Herbert Gajewski

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ON THE ITERATIVE SOLUTION OF SOME NONLINEAR EVOLUTION EQUATIONS

H. Gajewski, Berlin

The purpose of this paper is to show by three examples of nonlinear evolution equations arising from mathematical physics how a priori estimates can be used to establish globally convergent iteration processes. An important feature of these iteration processes is that one proceeds by solving <u>linear</u> evolution equations with <u>constant</u> coefficients.

We shall start our discussion with Burgers' equation. As further examples the spatially two-dimensional Navier-Stokes equations and the nonlinear Schrödinger equation will be considered. We shall conclude with some remarks concerning the numerical realisation of the iteration processes.

At first we introduce some notations. Let X be a Banach space and $S=[0,T]$ a bounded time interval. Then $C(S;X)$ is the Banach space of continuous mappings from S into X provided with the maximum norm. $L^p(S;X)$, $1\leq p\leq\infty$, denotes the Banach space of Bochner integrable functions $u:(0,T)\to X$ with the norms

$$\Big(\int_S \|u(t)\|_X^p\,dt\Big)^{1/p}\ ,\quad 1\leq p<\infty\ ,\qquad \operatorname*{ess\,sup}_{t\in S}\|u(t)\|_X\ ,\quad p=\infty .$$

1. Burgers' equation

Let $H=L^2(0,1)$, $V=H_0^1(0,1)$ and $V^*=H^{-1}(0,1)$ be the usual spaces with the norms $|\cdot|$, $\|\cdot\|$ and $\|\cdot\|_*$, respectively. We consider the initial-boundary value problem

$$\text{(1.1)}\qquad \begin{gathered} u_t-\nu u_{xx}+u\,u_x=f \quad \text{in} \quad (0,T)\times(0,1)\ , \\ u(0,x)=a(x)\ ,\ x\in(0,1)\ ,\quad u(t,0)=u(t,1)=0\ ,\ t\in(0,T]\ . \end{gathered}$$

Here the subscripts t and x indicate partial differentiation, ν is a positive constant. We suppose up to the end of this section that

$$\text{(1.2)}\qquad f\in L^2(S;V^*)\ ,\quad a\in H\ .$$

Then, as is well known, the problem (1.1) has a unique solution $u\in L^2(S;V)\cap C(S;H)$ with $u_t\in L^2(S;V^*)$ satisfying the a priori estimate (cf. /3/)

$$(1.3) \qquad \|u\|_{C(S;H)} \leq r \quad , \quad r^2 = |a|^2 + \frac{1}{2\nu}\|f\|^2_{L^2(S;V^*)} \ .$$

For constructing the solution of (1.1) Carasso /1/ proposed the following iteration procedure

$$(1.4) \qquad \begin{aligned} &u^j_t - \nu u^j_{xx} = f - u^{j-1}u^{j-1}_x \ , \quad j=1,2,\ldots, \ u^0=0 \ , \\ &u^j(0,x) = a(x) \ , \ u^j(t,0) = u(t,1) = 0 \ . \end{aligned}$$

A corresponding method has been used by Fujita and Kato /2/ as a means of proving existence and uniqueness theorems for the Navier-Stokes equations. Carasso /1/ stated the following sufficient convergence condition for (1.4)

$$(1.5) \qquad \left(\frac{64T}{\nu}\right)^{1/2} \left(\|a\| + \int_S \|f(t)\|\,dt\right) < 1 \ .$$

Possibly this condition could be weakened but it cannot be replaced by a global condition because counter-examples show (cf. /1/) that the convergence of the procedure (1.4) is in fact only local in time, even if the global solution of (1.1) is smooth.

We want now to show that the iteration method (1.4) can be easily modified in such a way that we get a globally convergent process. For that we define the projector of H onto the r-ball in H by

$$(1.6) \qquad P\,v = \begin{cases} v & \text{if } |v| \leq r \\ r\dfrac{v}{|v|} & \text{if } |v| > r \ , \end{cases}$$

where r is the constant from (1.3). We suggest replacing (1.2) by

$$(1.7) \qquad \begin{aligned} &u^j_t - \nu u^j_{xx} = f - (Pu^{j-1})u^{j-1}_x \ , \quad j = 1,2,\ldots, \\ &u^j(0,x) = a(x) \ , \quad u^j(t,0) = u^j(t,1) = 0 \ . \end{aligned}$$

The following global convergence theorem holds.

