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In: Jiří Fábera (ed.): Equadiff IV, Czechoslovak Conference on Differential Equations and Their Applications. Proceedings, Prague, August 22-26, 1977. Springer-Verlag, Berlin, 1979. Lecture Notes in Mathematics, 703. pp. [86]--92.

Persistent URL: http://dml.cz/dmlcz/702207

SINGULAR PERTURBATIONS AND LINEAR FEEDBACK CONTROL

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1. Introduction

The results reported here center around the classical regulator problem: a linear control system with a quadratic cost function. We shall consider two situations; the first one corresponds to the case of fast variables in the control system, the second one to "cheap control". A crucial point in solving these problems is the behaviour of the optimal cost and to study it one has to consider singularly perturbed matrix Riccati differential equations. References for these problems are [1] , [2] , [3] .

2. The control problems and the associated Riccati differential equations

A. Let the control system be

$$\varepsilon\dot{x} = A(t)x + B(t)u, \quad x(t_0) = x_0$$

$$J(u) = x^*(T)Gx(T) + \int_{t_0}^{T}\left[x^*(t)F(t)x(t)+u^*(t)H(t)u(t)\right]dt$$

$$G \geq 0, \ F(t) \geq 0, \ H(t) > 0 .$$

The matrix Riccati equation giving the optimal cost is

$$\dot{P} = -\frac{1}{\varepsilon}A^*(t)P - \frac{1}{\varepsilon}PA(t) + \frac{1}{\varepsilon^2}PB(t)H^{-1}(t)B^*(t)P - F(t)$$

$$P(T, \varepsilon) = G$$

and for $P(t, \varepsilon) = \varepsilon R(t, \varepsilon)$ we get

(1) $$\varepsilon\dot{R} = -A^*(t)R - RA(t) + RM(t)R - F(t), \quad R(T, \varepsilon) = \frac{1}{\varepsilon}G .$$

The problem is to study the behaviour of

$$R(t, \varepsilon) \ \text{as} \ \varepsilon \to 0 .$$

B. The "cheap control" problem is defined by

$$\dot{x} = A(t)x + B_0(t)u_0 + B_1(t)u_1 , \quad x(t_0) = x_0$$

$$J(u) = x^*(T)Gx(T) + \int_{t_0}^{T}\left[x^*(t)F(t)x(t) + u_0^*(t)H_0(t)u_0(t) + \varepsilon^2 u_1^*(t)H_1(t)u_1(t)\right]dt .$$

The associated Riccati equation is

$$\varepsilon^2\dot{P} = -\varepsilon^2\left[A^*(t)P + PA(t) - PM_0(t)P + F(t)\right] + PB_1(t)H_1^{-1}(t)B_1(t)P ,$$

$$P(T, \varepsilon) = G .$$

The problem is again to study the behaviour of $P(t,\varepsilon)$ as $\varepsilon \longrightarrow 0$.

3. The singularly perturbed Riccati equations

A. Let $M(t) = D^*(t)D(t)$, $F(t) = C^*(t)C(t)$, $(A^*(t),C(t))$ completely controllable, $(A(t),D^*(t))$ stabilizable, A,C,D, Lipschitz. Let $\hat{R}(t)$ be the unique positive definite solution of the equation

$$A^*(t)R + RA(t) - RM(t)R + F(t) = 0 .$$

Let $R(t,\varepsilon)$ be the solution of the Cauchy problem (1). Then $\lim_{\varepsilon \to 0} R(t,\varepsilon) = \hat{R}(t)$ for $t < T$.

To prove this result we first consider the solution $\tilde{R}(t,\varepsilon)$ of the equation in (1) with $\tilde{R}(T,\varepsilon) = \hat{R}(T)$ and the solution $R_0(t,\varepsilon)$ of the same equation with $R_0(T,\varepsilon) = 0$.

Denote by $\hat{C}(s,t,\varepsilon)$ the fundamental matrix solution of the system $\varepsilon x' = \hat{A}(s)x$ where $\hat{A}(t) = A(t) - M(t)R(t)$ is Hurwitz for every t .

