_\alpha(x;\xi) \geqq c_1 \sum_{\alpha \in E} |\xi_\alpha|^p - c_2$$

for every $\xi \in R^{\varkappa(E)}$ with $c_1 > 0$, $c_2 \geqq 0$.

Then there exists at least one weak solution $u \in W^{E,p}(\Omega)$ of the boundary value problem $(\{a_\alpha\}\,;\, W^{E,p}(\Omega);\, V,Q)$.

P r o o f : We use the Leray-Lions theory concerning general equations in monotone operators (see [2]) and the properties of the anisotropic Sobolev spaces $W^{E,p}(\Omega)$ developed in [3].

Let us define an operator $A(u,v)$ on $V \times V$ by

$$A(u,v) = A_1(u,v) + A_2(u),$$

where

$$\langle A_1(u,v),w\rangle = \sum_{\alpha \in B} \int_\Omega a_\alpha(x;\, \delta_{E-B}(v(x) + \varphi(x)),\, \delta_B(u(x) + \varphi(x)))\, D^\alpha w(x)\, dx,$$

$$\langle A_2(u),w\rangle = \sum_{\alpha \in E-B} \int_\Omega a_\alpha(x;\, \delta_E(u(x) + \varphi(x)))\, D^\alpha w(x)\, dx$$

for $w \in V$, and let the operator $\mathcal{A}$ on V be defined by

$$\mathcal{A}(u) = A(u,u).$$

The growth conditions (3) imply that A maps $V \times V$ into V^* and that A is bounded. From conditions (6) and (7) and from the compactness of the imbedding

$$W^{E,p}(\Omega) \hookrightarrow W^{E-B+\vartheta}(\Omega)$$

proved in [3] it follows that the operator $\mathcal{A}$ is pseudomonotone (in the sense of [2], Chap. 2, Prop. 2.6). The coerciveness condition (8) implies that $\mathcal{A}$ is coercive. Therefore, the equation

$$\mathcal{A}w = F$$

has at least one solution $w \in V$ for every $F \in V^*$ by [2], Chap. 2, Theorem 2.7.

Since the identity (5) can be rewritten as

$$\langle \mathcal{A}(u - \varphi),\, v\rangle = \langle F,\, v\rangle$$

with $\langle F,v\rangle = \langle f,v\rangle + \langle g,v\rangle$, we conclude that the element

$$u = w + \varphi \in W^{E,p}(\Omega)$$

is a weak solution of the boundary value problem $(\{a_\alpha\}\,;\, W^{E,p}(\Omega);\, V,Q)$.

<u>1.9. Remark.</u> It is also possible to introduce a "variational formulation" of the boundary value problem $(\{a_\alpha\}\,;\, W^{E,p}(\Omega);\, V,Q)$:

assuming in addition that the coefficients a_α are symmetric in the following sense:

$$\frac{\partial a_\alpha}{\partial \xi_\beta} = \frac{\partial a_\beta}{\partial \xi_\alpha} \quad \text{for} \quad \alpha, \beta \in E$$

in the sense of distributions in $\xi \in R^{\varkappa(E)}$ for almost every $x \in \Omega$, we can show that the functional

$$\Phi(v) = \int_0^1 \Big(\sum_{\alpha \in E} \int_\Omega a_\alpha(x;\ t\,\delta_E v(x) + \delta_E \varphi(x))\, D^\alpha v(x)\,dx\Big)dt - \langle f,v \rangle - \langle g,v \rangle$$

defined on V attains (under the assumptions of Theorem 2.4) its minimum on V at a certain "point" u_0 and that this element u_0 determines the weak solution u of the boundary value problem $(\{a_\alpha\};\ W^{E,p}(\Omega);\ V,Q)$: it is $u = u_0 + \varphi$.