<u>Theorem 1.</u> Let u be the solution of (1.1) and $u^0 \in L^2(S;V) \cap C(S;H)$ an arbitrary starting function. Then the sequence (u^j) defined by (1.7) converges to u in $C(S;H)$ and $L^2(S;V)$.

<u>Proof.</u> First we note the simple inequalities

$$|P\,v - P\,w| \leq |v - w| \ , \ v, \ w \in H \quad \text{and} \quad |v|_\infty \leq 2\,|v|\,\|v\| \ , \ v \in V \ ,$$

where $|\cdot|_\infty$ is the norm in $L^\infty(0,1)$. Next we define by

$$\|v\|^2_{C,k} = \sup_{t\in S}\,(e^{-k(t)}|v(t)|^2) \quad , \quad \|v\|^2_{X,k} = \tfrac{1}{2}\|v\|^2_{C,k} + \sup_{t\in S}\left(e^{-k(t)}\int_0^t\|v\|^2 ds\right)$$

norms being equivalent to the basic norms in $C(S;H)$ and $X=C(S;H)\cap L^2(S;V)$, respectively. Here the function k is defined by

$$k(t) = 2\int_0^t (\|u(s)\|^2 + \frac{16}{\nu}(\frac{r^2}{\nu}+1))\,ds .$$

Now we see from (1.3) and (1.6) that $Pu(t)=u(t)$ for $t\in S$. Consequently, (1.2) may be written in the form

(1.8) $$u_t - \nu u_{xx} = f - (Pu)\,u_x .$$

Denoting the scalar product in H by $(.,.)$ and setting $v^j = u^j - u$, we obtain from (1.7) and (1.8)

$$\begin{aligned}
(\tfrac{1}{2}|v^j|^2)_t + \nu\|v^j\|^2 &\leq |(Pu^{j-1}u_x^{j-1} - Pu\cdot u_x\,,\, v^j)| \\
&= |(Pu^{j-1}v_x^{j-1} + (Pu^{j-1} - Pu)u_x\,,\, v^j)| \\
&\leq (|Pu^{j-1}|\,\|v^{j-1}\| + |Pu^{j-1} - Pu|\,\|u\|)\,|v^j|_\infty \\
&\leq (r\|v^{j-1}\| + |v^{j-1}|\,\|u\|)\,|v^j|_\infty \\
&\leq \tfrac{\nu}{8}\|v^{j-1}\|^2 + \tfrac{2r^2}{\nu}|v^j|_\infty^2 + \tfrac{1}{8}\|u\|^2|v^{j-1}|^2 + 2|v^j|_\infty^2 \\
&\leq \tfrac{\nu}{8}\|v^{j-1}\|^2 + \tfrac{1}{8}\|u\|^2|v^{j-1}|^2 + 4(\tfrac{r^2}{\nu}+1)|v^j|\,\|v^j\| \\
&\leq \tfrac{\nu}{8}\|v^{j-1}\|^2 + \tfrac{1}{8}\|u\|^2|v^{j-1}|^2 + \tfrac{8}{\nu}(\tfrac{r^2}{\nu}+1)^2|v^j|^2 + \tfrac{\nu}{2}\|v^j\|^2
\end{aligned}$$

or

$$\begin{aligned}
(|v^j|^2)_t + \nu\|v^j\|^2 &\leq \tfrac{\nu}{4}\|v^{j-1}\|^2 + \tfrac{1}{4}\|u\|^2|v^{j-1}|^2 + \tfrac{16}{\nu}(\tfrac{r^2}{\nu}+1)^2|v^j|^2 \\
&\leq \tfrac{\nu}{4}\|v^{j-1}\|^2 + \tfrac{k'}{2}(\tfrac{1}{4}\|v^{j-1}\|^2 + |v^j|^2) .
\end{aligned}$$

Integration with respect to t yields

$$\begin{aligned}
|v^j(t)|^2 + \nu\int_0^t\|v^j\|^2 ds &\leq \int_0^t(\tfrac{\nu}{4}\|v^{j-1}\|^2 + (\tfrac{1}{8}|v^{j-1}|^2 + \tfrac{1}{2}|v^j|^2)e^{-k}k'e^k)\,ds \\
&\leq \tfrac{\nu}{4}\int_0^t\|v^{j-1}\|^2 ds + (\tfrac{1}{8}\|v^{j-1}\|_{C,k}^2 + \tfrac{1}{2}\|v^j\|_{C,k}^2)(e^{k(t)} - 1).
\end{aligned}$$

We divide by $e^{k(t)}$ and obtain

$$e^{-k(t)}(\|v^j(t)\|^2 + \nu\int_0^t\|v^j\|^2 ds) \leq \tfrac{1}{4}\|v^{j-1}\|_{X,k}^2 + \tfrac{1}{2}\|v^j\|_{C,k}^2$$

and hence

$$\|v^j\|_{X,k}^2 \leq \tfrac{3}{4}\|v^{j-1}\|_{X,k}^2 \leq \dots \leq (\tfrac{3}{4})^j\|v^0\|_{X,k}^2 .$$

From this our theorem follows.