We use the representation formulae

$$\tilde{R}(t,\varepsilon) = \hat{C}^*(T,t,\varepsilon)\hat{R}(T)\hat{C}(T,t,\varepsilon) +$$
$$+ \frac{1}{\varepsilon}\int_t^T \hat{C}^*(s,t,\varepsilon)\left[F(s) + \hat{R}(s)M(s)\hat{R}(s)\right]\hat{C}(s,t,\varepsilon)ds -$$
$$- \frac{1}{\varepsilon}\int_t^T \hat{C}^*(s,t,\varepsilon)\left[\tilde{R}(s,\varepsilon) - \hat{R}(s)\right]M(s) .$$
$$.\left[\tilde{R}(s,\varepsilon) - \hat{R}(s)\right]\hat{C}(s,t,\varepsilon)ds ,$$

$$\hat{R}(t) = \exp\left(\hat{A}^*(t)\frac{T-t}{\varepsilon}\right)\hat{R}(t)\exp\left(\hat{A}(t)\frac{T-t}{\varepsilon}\right) +$$
$$+ \frac{1}{\varepsilon}\int_t^T \exp\left(\hat{A}^*(t)\frac{s-t}{\varepsilon}\right)\left[F(t) + \hat{R}(t)M(t)\hat{R}(t)\right] .$$
$$.\exp\left(\hat{A}(t)\frac{s-t}{\varepsilon}\right)ds$$

and

$$\hat{C}(s,t,\varepsilon) = \exp\left(\hat{A}(t)\frac{s-t}{\varepsilon}\right) + \omega(s,t,\varepsilon) ,$$

$$|\omega(s,t,\varepsilon)| \le \frac{\varepsilon \ell k^2}{2}\left(\frac{s-t}{\varepsilon}\right)^2 \exp\left(-\alpha\left(\frac{s-t}{\varepsilon}\right)\right)$$

to obtain $|\tilde{R}(t,\varepsilon) - \hat{R}(t)| \le k\varepsilon$.

Denote next $S(t,\varepsilon) = R_0(t,\varepsilon) - \tilde{R}(t,\varepsilon)$, and let $\tilde{C}(t,t_0,\varepsilon)$ be the fundamental matrix solution of the system $\varepsilon\dot{x} = \tilde{A}(t,\varepsilon)x$, where $\tilde{A}(t,\varepsilon) = A(t) - M(t)\tilde{R}(t,\varepsilon)$.

We obtain the representation formula

$$S(t,\varepsilon) = -\widetilde{C}^*(T,t,\varepsilon)\Big[\hat{R}^{-1}(T) - \frac{1}{\varepsilon}\int_t^T \widetilde{C}(T,s,\varepsilon)M(s)\cdot$$
$$\cdot\widetilde{C}^*(T,s,\varepsilon)ds\Big]^{-1}\widetilde{C}(T,t,\varepsilon)$$

and a lemma in singular perturbations [1] gives

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_t^T \widetilde{C}(T,s,\varepsilon)M(s)\widetilde{C}^*(T,s,\varepsilon)ds = \hat{L}$$

where $\hat{A}(T)\hat{L} + \hat{L}\hat{A}^*(T) = -M(T)$.-

The representation formula gives then the estimate

$$|S(t,\varepsilon)| \le k' \exp\left(-\alpha'\left(\frac{T-t}{\varepsilon}\right)\right).$$

To perform the last step denote $S_0(t,\varepsilon) = R(t,\varepsilon) - R_0(t,\varepsilon)$ and let $C_0(s,t,\varepsilon)$ be the fundamental matrix solution of the system $\varepsilon x'(s) = A_0(s,\varepsilon)x(s)$ with $A_0(t,\varepsilon) = A(t) - M(t)R_0(t,\varepsilon)$.

Then a representation formula for S_0 gives

$$S_0(t,\varepsilon) \le C_0^*(T,t,\varepsilon)\frac{1}{\varepsilon}GC_0(T,t,\varepsilon)$$

and since $|C_0(s,t,\varepsilon)| \le k\exp\left(-\alpha\left(\frac{s-t}{\varepsilon}\right)\right)$ the result is proved.

B. For the "cheap control" problem assume

$$\widetilde{F}(t) = B_1^*(t)F(t)B_1(t) > 0, \quad \widetilde{G} = B_1^*(T)GB_1(T) > 0 .$$

Define

$$Q(t,\varepsilon) = \frac{1}{\varepsilon}B_1^*(t)P(t,\varepsilon), \quad R(t,\varepsilon) = \frac{1}{\varepsilon}B_1^*(t)P(t,\varepsilon)B_1(t),$$
$$\hat{F}(t) = B_1^*(t)F(t), \quad \hat{G} = B_1^*(T)G .$$