1.10. At the first sight, it is not clear how the boundary value problem $(\{a_\alpha\};\ W^{E,p}(\Omega);\ V,Q)$ is to be interpreted, i.e., to which "classical" boundary value problem it corresponds. Therefore we shall give here one example which illustrates the difference between the isotropic and anisotropic cases:

If the set E is defined by

$$E = \{\alpha\ ;\ |\alpha| \leq k\}$$

with k a positive integer, then the corresponding space $W^{E,p}(\Omega)$ is the "usual" Sobolev space $W^{k,p}(\Omega)$. In this case, the choice

$$V = W_0^{E,p}(\Omega) \quad (\text{i.e.,} \quad V = W_0^{k,p}(\Omega)\ )$$

corresponds to the Dirichlet problem for the equation (2). Now, let us show what the Dirichlet problem means in the a n i s o t r o p i c case.

1.11. Example. Let $N = 2$, let Ω be the square $]0,1[\times]0,1[$ and E the set $\{(2,0),\ (1,1),\ (1,0),\ (0,1),\ (0,0)\}$. The equation (2) then assumes the form

$$(9) \quad \frac{\partial^2}{\partial x^2} a_{(2,0)}(x,y;\ \delta_E u) + \frac{\partial^2}{\partial x\,\partial y} a_{(1,1)}(x,y;\ \delta_E u) - \frac{\partial}{\partial x} a_{(1,0)}(x,y;\ \delta_E u) - \frac{\partial}{\partial y} a_{(0,1)}(x,y;\ \delta_E u) + a_{(0,0)}(x,y;\ \delta_E u) = f(x,y)$$

where $\delta_E u = \left\{ \frac{\partial^2 u}{\partial x^2}, \frac{\partial^2 u}{\partial x\, \partial y}, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, u \right\}$. Let us choose $V = W_0^{E,p}(\Omega)$, $Q = L_r(\Omega)$ with a suitable value r, $\varphi \equiv 0$ (which corresponds to homogeneous boundary conditions) and let $f \in Q^*$ be defined by the function $f(x,y)$ in (9); in view of the choice of V and of the condition (4), it is not necessary to choose g. If the smoothness of the weak solution and of the data of our boundary value problem allow to introduce the concept of a classical solution, then it can be shown that the "abstract" boundary value problem $(\{a_\alpha\}; W^{E,p}(\Omega); V,Q)$ corresponds to the following "Dirichlet problem for the equation (9)" : The solution u has to satisfy equation (9) on Ω and the following boundary conditions on $\partial\Omega$:

$$u\big|_{\partial\Omega} = 0 , \quad \frac{\partial u}{\partial x}\Big|_{\partial\Omega} = 0 , \quad \frac{\partial u}{\partial y}\Big|_{\Gamma} = 0$$

where Γ is the part of $\partial\Omega$ described by the conditions $\{x = 0$ or $x = 1\}$. In other words, the "Dirichlet problem" (with homogeneous boundary conditions) means that $u = 0$ on the whole boundary $\partial\Omega$ while the normal derivative $\frac{\partial u}{\partial n} = 0$ only on Γ and <u>no</u> values for $\frac{\partial u}{\partial n}$ are prescribed on $\partial\Omega - \Gamma$.

For a comparison, let us note that if we add the multiindex (0,2) to the set E, we obtain the space $W^{2,p}(\Omega)$ and the boundary conditions corresponding to the Dirichlet problem for this choice of multiindices are the usual ones:

$$u = 0 \quad \text{and} \quad \frac{\partial u}{\partial n} = 0 \quad \text{on the } \underline{\text{whole}} \text{ boundary} \quad \partial\Omega .$$

2. Sobolev weight spaces

<u>2.1.</u> Spaces of this type are useful for the investigation of uniformly elliptic as well as degenerate elliptic equations. In the case of a degenerate equation, the weight function σ is prescribed by the degeneration; in the case of a uniformly elliptic equation, the application of a Sobolev weight space is motivated by the desire of having a possibility of extending the class of solvable boundary value problems, e.g., by extending the class of admissible right-hand sides of the equation or the class of boundary conditions.