Remark 1.1. Of course, the constant r in (1.6) can be replaced by any other $C(S;H)$ a priori estimate for u. So in the special case $f=0$, $a\ L\in(0,1)$ one can set $r=\|a\|_\infty$ because of the maximum principle. If $f\in L^2(S;H)$, it is easy to see that $r=\sqrt{2}(|a| + \sqrt{2T}\|f\|_{L^2(S;H)})$

is a suitable bound. It is worth noticing that both these estimates are independent of the viscosity ν.

Remark 1.2. Evidently, there are other possibilities to introduce a projector like P in order to obtain a globally convergent version of the Kato-Fujita method. However, the operator P defined by (1.6) turns out to be favourable with respect to the numerical realisation of the iteration process.

2. The Navier-Stokes equations in two space dimensions

Let G be a bounded domain in R^2 with smooth boundary Γ and let $L^2(G)$, $H_0^1(G)$, $H^2(G)$, $H^{-1}(G)$ be the usual Hilbert spaces. We set

$$H = (H_0^1(G))^2 = H_0^1(G)\times H_0^1(G)\ ,\quad V = (H^2(G)\cap H_0^1(G))^2\ ,\quad V^* = (L^2(G))^2$$

and use again the symbols $|\cdot|$, $\|\cdot\|$, $\|\cdot\|_*$ to denote the norms in H, V and V^*, respectively.

Let us consider the spatially two-dimensional Navier-Stokes equations

$$\begin{aligned} &u_t - \nu\Delta u + u\cdot\nabla u + \nabla p = f\ ,\quad \nabla\cdot u = 0 \quad \text{in} \quad G\ , \\ &u(0,x) = a(x)\ ,\quad u|_\Gamma = 0\ . \end{aligned} \tag{2.1}$$

Throughout this section we assume that

$$f\in L^2(S;V^*)\ ,\quad a\in H\ ,\quad \nabla\cdot a = 0\ .$$

Then (cf. /5/), (2.1) has a unique solution (u,p) with

$$u\in L^2(S;V)\cap C(S;H)\ ,\quad u_t\in L^2(S;V^*)\ ,\quad p\in L^2(S;V^*)$$

and the following a priori estimate holds

$$\|u\|_{C(S;H)} \le r\ ,\quad r^2 = c\,(\,|a|^2 + \|f\|^2_{L^2(S;V^*)})\ , \tag{2.2}$$

where the constant c depends only on ν and G.

We now turn to the formulation of a globally convergent iteration procedure for solving (2.1). To this purpose we introduce the projector of V^* onto the r-ball in V^* which is defined by

$$P\,v = \begin{cases} v & \text{if } \|v\|_* \le r \\ \dfrac{r\,v}{\|v\|_*} & \text{if } \|v\|_* > r\ , \end{cases}$$

where r is the constant from (2.2). Now we are able to present the announced iteration procedure

$$(2.3)\quad \begin{aligned} &u_t^j - \nu\Delta u^j + \nabla p^j = f - u^{j-1}\cdot P\nabla u^{j-1}\ , \nabla\cdot u^j = 0\ ,\ j=1,2,\ldots,\\ &u^j(0,x) = a(x)\ ,\quad u^j\big|_\Gamma = 0\ . \end{aligned}$$

Theorem 2. Let (u,p) be the solution of (2.1), $u^0 \in L^2(S;V)\cap C(S;H)$ an arbitrary starting function and $((u^j,p^j))$ the iteration sequence defined by (2.3). Then the following assertions hold

$$u^j \longrightarrow u \ \text{ in } \ C(S;V^*) \ \text{ and } \ L^2(S;H)\ , \qquad \nabla p^j \longrightarrow \nabla p \ \text{ in } \ (L^2(S;H^{-1}))^2.$$