Then $P(t,\varepsilon)$, $Q(t,\varepsilon)$, $R(t,\varepsilon)$ is a solution of the problem

$$\dot{P} = -A^*(t)P - PA(t) + PM_0(t)P + QH_1^{-1}(t)Q - F(t),$$
$$\varepsilon\dot{Q} = RH_1^{-1}(t)Q - \varepsilon QA(t) + \varepsilon QM_0(t)P - B_2^*(t)P - \hat{F}(t),$$
$$\varepsilon\dot{R} = RH_1^{-1}(t)R - \widetilde{F}(t) - \varepsilon B_2^*(t)Q - \varepsilon QB_2(t) +$$
$$+ \varepsilon^2 QM_0(t)Q,$$
$$P(T,\varepsilon) = G,\ Q(T,\varepsilon) = \frac{1}{\varepsilon}\hat{G},\quad R(T,\varepsilon) = \frac{1}{\varepsilon}\widetilde{G} .$$

We start with the equation

$$\varepsilon\dot{R} = RH_1^{-1}(t)R - \widetilde{F}(t) .$$

Denote $\widetilde{R}(t)$ the unique positive definite stabilizing solution of the algebraic equation $RH_1^{-1}(t)R = \widetilde{F}(t)$.

Denote $R_0(t,\varepsilon)$ the solution of the differential equation with $R_0(T,\varepsilon) = 0$ and $\widetilde{R}_0(t,\varepsilon)$ the solution of the same equation with $\widetilde{R}_0(T,\varepsilon) = \frac{1}{\varepsilon}\widetilde{G}$.

We have the representation

$$\widetilde{R}_0(t,\varepsilon) - R_0(t,\varepsilon) = C_0^*(T,t,\varepsilon)U^{-1}(t,\varepsilon)C_0(T,t,\varepsilon) ,$$

$$U(t,\varepsilon) = \varepsilon\widetilde{G}^{-1} + \frac{1}{\varepsilon}\int_t^T C_0(T,\tau,\varepsilon)H_1^{-1}(\tau)C_0^*(T,\tau,\varepsilon)d\tau$$

where C_0 is defined by

$$\varepsilon \frac{dC}{ds} = -H_1^{-1}(s)R_0(s,\varepsilon)C, \quad C_0(t,t,\varepsilon) = E .$$

A crucial point in the proof is the estimate

$$|U^{-1}(t,\varepsilon)| \le \frac{2\beta}{2\beta\rho\varepsilon + 1 - \exp\left(-2\alpha\left(\frac{T-t}{\varepsilon}\right)\right)} .$$

Define $\widetilde{Q}_0$ as the solution of the Cauchy problem

$$\varepsilon\dot{Q} = \widetilde{R}_0(t,\varepsilon)H_1^{-1}(t)Q - \hat{F}_1(t) , \quad \widetilde{Q}_0(T,\varepsilon) = \frac{1}{\varepsilon}\widetilde{G} .$$

We prove that

$$\lim_{\varepsilon\to 0} \widetilde{Q}_0(t,\varepsilon) = \hat{Q}_0(t) \quad \text{for} \quad t < T$$

where $\hat{Q}_0(t) = H_1(t)\widetilde{R}^{-1}(t)\hat{F}_1(t)$.

Consider now $\widetilde{P}$ defined by

$$\varepsilon\frac{d}{dt}\widetilde{P}(t,\varepsilon) = \widetilde{Q}_0^*(t,\varepsilon)H_1^{-1}(t)\widetilde{Q}_0(t,\varepsilon) - \hat{Q}_0^*(t)H_1^{-1}(t)\hat{Q}_0(t).$$

A long series of estimates give finally

$$|\widetilde{P}(t,\varepsilon)| \le \mu + \frac{\gamma}{2\beta\rho\varepsilon + 1 - \exp\left(-2\alpha\left(\frac{T-t}{\varepsilon}\right)\right)} .$$

Let us state now the final result.

Define P_0 from the Cauchy problem

$$\begin{aligned}\dot{P}_0 = &-[A^*(t) - \hat{F}^*(t)\widetilde{F}^{-1}(t)B_2^*(t)]P_0 - P_0[A(t) - \\ &- B_2(t)\widetilde{F}^{-1}(t)\hat{F}_1(t) + P_0[M_0(t) + B_2(t)\widetilde{F}^{-1}(t)B_2^*(t)]P_0 - \\ &- [F(t) - \hat{F}^*(t)\widetilde{F}^{-1}(t)\hat{F}(t)] ,\end{aligned}$$

$$P_0(T) = G - \hat{G}^*\widetilde{G}^{-1}\hat{G} .$$

Let $\widetilde{Q}_0$ and $\widetilde{P}$ be defined as above with $\hat{F}_1$ replaced by $B_2^*(t)P_0(t) + \hat{F}(t)$.