We shall deal here with the latter case. Then the question arises for what type of weight functions one can obtain assertions about existence and uniqueness of a (weak) solution of a boundary value problem for an equation of order $2k$ in the space $W^{k,p}(\Omega;\sigma)$. This question was partially answered by some authors in the case of linear differential equations (i.e., $p = 2$) and of the Dirichlet problem and the mixed problem: In [4], [5], [6] weight functions of the type

$$\sigma(x) = [\mathrm{dist}\,(x,M)]^{\varepsilon} \tag{10}$$

with $M \subset \partial\Omega$ and ε a real number were investigated and it was shown that there exist positive numbers c_1, c_2 such that if $-c_1 < \varepsilon < c_2$ then the Dirichlet problem (or the mixed problem) is uniquely solvable in the space $W^{k,2}(\Omega;\sigma)$ with weight functions σ from (10). Further, in [7] weight functions of the type

$$\sigma(x) = s(\mathrm{dist}\,(x,M)) \tag{11}$$

with $s = s(t)$ defined for $t \geqq 0$ were investigated and conditions on s were given which guarantee again the existence and uniqueness of the solution of the (linear) Dirichlet problem in the space $W^{k,2}(\Omega;\sigma)$.

2.2. Here, we shall deal with non-linear equations of the type

$$\sum_{|\alpha|\leqq k} (-1)^{|\alpha|} D^{\alpha} a_{\alpha}(x;\, \delta_k u(x)) = f(x)\ , \quad x \in \Omega\ , \tag{12}$$

and with weight functions of the type (10). The symbol $\delta_k u$ denotes the vector function

$$\{D^{\beta}u;\ \ |\beta| \leqq k\}\ ,$$

$\varkappa$ is the number of components of this vector and it is assumed that the "coefficients" $a_{\alpha}(x;\xi)$ in the equation (3) are defined for $x \in \Omega$ and $\xi \in R^{\varkappa}$, satisfy the Carathéodory condition, the growth conditions

$$|a_{\alpha}(x;\xi)| \leqq c_{\alpha}\Big(1 + \sum_{|\beta|\leqq k} |\xi_{\beta}|^{p-1}\Big) \tag{13}$$

for $|\alpha| \leqq k$, $x \in \Omega$, $\xi \in R^{\varkappa}$ with constants $c_{\alpha} > 0$, the monotonicity condition

$$\sum_{|\alpha|\leqq k} (a_{\alpha}(x;\xi) - a_{\alpha}(x;\eta))(\xi_{\alpha} - \eta_{\alpha}) > 0 \tag{14}$$

for every choice of $\xi, \eta \in R^{\varkappa}$, $\xi \neq \eta$, and the <u>coerciveness condition</u>

$$(15) \qquad \sum_{|\alpha| \leqq k} \xi_\alpha a_\alpha(x; \xi) \geqq c_1 \sum_{|\beta|=k} |\xi_\beta|^p + c_2 |\xi_\theta|^p - c_3$$

for every $\xi \in R^{\varkappa}$ with constants $c_1 > 0$, $c_2 \geqq 0$, $c_3 \geqq 0$.

<u>2.3.</u> In the following, we shall investigate the D i r i c h l e t problem only; therefore it is necessary to introduce the space

$$W_0^{k,p}(\Omega; \sigma)$$

as the closure of the space $C_0^\infty(\Omega)$ with respect to the norm $\|\cdot\|_{k,p,\sigma}$. - Further, we shall consider weight functions σ of the type (10) where M is a subset of the boundary $\partial\Omega$ (usually it is $M = \partial\Omega$ or $M = \{x_0\}$ with $x_0 \in \partial\Omega$).

<u>2.4. Definition.</u> Let φ be a given function from $W^{k,p}(\Omega;\sigma)$, f a given functional from the dual space $(W_0^{k,p}(\Omega;\sigma))^*$. A function $u \in W^{k,p}(\Omega;\sigma)$ is said to be a <u>weak solution of the Dirichlet problem</u> for the equation (12) if the following conditions are satisfied:

(i) $u - \varphi \in W_0^{k,p}(\Omega;\sigma)$,

(ii) the identity

$$(16) \qquad \sum_{|\alpha| \leqq k} \int_\Omega a_\alpha(x; \delta_k u(x)) D^\alpha v(x)\,dx = \langle f, v \rangle$$

holds for every $v \in C_0^\infty(\Omega)$.