Proof. We need the following well known inequalities

$$\|Pv - Pw\|_* \leq \|v-w\|_*\ ,\ v,w\in V\ ,\qquad \|v\|_4^2 \leq c\|v\|_*\,|v|\ ,\quad v\in H\ .$$

Here the constant c depends on G and ν, $|\cdot|_4$ denotes the $L^4(G)$-norm. Let $(.,.)$ be the scalar product in V^*. Then, using $P\nabla u = \nabla u$ and setting $v^j = u^j - u$, we find from (2.1) and (2.3)

$$\begin{aligned}
(\tfrac{1}{2}\|v^j\|_*^2)_t + \nu|v^j|^2 &= |(v^{j-1}P\nabla u^{j-1} + u\cdot(P\nabla u^{j-1} - P\nabla u)\ ,\ v^j)|\\
&\leq (\|v^{j-1}\|_4\|P\nabla u^{j-1}\|_* + \|u\|_4\|P\nabla v^{j-1} - P\nabla u\|_*)\|v^j\|_4\\
&\leq c_1(\|v^{j-1}\|_4 + \|\nabla v^{j-1}\|_*)\,\|v^j\|_4\\
&\leq c_1(\|v^{j-1}\|_*^{1/2}|v^{j-1}|^{1/2} + |v^{j-1}|)\|v^j\|_*^{1/2}|v^j|^{1/2}\\
&\leq \tfrac{\nu}{2}(\tfrac{1}{4}|v^{j-1}|^2 + |v^j|^2) + \tfrac{c_2}{4}(\tfrac{1}{4}\|v^{j-1}\|_*^2 + \|v^j\|_*^2\ )
\end{aligned}$$

or

$$(\|v^j\|_*^2)_t + \nu|v^j|^2 \leq \tfrac{\nu}{4}|v^{j-1}|^2 + \tfrac{c_2}{2}(\tfrac{1}{4}\|v^{j-1}\|_*^2 + \|v^j\|_*^2)\ .$$

Now we introduce the norms

$$\|v\|_{C,k} = \sup_{t\in S}(e^{-k(t)}\|v(t)\|_*)\ ,\quad \|v\|_{X,k}^2 = \tfrac{1}{2}\|v\|_{C,k}^2 + \nu\sup_{t\in S}(e^{-k(t)}\int_0^t|v|^2\,ds\ ,$$

being equivalent to the usual norms in $C(S;V^*)$ and $X=C(S;V^*)\cap L^2(S;H)$, respectively. Here $k(t) = c_2 t$. As in the proof of Theorem 1 we then obtain

$$\|v^j\|_{X,k} \leq (\tfrac{3}{4})^j\|v^0\|_{X,k}$$

and hence $u^j \longrightarrow u$ in $C(S;V^*)$ and $L^2(S;H)$. Using (2.1) and (2.3), we conclude from the last convergence statement firstly $u_t^j \longrightarrow u_t$ in $L^2(S;H^{-1}(G))$ and after that $\nabla p_j \longrightarrow \nabla p$ in $L^2(S;H^{-1}(G))$.

3. The nonlinear Schrödinger equation

In this section $L^2(0,1)$ denotes the space of complex-valued quadratically integrable functions on $(0,1)$. We set

$$H = L^2(0,1)\ ,\quad V = \{\ v\in H\ \mid v_x\in H\ ,\ v(0) = v(1)\}$$

and use now the symbol $\|\cdot\|$ to denote the norm in H, whereas $|z|$

is the modulus of the complex number z .

We consider the nonlinear Schrödinger equation with spatially periodic boundary conditions

$$(3.1)\quad \begin{aligned} & i\,u_t + u_{xx} + k|u|^2 u = 0\ , \quad i^2 = -1\ , \\ & u(0,x) = a(x)\ ,\ u(t,0) = u(t,1)\ ,\ u_x(t,0) = u_x(t,1)\ . \end{aligned}$$

Here k is a real constant.