Then

$$\begin{aligned}P(t,\varepsilon) &= P_0(t) + \varepsilon\widetilde{P}(t,\varepsilon) + \varepsilon P_1(t,\varepsilon)\\ Q(t,\varepsilon) &= \widetilde{Q}_0(t,\varepsilon) + \sqrt{\varepsilon}Q_1(t,\varepsilon)\\ R(t,\varepsilon) &= \widetilde{R}_0(t,\varepsilon) + \varepsilon R_1(t,\varepsilon)\end{aligned}$$

where P_1, Q_1, R_1 are bounded for $t \le T$.

To prove this result we denote

$$\mathcal{P} = \begin{pmatrix} P_1 & Q_1^* \\ Q_1 & R_1 \end{pmatrix}$$

and show that $\mathcal{P}$ is the solution of the problem

$$\varepsilon\dot{\mathcal{P}} = -\mathcal{A}^*(t,\varepsilon)\mathcal{P} - \mathcal{P}\mathcal{A}(t,\varepsilon) + \mathcal{P}\mathcal{M}(t,\varepsilon)\mathcal{P} - \mathcal{F}(t,\varepsilon),$$

$$\mathcal{P}(T,\varepsilon) = 0$$

where

$$\mathcal{A} = \begin{pmatrix} \mathcal{A}_{11} & \mathcal{A}_{12} \\ \mathcal{A}_{21} & \mathcal{A}_{22} \end{pmatrix}, \quad \mathcal{M} = \begin{pmatrix} \mathcal{M}_{11} & 0 \\ 0 & \mathcal{M}_{22} \end{pmatrix}, \quad \mathcal{F} = \begin{pmatrix} \mathcal{F}_{11} & \mathcal{F}_{12} \\ \mathcal{F}_{12}^* & \mathcal{F}_{22} \end{pmatrix},$$

$$\mathcal{A}_{11}(t,\varepsilon) = \varepsilon[A(t) - M_0(t)(P_0(t) + \varepsilon\widetilde{P}(t,\varepsilon))],$$
$$\mathcal{A}_{12}(t,\varepsilon) = \sqrt{\varepsilon}[B_2^*(t) - M_0(t)\widetilde{Q}_0^*(t,\varepsilon)],$$
$$\mathcal{A}_{21}(t,\varepsilon) = \sqrt{\varepsilon}H_1^{-1}(t)\widetilde{Q}_0(t,\varepsilon),$$
$$\mathcal{A}_{22}(t,\varepsilon) = -H_1^{-1}(t)\widetilde{R}_0(t,\varepsilon),$$
$$\mathcal{M}_{11}(t,\varepsilon) = \varepsilon^2 M_0(t), \qquad \mathcal{M}_{22}(t,\varepsilon) = \varepsilon H_1^{-1}(t),$$
$$\mathcal{F}_{11}(t,\varepsilon) = \varepsilon A^*(t)\widetilde{P}(t,\varepsilon) + \varepsilon\widetilde{P}(t,\varepsilon)A(t) - \varepsilon\widetilde{P}(t,\varepsilon)M_0(t)(P_0(t) + \varepsilon\widetilde{P}(t,\varepsilon)) - \varepsilon P_0(t)M_0(t)\widetilde{P}(t,\varepsilon),$$
$$\mathcal{F}_{12}(t,\varepsilon) = \sqrt{\varepsilon}B_2^*(t)\widetilde{P}(t,\varepsilon) - \sqrt{\varepsilon}\widetilde{Q}_0(t,\varepsilon)[M_0(t)P_0(t) + \varepsilon M_0(t)\widetilde{P}(t,\varepsilon) - A(t)],$$
$$\mathcal{F}_{22}(t,\varepsilon) = B_2^*(t)\widetilde{Q}_0^*(t,\varepsilon) + \widetilde{Q}_0(t,\varepsilon)B_2(t) - \varepsilon\widetilde{Q}_0(t,\varepsilon)M_0(t)\widetilde{Q}_0^*(t,\varepsilon).$$

We write for $\mathcal{P}$ a nonlinear integral equation, which shows $\mathcal{P}$ is a fixed point of a certain nonlinear integral operator and we get the estimates for $\mathcal{P}$ by proving that this operator maps a ball into itself.