Now we are able to formulate the following existence and uniqueness theorem:

<u>2.5. Theorem.</u> Let the assumptions in 2.2, 2.3 and 2.4 be fulfilled. Then there are positive numbers d_1, d_2 such that if we consider the space $W^{k,p}(\Omega;\sigma)$ where σ is a weight function of the type (10) with $\varepsilon \in]-d_1, d_2[$, then there exists one and only one weak solution $u \in W^{k,p}(\Omega;\sigma)$ of the Dirichlet problem for the equation (12). Further, a constant $c > 0$ exists such that

$$\|u\|_{k,p,\sigma} \leqq c(\|\varphi\|_{k,p,\sigma} + \|f\|) .$$

P r o o f : Analogously as in the proof of Theorem 1.8, we use the Leray-Lions theory (see [2]) and the properties of the Sobolev weight spaces $W^{k,p}(\Omega;\sigma)$ with weights of the type (10) (see e.g. [4], [8]): If we define the operator A on $V = W_0^{k,p}(\Omega;\sigma)$ by

$$\langle Au,w\rangle = \sum_{\alpha\leq k}\int_\Omega a_\alpha(x;\ \delta_k(u(x)+\varphi(x)))D^\alpha w(x)dx$$

for $w \in C_0^\infty(\Omega)$, then the growth, monotonicity and coerciveness conditions (13), (14) and (15) imply that A is a bounded, hemicontinuous, monotone and coercive operator from V into V^*. Therefore, the equation

$$Aw = f$$

has one and only one solution $w \in V$ for every $f \in V^*$ by [2], Chap. 2, Theorems 2.1 and 2.2. Consequently, the element $u = w + \varphi \in W^{k,p}(\Omega;\sigma)$ is the (unique) weak solution of the Dirichlet problem.

2.6. Remark. The foregoing theorem states only the existence of admissible values ε such that the Dirichlet problem is uniquely solvable in the space $W^{k,p}(\Omega;\sigma)$ with σ of the type (1) without explicit bounds for these values of ε. A computation of the values d_1, d_2 which determine the admissible interval $]-d_1,d_2[$ for ε shows that they depend mainly on the geometrical properties of $\partial\Omega$, more precisely on the properties of the subset $M\subset\partial\Omega$ (of course, in addition to their dependence on the other data of the boundary value problem).

In the conclusion, it should be noted that the results of Section 1 were obtained together with J. RÁKOSNÍK.

References:

[1] NIKOL'SKIǏ, S. M.: Some properties of differentiable functions defined on an n-dimensional open set. Izv. Akad. Nauk SSSR Ser. Mat. 23(1959), 213-242 (Russian).

[2] LIONS, J.-L.: Quelques méthodes de résolution des problèmes aux limites non linéaires. Dunod, Gauthier-Villars, Paris 1969.

[3] RÁKOSNÍK, J.: Some remarks to anisotropic Sobolev spaces. Beiträge zur Analysis 13 (to appear).

[4] NEČAS, J.: Les méthodes directes en théorie des équations elliptiques. Academia, Prague 1967.

[5] KUFNER, A.: Lösungen des Dirichletschen Problems für elliptische Differentialgleichungen in Räumen mit Belegungsfunktionen. Czech. Math. J. 15(90) (1965), 621-633.

[6] KADLEC, J.; KUFNER, A.: On the solution of the mixed problem. Comment. Math. Univ. Carolinae 7 (1966), 1, 75-84.

[7] KUFNER, A.; OPIC, B.: Solution of the Dirichlet problem in Sobolev space with weight of a general typ. Trudy Sem. S.L. Soboleva, No 2, Novosibirsk 1976, 35-48 (Russian).

[8] KUFNER, A.: Einige Eigenschaften der Sobolevschen Räume mit Belegungsfunktionen. Czech.Math.J. 15(90) (1965), 597-620.

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