We suppose $a \in V$. Then (3.1) has a unique solution $u \in C(S;H) \cap L^{\infty}(S;V)$ with $u_t \in L^{\infty}(S;V^*)$. Moreover, u satisfies the a priori estimate (cf. /4/)

$$\|u\|_{L^{\infty}((0,T)\times(0,1))} \leq r\ ,$$

where

$$r^2 = \|a\|(\|a\| + 2(|k|\,\|a\|^4(1 + |k|\,\|a\|^2) + \left|2\|a_x\|^2 - k\|a\|^4_{L^4(0,1)}\right|)^{1/2})\ .$$

This time we choose as the operator P the projector of the complex plane onto the r-circle, i.e.

$$P\,z = \begin{cases} z & \text{if } |z| \leq r \\ r\dfrac{z}{|z|} & \text{if } |z| > r\ . \end{cases}$$

Now we can formulate a globally convergent iteration method for solving (3.1).

$$(3.2)\quad \begin{aligned} & i\,u^j_t + u^j_{xx} = -k\,|Pu^{j-1}|^2 u^{j-1}\ , \qquad j=1,2,\dots, \\ & u^j(0,x) = a(x)\ ,\ u^j(t,0) = u^j(t,1)\ ,\ u^j_x(t,0) = u^j_x(t,1)\ . \end{aligned}$$

Theorem 3. Let u be the solution of (3.1), $u^0 \in L^2(S;V) \cap C(S;H)$ an arbitrary starting function. Then the sequence (u^j) defined by (3.2) converges to u in $C(S;H)$.

The proof of this theorem as well as proofs of further convergence statements concerning the iteration process (3.2) may be found in /4/.

4. Numerical realisation

The iteration processes under consideration reduce the problem of solving nonlinear evolution equations to the successive solution of sequences of linear evolution equations with constant coefficients. Nevertheless for numerical purposes it is necessary to combine them with other approximation methods. We have made some good numerical experience by combining iteration processes with a time-discrete Galerkin method. Let us briefly discuss this point. We confine ourself to Burgers' equation and use the notation introduced in section 1.

As basis functions we choose

$$h_l = h_l(x) = \sqrt{2}\sin l\pi x \, , \quad l=1,2,\ldots \, .$$

The initial value a has then the representation

$$a = \sum_{l=1}^{n} a_l h_l \, , \quad a_l = \int_0^1 a h_l \, dx \, .$$

We set $u_n = \sum_{l=1}^{n} c_l h_l$ and determine the coefficients $c_l = c_l(t)$ according to Galerkin's method by the following system of nonlinear ordinary differential equations

$$\text{(4.1)} \qquad \begin{aligned} & c_l' + p_l c_l + \int_0^1 (u_n(u_n)_x - f) h_l \, dx = 0 \, , \quad p_l = \nu(l\pi)^2 \, , \\ & c_l(0) = a_l \, , \quad l = 1,\ldots,n \, . \end{aligned}$$

Taking into account (1.2), it is easy to show that the sequence (u_n) of Galerkin approximations converges to the solution u of Burgers' equation in $L^2(S;H_0^1(0,1))$ and $C(S;L^2(0,1))$.

In order to calculate u_n we use an iteration process like (1.7). We set $u_n^j = \sum_{l=1}^{n} c_l^j h_l$ and determine the coefficients $c_l^j = c_l^j(t)$ by the system of linear ordinary differential equations

$$\begin{aligned} & (c_l^j)' + p_l c_l^j = \int_0^1 (f - P u_n^{j-1}(u_n^{j-1})_x) h_l \, dx \, , \quad j = 1,2,\ldots \, , \\ & c_l^j(0) = a_l \, , \quad l = 1,\ldots,n \, . \end{aligned}$$

The solution of this system is

$$\text{(4.2)} \qquad c_l^j(t) = \exp(-p_l t)(a_l + \int_0^t \exp(p_l s) F_l^{j-1}(s) \, ds) \, , \quad l = 1,\ldots,n \, ,$$

where

$$F_l^{j-1}(s) = F_l(s, u_n^{j-1}(s))$$

and the function $F_l(s,.)$ is defined by

$$F_l(s,v) = \int_0^1 (f(s) - v v_x) h_l \, dx \cdot \begin{cases} 1 & \text{if } |v| \le r \\ \frac{r}{|v|} & \text{if } |v| > r \, . \end{cases}$$

Here $|.|$ denotes the norm in $H = L^2(0,1)$ and r is the a priori bound given in (1.3). We see that in order to get u_n^j from u_n^{j-1} we have only to calculate definite integrals. This can be done by using suitable rules for numerical integration. In our calculations it turned out to be adventageous to divide the time interval in smaller intervals S_k , $S = \bigcup_{k=1}^{m} S_k$, and to carry out the iteration successively in S_k , $k=1,\dots,m$.

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Author's address: Akademie der Wissenschaften der DDR,
Zentralinstitut für Mathematik und Mechanik,
DDR - 108 Berlin, Mohrenstr. 39