Let us mention also the esimate

$$|\widetilde{R}_0(t,\varepsilon) - \widetilde{R}(t)| \le \varepsilon k_0 + k_1 \exp\left(-\alpha\left(\frac{T-t}{\varepsilon}\right)\right) + \frac{k_2 \exp\left(-2\alpha\left(\frac{T-t}{\varepsilon}\right)\right)}{2\beta\rho\varepsilon + 1 - \exp\left(-2\alpha\left(\frac{T-t}{\varepsilon}\right)\right)},$$

$$|\widetilde{Q}_0(t,\varepsilon) - \hat{Q}_0(t)| \le \frac{k_3 \exp\left(-\alpha\left(\frac{T-t}{\varepsilon}\right)\right)}{2\beta\rho\varepsilon + 1 - \exp\left(-2\alpha\left(\frac{T-t}{\varepsilon}\right)\right)},$$

$$\hat{Q}_0(t) = \widetilde{R}^{-1}(t)[B_2^*(t)P_0(t) + \hat{F}(t)].$$

We obtained in this way the required information concerning the asymptotic behaviour of $P(t,\varepsilon)$. Remark that O'Malley [5] considered the case $\widetilde{F}(t) > 0$, $G = 0$; this case is much simpler since it implies $Q(T,\varepsilon) = 0$, $R(T,\varepsilon) = 0$.

4. Complementary remarks

In the "cheap control" problem if we want to use a suboptimal control that might be simpler to compute we have to consider the behaviour of the solutions of a system of the form

$$\varepsilon\dot{x} = [\varepsilon A(t) + B(t)K^*(t)]x$$

where $K^*(t)B(t)$ is Hurwitz for fixed t.

Let $C_A(t,s)$ be the fundamental matrix solution of the system

$\dot{x} = A(t)x$ and let $y(t,\varepsilon) = C_A(s,t)x(t,\varepsilon)$. Then

$$\begin{aligned}\varepsilon\dot{y}(t,\varepsilon) &= C_A(s,t)[\varepsilon A(t) + B(t)K^*(t)]x(t,\varepsilon) - \\ &- \varepsilon C_A(s,t)A(t)x(t,\varepsilon) = \\ &= C_A(s,t)B(t)K^*(t)C_A(t,s)y(t,\varepsilon)\end{aligned}$$

hence

$$\varepsilon\dot{y}(t,\varepsilon) = \widetilde{B}(t)\widetilde{K}^*(t)y(t,\varepsilon)$$

where

$$\widetilde{B}(t) = C_A(s,t)B(t),\ K^*(t) = K^*(t)C_A(t,s)$$

hence

$$\widetilde{K}^*(t)\widetilde{B}(t) = K^*(t)B(t) \quad \text{is Hurwitz.}$$

Denote by $C(t,s,\varepsilon)$ the fundamental matrix of the given system and by $\widetilde{C}(t,s,\varepsilon)$ the fundamental matrix of the transformed one.

Then $C(t,s,\varepsilon) = C_A(t,s)\widetilde{C}(t,s,\varepsilon)$.

We next prove that

$$|\widetilde{K}^*(t)\widetilde{C}(t,s,\varepsilon)| \le \varepsilon k_2 + (k_1 - \varepsilon k_2)\exp\left(-\alpha\left(\frac{t-s}{\varepsilon}\right)\right)$$

and then

$$|\widetilde{C}(t,s,\varepsilon) - \widetilde{L}(t,s)| \le \varepsilon k + M\exp\left(-\alpha\left(\frac{t-s}{\varepsilon}\right)\right)$$

where $\widetilde{L}(t,s)$ is the solution of the Cauchy problem

$$\dot{y} = -\widetilde{B}(t)[K^*(t)B(t)]^{-1}\widetilde{K}^*(t)y$$

$$\widetilde{L}(s,s) = E - B(s)[K^*(s)B(s)]^{-1}K^*(s)\ .$$

If we denote $L(t,s) = C_A(t,s)L(t,s)$

we have

$$|C(t,s,\varepsilon) - L(t,s)| \le \varepsilon k' + M\exp\left(-\alpha\left(\frac{t-s}{\varepsilon}\right)\right)$$

where $L(t,s)$ is the solution of the Cauchy problem

$$\dot{x} = [A(t) - B(t)(K^*(t)B(t))^{-1}K^*(t)]x\ , \quad L(s,s) = \widetilde{L}(s,s).$$

Let us mention finally the formula

$$\begin{aligned}\exp\left(\frac{1}{\varepsilon}BK^*(t-s)\right) &= E - B(K^*B)^{-1}K^* + \\ &+ B(K^*B)^{-1}\exp\left(\frac{1}{\varepsilon}K^*B(t-s)\right)K^*\ .\end{aligned}$$

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